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SET-GENERATED SOFT SUBRINGS OF RINGS

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Abstract. This paper focuses on the set-oriented operations and set-oriented algebraic structures of soft sets. Relatedly, in this paper, firstly some essential properties of α-intersection of soft set are investigated, where α is a non-empty subset of the universal set. Later, by using α-intersection of soft set, the notion of set-generated soft subring of a ring is introduced. The generators of soft intersections and products of soft subrings are given. Some related properties about generators of soft subrings are investigated and illustrated by several examples.

1. Introduction

Since the modeling of uncertain data in medical science, economics, sociology, environmental science, engineering and many other fields is very complex, it is difficult to successfully deal with them by classical methods. In the last century, many approaches that are useful in modeling uncertainties have been proposed. The fuzzy set theory [1, 2], the interval mathematics [3], vague set theory [4] and rough set theory [5, 6] and are favorable approaches to describing uncertain data, but each of these theories has its own difficulties in classifying data parametrically. To fill this gap, Molodtsov [7] proposed a completely new approach named soft set theory. This approach allowed the uncertain data frequently encountered in many areas to be classified parametrically, thereby providing a better representation of them. In the years following the budding of soft sets, the theoretical and practical aspects of these sets were discussed. Maji et al. conceptualized the some set operations of soft sets [8] and made further efforts to show the implementation of soft sets in

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decision making [9]. Ali et al. [10] introduced some new soft set operations such as the restricted difference, the restricted intersection, the extended intersection and the restricted union. Çağman and Enginoğlu [11] revisited some basic operations of soft sets to make them more efficient in some cases. In [12–14], the authors studied the operations of difference and symmetric difference of soft sets. Aygün and Kamacı [15] developed some functional operations of soft sets and then demonstrated their efficiency in handling decision making problems. Also, they defined XOR and XNOR products of soft sets and derived new soft algebraic structures by using these soft set products [16]. Çağman and Enginoğlu [17] introduced the soft matrices representing soft sets and their handy operations to create a soft max-min decision making procedure which can be successfully applied to the problems containing uncertainties. In [18–21], the researchers discussed specific kinds of soft matrices and construct new improved types of soft max-min decision making procedure. Moreover, the inverse types of soft matrices were investigated and their applications to decision making were presented [22, 23]. Recently, the works on the operations of soft sets and soft matrices are progressing rapidly.

On the other hand, many algebraic structures based on the basic principles and operations of soft sets have been proposed. In 2007, Aktaş and Çağman [24] introduced the rudiments of soft groups and studied their basic properties. Uluçay et al. [25] studied soft representation of soft groups. Feng et al. [26] defined the concepts of soft subsemirings, soft semirings, soft semiring homomorphisms, soft ideals and idealistic soft semirings. In , the authors [27, 28] introduced the fundamentals of soft rings and soft normed rings. In [29], Atagün and Sezgin discussed the algebraic soft substructures of rings and defined soft subring of a ring, soft ideal of a ring, soft submodule of a module and soft subfield of a field. Sezgin et al. [30] expanded the study of soft near-rings, especially according to the idealistic soft near-rings. Ostadhadi-Dehkordi and Shum [31] investigated regular and strongly regular relations on the soft hyperrings. Tahat et al. [32] discussed the characterizations of soft topological soft groups and soft rings. Karaaslan [33] investigated some outstanding properties of collection of soft sets over AG-groupoid, AG-band and AG*-groupoid. In [34], Yousafzai et al. introduced the notion of soft sets in an ordered AG-groupoid and they studied different type ideals and strongly regular elements. Zhan et al. [35] defined some new soft algebraic structures such as (M,N)-soft union hemiring and (M,N)-soft union h-ideal, which are generalisations of soft union hemiring and soft union h-ideal to tackle many uncertainty problems. Atagün and Sezgin [36] described the notions of soft N-subgroups, soft subnear-rings and soft ideals of near-rings and also derived the product operation and bi-intersection of soft N-groups, soft subnear-rings and soft ideals of near-rings. On the other hand, some authors developed soft topology in various aspects and discussed real life examples [37–39].

In [40], Sezer et al. argued that the set-oriented approaches based on inclusion of soft set can be extended the range of operations, algebraic structures, topological structures, application aspects of soft sets. Thus, they defined the lower α-inclusion and upper α-inclusion of a soft set over the universal set U, where $\alpha \subseteq U$. Moreover, by using the upper α-inclusion of a soft set, they proposed the idea of upper α-semigroups for the soft sets. In [41], the authors made some analyzes with respect to group theory and showed that some subgroups of a group can be achieved easily by means of the notions of upper and lower α-inclusions of soft sets. They also demonstrated that a soft uni-group and a soft int-group can be derived by its lower α-subgroup and upper α-subgroup, respectively. In [40, 41], the authors focused on the α-oriented subgroup structures of soft sets. However, the α-oriented subring structure of soft sets is a gap in the literature. By filling this gap, both the theoretical aspects and practical aspects of the soft sets will be contributed. Relatedly, this paper aims to introduce soft subrings of a ring generated by the set α and to investigate their fundamental properties.

This paper is organized as follows. Section 2 recalls the rudiments of soft sets. Section 3 presents a detailed theoretical study for the α-intersection of a soft set. Section 4 introduces a new concept namely a soft subring of a ring generated by a set and gives many remarkable properties of this concept. Also, this section includes our main theorems, in which we examine generator sets under operations soft intersection and product. Some theoretical results are illustrated by several examples. Section 5 consists of the conclusions of the paper and the direction for future studies.

2. Preliminaries

In this section, we recall the rudiments of rings, soft sets and soft subrings.

By a ring, we mean an algebraic system $(\Re, +, .)$, where (the multiplication . will be omitted in formulas)

i) $(\Re, +)$ is a abelian group,
ii) $(\Re, .)$ is a semi-group,
iii) $a.(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in \Re$ (i.e., left and right distributive rules hold)

Throughout this paper, $\Re$ denotes a ring and the zero of $\Re$ is symbolized by $0_\Re$.

A subgroup S of $(\Re, +)$ with $SS \subseteq S$ is named a *subring* of $\Re$ and symbolized by $S < \Re$. Therefore, $S < \Re$ if and only if

i) $S \subseteq \Re$,
ii) $0_\Re \in S$,
iii) $a - b \in S$ for all $a, b \in S$,
iv) $ab \in S$ for all $a, b \in S$.

Molodtsov [7] described the soft set in the following manner:

Let U be a universal set and its power set be $P(U)$, $\mathcal{T}$ be a set of parameters and $\mathcal{X} \subseteq \mathcal{T}$.

Definition 1. *([7]) A pair $(\Psi, \mathcal{X})$ (or simply $\Psi_{\mathcal{X}}$) is termed to be a soft set over U, where Ψ is a mapping described by*

$$\Psi : \mathcal{X} \to P(U).$$

Stated in other words, a soft set over the universal set U can be considered as a parameterized family of the subsets of universal set U. For $t \in \mathcal{X}$, $\Psi(t)$ is the set of t-elements of the soft set $(\Psi, \mathcal{X})$, or simplistically the set of t-approximate elements of this soft set. To support this idea, Molodtsov presented various examples (see [7]). Indeed, there is a mutual correspondence among soft sets and binary relations as given in [42,43]. Namely, let $\mathcal{T}$ and U be non-empty sets and suppose that σ refers to an arbitral binary relation between an element of $\mathcal{T}$ and an element of U. A set-valued function $\Psi : \mathcal{T} \to P(U)$ can be described as $\Psi(t) = \{u \in U \mid (t, u) \in \sigma\}$ for all $t \in \mathcal{T}$. Hence, the pair $(\Psi, \mathcal{T})$ is a soft set over U, which is derived from the relation σ.

Definition 2. *([8]) A soft set $(\Psi, \mathcal{X})$ over U is termed to be a null soft set symbolized by $\Phi_{\mathcal{X}}$, if for all $t \in \mathcal{X}$, $\Psi(t) = \emptyset$ (null set).*

Definition 3. *([8]) A soft set $(\Psi, \mathcal{X})$ over U is termed to be an absolute soft set, if for all $t \in \mathcal{X}$, $\Psi(t) = U$.*

Note that we denote the absolute soft set $(\Psi, \mathcal{X})$ over U by $\mathcal{U}_{\mathcal{X}}$ throughout this paper.

Definition 4. *([10]) The relative complement of a soft set $(\Psi, \mathcal{X})$ is symbolized by $(\Psi, \mathcal{X})^c$ and is defined as $(\Psi, \mathcal{X})^c = (\Psi^c, \mathcal{X})$, where $\Psi^c : \mathcal{X} \to P(U)$ is a mapping given by $\Psi^c(t) = U \setminus \Psi(t)$ for all $t \in \mathcal{X}$.*

Definition 5. *([8, 10]) Let $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ be two soft sets over the universal set U.*

a): *The restricted intersection of $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ is denoted and defined as $(\Psi, \mathcal{X}) \Cap (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$, where $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y} \neq \emptyset$ and $\Theta(t) = \Psi(t) \cap \Upsilon(t)$ for all $t \in \mathcal{Z}$.*

b): *The extended intersection of $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ is denoted and defined as $(\Psi, \mathcal{X}) \sqcap (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$, where $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ and for all $t \in \mathcal{Z}$*

$$\Theta(t) = \begin{cases} \Psi(t), & \text{if } t \in \mathcal{X} \setminus \mathcal{Y} \\ \Upsilon(t), & \text{if } t \in \mathcal{Y} \setminus \mathcal{X} \\ \Psi(t) \cap \Upsilon(t), & \text{if } t \in \mathcal{X} \cap \mathcal{Y} \end{cases}$$

c): *The union intersection of $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ is denoted and defined as $(\Psi, \mathcal{X}) \Cup (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$, where $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y} \neq \emptyset$ and $\Theta(t) = \Psi(t) \cup \Psi(t)$ for all $t \in \mathcal{Z}$.*

d): *The extended union of* $(\Psi, \mathcal{X})$ *and* $(\Upsilon, \mathcal{Y})$ *is denoted and defined as* $(\Psi, \mathcal{X}) \sqcup (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$*, where* $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ *and for all* $t \in \mathcal{Z}$

$$\Theta(t) = \begin{cases} \Psi(t), \text{ if } t \in \mathcal{X} \setminus \mathcal{Y} \\ \Upsilon(t), \text{ if } t \in \mathcal{Y} \setminus \mathcal{X} \\ \Psi(t) \cup \Upsilon(t), \text{ if } t \in \mathcal{X} \cap \mathcal{Y} \end{cases}$$

In 2010, Çağman and Enginoğlu [11] redescribed the approximate function Ψ of soft set $(\Psi, \mathcal{X})$ from $\mathcal{T}$ to $P(U)$ such that $\Psi(t) = \emptyset$ if $t \notin \mathcal{X}$. Thus, they revisited the operations of intersection and union of soft sets as follows:

Definition 6. *([11]) Let* $\Psi_{\mathcal{X}}$ *and* $\Upsilon_{\mathcal{Y}}$ *be soft sets over* U*. Then,*

a): *the soft union of* $\Psi_{\mathcal{X}}$ *and* $\Upsilon_{\mathcal{Y}}$*, denoted by* $\Theta_{\mathcal{Z}} = \Psi_{\mathcal{X}} \widetilde{\cup} \Upsilon_{\mathcal{Y}}$*, is defined as* $\Theta(t) = \Psi(t) \cup \Upsilon(t)$ *for all* $t \in \mathcal{T}$*.*

b): *the soft intersection of* $\Psi_{\mathcal{X}}$ *and* $\Upsilon_{\mathcal{Y}}$*, denoted by* $\Theta_{\mathcal{Z}} = \Psi_{\mathcal{X}} \widetilde{\cap} \Upsilon_{\mathcal{Y}}$*, is defined as* $\Theta(t) = \Psi(t) \cap \Psi(t)$ *for all* $t \in \mathcal{T}$*.*

For more details, it can be reviewed the concepts in [11].

The following definition first introduced the soft substructures of an algebraic structure to the literature.

Definition 7. *([29]) Let* S *be a subring of* $\Re$ *and* (Ψ, S) *be a soft set over* $\Re$*. If for all* $t, v \in S$*,*

s1) $\Psi(t - v) \supseteq \Psi(t) \cap \Psi(v)$,
s2) $\Psi(tv) \supseteq \Psi(t) \cap \Psi(v)$,

then it is said to be a soft subring of $\Re$ *and symbolized by* $(\Psi, S) \widetilde{<} \Re$ *or simplistically* $\Psi_S \widetilde{<} \Re$*.*

Proposition 1. *([29]) If* $\Psi_S \widetilde{<} \Re$*, then* $\Psi(0) \supseteq \Psi(t)$ *for all* $t \in S$*.*

Theorem 1. *([29]) If* $\Psi_{S_1} \widetilde{<} \Re$ *and* $\Upsilon_{S_2} \widetilde{<} \Re$*, then* $\Psi_{S_1} \Cap \Upsilon_{S_2} \widetilde{<} \Re$*.*

Definition 8. *([26]) Let* $(\Psi, \mathcal{X})$ *be soft set over* U*. Then, the set*

$$supp(\Psi, \mathcal{X}) = \{t \in \mathcal{X} |\ \Psi(t) \neq \emptyset\}$$

is said to be the support of the soft set $(\Psi, \mathcal{X})$*. A soft set* $(\Psi, \mathcal{X})$ *is called non-null if* $supp(\Psi, \mathcal{X}) \neq \emptyset$*.*

3. Some Aspects on α-Intersection of Soft Sets

In this section, we present some theoretical findings for the α-intersection of soft sets.

Definition 9. *([44]) Let* $(\Psi, \mathcal{X})$ *be a soft set over* U *and* $\emptyset \neq \alpha \subseteq U$*. Then, the subset of* $\mathcal{X}$ *given by*

$$(\Psi, \mathcal{X})^{\cap \alpha} = \{t \in \mathcal{X} |\ \Psi(t) \cap \alpha \neq \emptyset\}$$

is called the α*-intersection of* $(\Psi, \mathcal{X})$*.*

It seen that if $\alpha = U$ and $\Psi(t) \neq \emptyset$ for all $t \in \mathcal{X}$, then $(\Psi, \mathcal{X})^{\cap U} = \mathcal{X}$.

Proposition 2. *Let $(\Psi, \mathcal{X})$ be a soft set over U and let $\emptyset \neq \alpha \subseteq U$.Then*

i) $(\Psi, \mathcal{X})^{\cap\alpha} \subseteq supp(\Psi, \mathcal{X})$.
ii) *If $\alpha \subseteq \Psi(t)$ for all $t \in \mathcal{X}$, then $(\Psi, \mathcal{X})^{\cap\alpha} = supp(\Psi, \mathcal{X}) = \mathcal{X}$.*
iii) *If $\Psi(t) \neq \emptyset$ and $\Psi(t) \subseteq \alpha$ for all $t \in \mathcal{X}$, $(\Psi, \mathcal{X})^{\cap\alpha} = supp(\Psi, \mathcal{X}) = \mathcal{X}$.*
iv) *If $(\Psi, \mathcal{X}) = \mathcal{U}_{\mathcal{X}}$, then $(\Psi, \mathcal{X})^{\cap\alpha} = supp(\Psi, \mathcal{X}) = \mathcal{X}$.*
v) *If $(\Psi, \mathcal{X}) = \Phi_{\mathcal{X}}$ or $supp(\Psi, \mathcal{X}) = \emptyset$, then $(\Psi, \mathcal{X})^{\cap\alpha} = \emptyset$.*

Proof. The proof of (i) is seen from the Definitions 8 and 9.
(ii) Since $\alpha \subseteq \Psi(t)$ for all $t \in \mathcal{X}$ and $\emptyset \neq \alpha \subseteq U$, then $\Psi(t) \neq \emptyset$ for all $t \in \mathcal{X}$ and $supp(\Psi, \mathcal{X}) = \mathcal{X}$. Under the assumption $\Psi(t) \cap \alpha \neq \emptyset$ for all $t \in \mathcal{X}$, then $supp(\Psi, \mathcal{X}) \subseteq (\Psi, \mathcal{X})^{\cap\alpha}$. Hence the equality obtained from (i).
The proof of (iii) is similar to proof of (ii). The rest of the proof is easily seen. □

Proposition 3. *Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha \subseteq U$.Then*

i) *If $\alpha \subseteq \beta$, then $(\Psi, \mathcal{X})^{\cap\alpha} \subseteq (\Psi, \mathcal{X})^{\cap\beta}$.*
ii) $(\Psi^c, \mathcal{X})^{\cap\alpha} = \{t \in \mathcal{X} |\ \alpha \setminus \Psi(t) \neq \emptyset\}$.
iii) $(\Psi, \mathcal{X})^{\cap(U\setminus\alpha)} = \{t \in \mathcal{X} |\ \Psi(t) \setminus \alpha \neq \emptyset\}$.
iv) $(\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)} = \{t \in \mathcal{X} |\ U \setminus (\Psi(t) \cup \alpha) \neq \emptyset\}$.

Proof. If $\alpha \subseteq \beta$, then $\Psi(t) \cap \alpha \neq \emptyset$ implies $\Psi(t) \cap \beta \neq \emptyset$. Hence the proof of (i) is done. The rest of proof is obtained using algebraic operations, easily. □

Proposition 4. *Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha \subseteq U$. If $(\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)} = \emptyset$ then $(\Psi, \mathcal{X})^{\cap\alpha} \cup (\Psi^c, \mathcal{X})^{\cap\alpha} \cup (\Psi, \mathcal{X})^{\cap(U\setminus\alpha)} = \mathcal{X}$.*

Proof. Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha \subseteq U$. We assume that $(\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)} = \emptyset$. Then, by Proposition 3 (iv), we have $\Psi(t) \cup \alpha = U$ for all $t \in \mathcal{X}$. Hence, the proof is obvious from Definition 9 and Proposition 3 (ii) and (iii). □

Proposition 5. *Let $(\Psi, \mathcal{X})$ be a soft set over U and let $\emptyset \neq \alpha \subsetneq U$. If $\Psi(t) \cup \alpha \neq U$ for all $t \in \mathcal{X}$, then*

i) $supp(\Psi, \mathcal{X}) \subseteq (\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)}$.
ii) $(\Psi, \mathcal{X})^{\cap\alpha} \subseteq (\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)}$.

Proof. *(i)* Let $t \in supp(\Psi, \mathcal{X})$. Since $\Psi(t) \cup \alpha \neq U$ for all $t \in \mathcal{X}$, then $U \setminus (\Psi(t) \cup \alpha) \neq \emptyset$, which implies $t \in (\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)}$ by Proposition 3 (iv).
(ii) It is seen from the assertion (i) and Proposition 2 (i). □

Proposition 6. *Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha, \beta \subseteq U$. Then*

i) $(\Psi, \mathcal{X})^{\cap\alpha} \cap (\Psi, \mathcal{X})^{\cap\beta} \subseteq (\Psi, \mathcal{X})^{\cap(\alpha\cup\beta)}$. *Here the equality does not hold in general, even if $\alpha \cap \beta = \emptyset$.*
ii) $(\Psi, \mathcal{X})^{\cap\alpha} \cup (\Psi, \mathcal{X})^{\cap\beta} = (\Psi, \mathcal{X})^{\cap(\alpha\cup\beta)}$.
iii) $(\Psi, \mathcal{X})^{\cap(\alpha\cap\beta)} \subseteq (\Psi, \mathcal{X})^{\cap\alpha} \cap (\Psi, \mathcal{X})^{\cap\beta}$.

Proof. *(i)* Let $t \in (\Psi, \mathcal{X})^{\cap\alpha} \cap (\Psi, \mathcal{X})^{\cap\beta}$. Then $\Psi(t) \cap \alpha \neq \emptyset$ and $\Psi(t) \cap \beta \neq \emptyset$, which implies $\Psi(t) \cap (\alpha \cup \beta) \neq \emptyset$. For the rest of the proof, we have the Example 1.
(ii) □

$$\begin{aligned}(\Psi, \mathcal{X})^{\cap\alpha} \cup (\Psi, \mathcal{X})^{\cap\beta} &= \{t \in supp(\Psi, \mathcal{X}) |\ (\Psi(t) \cap \alpha \neq \emptyset) \vee (\Psi(t) \cap \beta \neq \emptyset)\} \\ &= \{t \in supp(\Psi, \mathcal{X}) |\ \Psi(t) \cap (\alpha \cup \beta) \neq \emptyset)\} \\ &= (\Psi, \mathcal{X})^{\cap(\alpha\cup\beta)}\end{aligned}$$

(iii) Let $t \in (\Psi, \mathcal{X})^{\cap(\alpha\cap\beta)}$. Then $\Psi(t) \cap (\alpha \cap \beta) \neq \emptyset$, which implies $\Psi(t) \cap \alpha \neq \emptyset$ and $\Psi(t) \cap \beta \neq \emptyset$. Therefore $t \in (\Psi, \mathcal{X})^{\cap\alpha} \cap (\Psi, \mathcal{X})^{\cap\beta}$.

Example 1. *Let the universe* $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$*, the parameter set* $\mathcal{T} = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$*, and* $\mathcal{X} = \{t_1, t_3, t_4, t_5\}$ *and* $\mathcal{Y} = \{t_1, t_2, t_3, t_4\}$ *be two subsets of* $\mathcal{T}$*. Suppose that corresponding soft sets of* $\mathcal{X}$ *and* $\mathcal{Y}$ *are*

$$(\Psi, \mathcal{X}) = \{(t_1, \{u_1, u_2\}), (t_3, \{u_1, u_4, u_5\}), (t_4, \{u_6\}), (t_5, \emptyset)\}$$

and

$$(\Upsilon, \mathcal{Y}) = \{(t_1, \{u_3, u_4\}), (t_2, \{u_1, u_2, u_5\}), (t_3, \{u_3, u_5\}), (t_4, \{u_1, u_5, u_6\})\}.$$

If $\alpha = \{u_4, u_6\}$ *and* $\beta = \{u_1\}$*, then it is seen that* $(\Psi, \mathcal{X})^{\cap\alpha} = \{t_3, t_4\}$*,* $(\Psi, \mathcal{X})^{\cap\beta} = \{t_1, t_3\}$ *and* $(\Psi, \mathcal{X}))^{\cap(\alpha\cup\beta)} = \{t_1, t_3, t_4\}$*. (Because, it is obtained that* $\Psi(t_3) \cap \alpha = \{u_4\} \neq \emptyset$*,* $\Psi(t_4) \cap \alpha = \{u_6\} \neq \emptyset$*,* $\Psi(t_1) \cap \beta = \{u_1\} \neq \emptyset$*,* $\Psi(t_3) \cap \beta = \{u_1\} \neq \emptyset$*,* $\Psi(t_1) \cap (\alpha \cup \beta) = \{u_1\} \neq \emptyset$*,* $\Psi(t_3) \cap (\alpha \cup \beta) = \{u_1, u_4\} \neq \emptyset$ *and* $\Psi(t_3) \cap (\alpha \cup \beta) = \{u_6\} \neq \emptyset$*). Thus, the proof of Proposition 6 (i) is completed. Since* $\alpha \cap \beta = \emptyset$*, we have* $(\Psi, \mathcal{X})^{\cap(\alpha\cap\beta)} = \emptyset$*.*
If $\alpha = \{u_3, u_6\}$ *and* $\beta = \{u_5, u_6\}$ *(i.e.,* $\alpha \cap \beta = \{u_6\}$*), then it is seen that* $(\Upsilon, \mathcal{Y})^{\cap\alpha} = \{t_1, t_3, t_4\}$*,* $(\Upsilon, \mathcal{Y})^{\cap\beta} = \{t_2, t_3, t_4\}$ *and* $(\Upsilon, \mathcal{Y})^{\cap(\alpha\cap\beta)} = \{t_4\}$*. So, we have* $(\Upsilon, \mathcal{Y})^{\cap(\alpha\cap\beta)} \subseteq (\Upsilon, \mathcal{Y})^{\cap\alpha} \cap (\Upsilon, \mathcal{Y})^{\cap\beta}$*.*

4. Set-Generated Soft Subrings of Rings

In this section, we propose the set-generated soft subrings of a ring by employing the α-intersection of soft sets. We also discuss some of the main properties and theoretical implications of this newly emerging soft algebraic structure.

Throughout this section, $\Re$ is a ring and $(\Psi, \Re)$ is a soft set over $\Re$. A subring S of $\Re$ denoted by $S < \Re$.

Definition 10. *Let* $\Re$ *be a ring,* $\emptyset \neq \alpha \subseteq \Re$ *and* $(\Psi, \Re)$ *be a soft set over* $\Re$*. If the soft set* $(\Psi, (\Psi, \Re)^{\cap\alpha})$ *is a soft subring of* $\Re$*, then this soft set is said to be a soft subring of* $\Re$ *generated by the set* α *and denoted by* $\langle\Psi^{\cap\alpha}\rangle_{\Re}$*. If the set* $\alpha = \{t\}$*,* $\langle\Psi^{\cap\alpha}\rangle_{\Re}$ *is a soft subring of* $\Re$ *generated by the element* $t \in \Re$*.*

As can be seen Definition 10, $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$ if and only if there exists at least an $\emptyset \neq \alpha \subseteq \Re$ such that $(\Psi,\Re)^{\cap\alpha}$ is a subring of $\Re$ and the conditions $s1, s2$ of the Definition 7 are satisfied for $S = (\Psi,\Re)^{\cap\alpha}$.

Example 2. *Given the ring $\Re = (\mathbb{Z}_6, +, .)$, a soft set $(\Psi,\Re)$ over $\Re$, where $\Psi : \Re \to P(\Re)$ is a set-valued function defined by $\Psi(0) = \{0,1,4,5\}, \Psi(1) = \{3\}, \Psi(2) = \{2\}, \Psi(3) = \{0,4,5\}, \Psi(4) = \{1,2\}$ and $\Psi(5) = \{3\}$. Let $\alpha = \{4,5\} \subseteq \Re$. Then, $(\Psi,\Re)^{\cap\alpha} = \{0,3\}$ is a subring of $\Re$ and the soft set $(\Psi,(\Psi,\Re)^{\cap\alpha}) = \{(0,\{0,1,4,5\}), (3,\{0,4,5\})\}$ satisfies the conditions $s1, s2$ of the Definition 7. (That is, $\Psi^{\cap\alpha}(t-v) \supseteq \Psi^{\cap\alpha}(t) \cap \Psi^{\cap\alpha}(v)$ and $\Psi^{\cap\alpha}(tv) \supseteq \Psi^{\cap\alpha}(t) \cap \Psi^{\cap\alpha}(v)$ for all $t, v \in \Re$). Hence $(\Psi,(\Psi,\Re)^{\cap\alpha}) = \langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$. If $\beta = \{4\}$ a single point set, then it is seen that $(\Psi,\Re)^{\cap\alpha} = (\Psi,\Re)^{\cap\beta}$ and then $\langle\Psi^{\cap\alpha}\rangle_{\Re} = \langle\Psi^{\cap\beta}\rangle_{\Re}$. Therefore, the soft set $\{(0,\{0,1,4,5\}, (3,\{0,4,5\})\}$ is a soft subring of $\Re$, generated by the element $4 \in \Re$.*

Let $(\Psi,\Re)$ be a soft set over $\Re$. Since $\{0_{\Re}\}$ is a subring of $\Re$, it is easily seen that $(\Psi,\{0_{\Re}\})\widetilde{<}\Re$.

Definition 11. *The soft subring $(\Psi,\{0_{\Re}\})$ of $\Re$ is called a trivial soft subring of $\Re$ and denoted by $\langle 0_{\Re}\rangle_{\Psi}$.*

It is important to note that the soft sets $\langle 0_{\Re}\rangle_{\Psi}$ and $(\Psi,(\Psi,\Re)^{\cap\{0_{\Re}\}})$ are different, in general. In Example 2, $\langle 0_{\Re}\rangle_{\Psi} = (\Psi,\{0_{\Re}\}) = \{(0,\{0,1,4,5\})\}$ and $(\Psi,(\Psi,\Re)^{\cap\{0_{\Re}\}}) = \{(0,\{0,1,4,5\}),(3,\{0,4,5\})\}$. Furthermore, $(\Psi,(\Psi,\Re)^{\cap\{0_{\Re}\}})$ does not have to be a soft subring of $\Re$.

Proposition 7. *If $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$, then the generator α doesn't have to be unique. Furthermore, if $\langle\Psi^{\cap\{t\}}\rangle_{\Re} = \langle\Psi^{\cap\{v\}}\rangle_{\Re}$ for $t, v \in \Re$, then $\langle\Psi^{\cap\{t,v\}}\rangle_{\Re} = \langle\Psi^{\cap\{t\}}\rangle_{\Re}$.*

Proof. In the example 2, if we take $\eta = \{5\} \subseteq \Re$, then it is seen that $(\Psi,\Re)^{\cap\eta} = (\Psi,\Re)^{\cap\beta}$ and then $\langle\Psi^{\cap\eta}\rangle_{\Re} = \langle\Psi^{\cap\beta}\rangle_{\Re}$. Hence the generator doesn't have to be unique, even if it is a single point set. Now, let $\langle\Psi^{\cap\{t\}}\rangle_{\Re} = \langle\Psi^{\cap\{v\}}\rangle_{\Re}$ for $t, v \in \Re$. Consider the sets $\mathcal{X} = \{t \in \Re : \Psi(t) \cap \{t\} \neq \emptyset\} = \{t \in \Re : \Psi(t) \cap \{v\} \neq \emptyset\}$ and $\mathcal{Y} = \{t \in \Re : \Psi(t) \cap \{t,v\} \neq \emptyset\}$. Obviously $\mathcal{X} \subseteq \mathcal{Y}$. Let $t \in \mathcal{Y}$. Then,

$$\begin{aligned}\Psi(t) \cap \{t,v\} \neq \emptyset &\Rightarrow \Psi(t) \cap \{t\} \neq \emptyset \text{ or } \Psi(t) \cap \{v\} \neq \emptyset \\ &\Rightarrow t \in \mathcal{X} \text{ or } t \in \mathcal{X} \\ &\Rightarrow t \in \mathcal{X}\end{aligned}$$

Hence $\mathcal{Y} \subseteq \mathcal{X}$. Therefore, $(\Psi,\Re)^{\cap\{t\}} = (\Psi,\Re)^{\cap\{v\}} = (\Psi,\Re)^{\cap\{t,v\}}$, which implies that $\langle\Psi^{\cap\{t,v\}}\rangle_{\Re} = \langle\Psi^{\cap\{t\}}\rangle_{\Re}$. □

Proposition 8. *If $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$, then $\Psi(0_{\Re}) \cap \alpha \neq \emptyset$. But the reverse implication is not true, in general.*

Proof. Let $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$. Then the set $(\Psi,\Re)^{\cap\alpha} = \{t \in \Re : \Psi(t) \cap \alpha \neq \emptyset\}$ is a subring of $\Re$. Then $\Psi(0_{\Re}) \supseteq \Psi(t)$ for all $t \in (\Psi,\Re)^{\cap\alpha}$ by Proposition 1. Since $\Psi(t) \cap \alpha \neq \emptyset$ and $\Psi(0_{\Re}) \supseteq \Psi(t)$ for all $t \in (\Psi,\Re)^{\cap\alpha}$, then $\Psi(0_{\Re}) \cap \alpha \neq \emptyset$. For the rest of the proof, let $\lambda = \{0,1,5\} \subseteq \Re$ in Example 2. Then it is seen that $\Psi(0_{\Re}) \cap \lambda \neq \emptyset$, but

$(\Psi,\Re)^{\cap\lambda}=\{0,3,4\}$ is not a subring of $\Re$. Therefore, $(\Psi,(\Psi,\Re)^{\cap\lambda})$ is not a soft subring of $\Re$. □

Proposition 9. *Let $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$ and $\langle\Psi^{\cap\beta}\rangle_{\Re}\widetilde{<}\Re$. If $\alpha\subseteq\beta$, then $\langle\Psi^{\cap\alpha}\rangle_{\Re}\subseteq\langle\Psi^{\cap\beta}\rangle_{\Re}$.*

Proof. Let $t\in(\Psi,\Re)^{\cap\alpha}$. Then $\Psi(t)\cap\alpha\neq\emptyset$. Since $\alpha\subseteq\beta$, $\Psi(t)\cap\beta\neq\emptyset$. Hence $(\Psi,\Re)^{\cap\alpha}\subseteq(\Psi,\Re)^{\cap\beta}$. Therefore $(\Psi,(\Psi,\Re)^{\cap\alpha})\subseteq(\Psi,(\Psi,\Re)^{\cap\beta})$, which completes the proof. □

The following Theorem shows that Theorem 1 is also true for the operation soft intersection instead of restricted intersection when taking the soft set $(\Psi,\Re)$ instead of (Ψ,S).

Theorem 2. *If $(\Psi,\Re)\widetilde{<}\Re$ and $(\Upsilon,\Re)\widetilde{<}\Re$, then $(\Psi,\Re)\widetilde{\cap}(\Upsilon,\Re)\widetilde{<}\Re$.*

Proof. By Definition 6 $(\Psi,\Re)\widetilde{\cap}(\Upsilon,\Re)=(\Theta,\Re)$, where $\Theta(t)=\Psi(t)\cap\Upsilon(t)$ for all $t\in\Re$. Then for all $t,v\in\Re$,

$$\begin{aligned}
\Theta(t-v) &= \Psi(t-v)\cap\Upsilon(t-v)\\
&\supseteq (\Psi(t)\cap\Psi(v))\cap(\Upsilon(t)\cap\Upsilon(v))\\
&= (\Psi(t)\cap\Upsilon(t))\cap(\Psi(v)\cap\Upsilon(v))\\
&= \Theta(t)\cap\Theta(v),\\
\Theta(tv) &= \Psi(tv)\cap\Upsilon(tv)\\
&\supseteq (\Psi(t)\cap\Psi(v))\cap(\Upsilon(t)\cap\Upsilon(v))\\
&= (\Psi(t)\cap\Upsilon(t))\cap(\Psi(v)\cap\Upsilon(v))\\
&= \Theta(t)\cap\Theta(v).
\end{aligned}$$

Therefore $(\Psi,\Re)\widetilde{\cap}(\Upsilon,\Re)\widetilde{<}\Re$. □

Now, some problems arise such that: Is the soft intersection of two set-generated soft subrings of $\Re$, again a set-generated soft subring of $\Re$? And, if $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$, $\langle\Psi^{\cap\beta}\rangle_{\Re}\widetilde{<}\Re$ such that $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle\Psi^{\cap\xi}\rangle_{\Re}$, then can the subset ξ be expressed using α and β? The answer of the first problem is "No", we have the following example:

Example 3. *Given the ring $\Re=(\mathbb{Z}_{12},+,.)$, a soft set $(\Psi,\Re)$ over $\Re$, where $\Psi:\Re\to P(\Re)$ is a set-valued function defined by $\Psi(0)=\{1,3,5,6,7,9,11\}$, $\Psi(1)=\{2,4\}$, $\Psi(2)=\{3,6,7,11\}$, $\Psi(3)=\{1,5,9\}$, $\Psi(4)=\{3,6,7,11\}$, $\Psi(5)=\{8,10\}$, $\Psi(6)=\{1,3,5,6,7,9,11\}$, $\Psi(7)=\{2,10\}$, $\Psi(8)=\{3,6,7,11\}$, $\Psi(9)=\{1,5,9\}$, $\Psi(10)=\{3,6,7,11\}$ and $\Psi(11)=\{2,8\}$. Let $\alpha=\{11\}$ and $\beta=\{5\}$. Then, $(\Psi,\Re)^{\cap\alpha}=\{0,2,4,6,8,10\}$ and $(\Psi,\Re)^{\cap\beta}=\{0,3,6,9\}$ are subrings of $\Re$ and the soft sets*

$$(\Psi,(\Psi,\Re)^{\cap\alpha})=\left\{\begin{array}{l}(0,\{1,3,5,6,7,9,11\}),(2,\{3,6,7,11\}),(4,\{3,6,7,11\}),\\(6,\{1,3,5,6,7,9,11\}),(8,\{3,6,7,11\}),(10,\{3,6,7,11\})\end{array}\right\}$$

and

$$(\Psi,(\Psi,\Re)^{\cap\beta})=\left\{\begin{array}{l}(0,\{1,3,5,6,7,9,11\}),(3,\{1,5,9\}),\\(6,\{1,3,5,6,7,9,11\}),(9,\{1,5,9\})\end{array}\right\}$$

satisfy the conditions $s1, s2$ of Definition 7. Hence $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$ and $\langle\Psi^{\cap\beta}\rangle_{\Re}\widetilde{<}\Re$. Then,

$\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\{(0,\{1,3,5,6,7,9,11\}),(6,\{1,3,5,6,7,9,11\})\}=(\Psi,S)\widetilde{<}\Re.$

But, there is no subset ξ of $\Re$ such that $(\Psi,S)=\langle\Psi^{\cap\xi}\rangle_{\Re}$.

Corollary 1. *The soft intersection of two set-generated soft subrings of $\Re$ is not a set-generated soft subring of $\Re$, in general.*

But, we have the following:

Theorem 3. *Let $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$ and $\langle\Psi^{\cap\beta}\rangle_{\Re}\widetilde{<}\Re$. Then, either $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle 0_{\Re}\rangle_{\Psi}$ trivial soft subring or if $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle\Psi^{\cap\xi}\rangle_{\Re}$, then there exists $\xi\subseteq\Re$ such that $\emptyset\neq\xi\subseteq\alpha\cup\beta$.*

Proof. If $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle 0_{\Re}\rangle_{\Psi}$, it is obvious. Assume that

$$\langle 0_{\Re}\rangle_{\Psi}\neq\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle\Psi^{\cap\xi}\rangle_{\Re}.$$

Then $\langle\Psi^{\cap\xi}\rangle_{\Re}\widetilde{<}\Re$ by Theorem 2. Since $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle\Psi^{\cap\xi}\rangle_{\Re}$, then we have

$$U\cap\alpha\neq\emptyset\wedge U\cap\beta\neq\emptyset\Leftrightarrow U\cap\xi\neq\emptyset$$

for $(t,U)\in\langle\Psi^{\cap\xi}\rangle_{\Re}$. The requirement (1) holds for:

i) $\alpha\subseteq\beta\Rightarrow\xi=\beta$,
ii) $\beta\subseteq\alpha\Rightarrow\xi=\alpha$,
iii) $\alpha\cap\beta=\emptyset\Rightarrow\xi=\alpha\cup\beta$,
iv) $\alpha\neq\beta$ and $\alpha\cap\beta\neq\emptyset\Rightarrow\xi=\alpha\cup\beta$.

Although the requirement (1) also holds for $\xi\supseteq\alpha\cup\beta$, it is enough to show existing $\xi\subseteq\Re$ such that $\emptyset\neq\xi\subseteq\alpha\cup\beta$ to complete the proof. □

Definition 12. *([29]) Let $\Re_1$ and $\Re_2$ be two rings, and (Ψ,S_1) and (Υ,S_2) be two soft subrings of $\Re_1$ and $\Re_2$, respectively. The product of soft subrings (Ψ,S_1) and (Υ,S_2) is defined as $(\Psi,S_1)\times(\Upsilon,S_2)=(\Omega,S_1\times S_2)$, where $\Omega(t,v)=\Psi(t)\times\Upsilon(v)$ for all $(t,v)\in S_1\times S_2$.*

Theorem 4. *([29]) If $\Psi_{S_1}\widetilde{<}\Re_1$ and $\Upsilon_{S_2}\widetilde{<}\Re_2$, then $\Psi_{S_1}\times\Upsilon_{S_2}\widetilde{<}\Re_1\times\Re_2$.*

Theorem 4 leads to the problem: Is the product of two set-generated soft subrings of two rings, again a set-generated soft subring of the ring of product of rings? The answer is "Yes", we have the following:

Theorem 5. *Let $\Re_1$ and $\Re_2$ be two rings and let $(\Psi,\Re_1),(\Upsilon,\Re_2)$ be two soft sets over $\Re_1$ and $\Re_2$, respectively. If there exist $\alpha\subseteq\Re_1$ and $\beta\subseteq\Re_2$ such that $\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\widetilde{<}\Re_1$ and $\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}\widetilde{<}\Re_2$, then*

$$\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\times\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}=\langle\Theta^{\cap(\alpha\times\beta)}\rangle_{\Re_1\times\Re_2}.$$

Proof. Let $\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\widetilde{<}\Re_1$ and $\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}\widetilde{<}\Re_2$. Then $(\Psi,\Re_1)^{\cap\alpha}$ is a subring of $\Re_1$ and $(\Upsilon,\Re_2)^{\cap\beta}$ is a subring of $\Re_2$. So $(\Psi,\Re_1)^{\cap\alpha}\times(\Upsilon,\Re_2)^{\cap\beta}$ is a subring of $\Re_1\times\Re_2$. Therefore, $\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\times\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}\widetilde{<}\Re_1\times\Re_2$ by Theorem 4. Now, let $(t,\Psi(t))\in\langle\Psi^{\cap\alpha}\rangle_{\Re_1}$ and $(v,\Upsilon(v))\in\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}$. Then $\Psi(t)\cap\alpha\neq\emptyset$ and $\Upsilon(v)\cap\beta\neq\emptyset$. Since

$$\Psi(t)\cap\alpha\neq\emptyset\wedge\Upsilon(v)\cap\beta\neq\emptyset\Leftrightarrow(\Psi(t)\times\Upsilon(v))\cap(\alpha\times\beta)\neq\emptyset,$$

then we have

$$(t,\Psi(t))\in\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\wedge(v,\Upsilon(v))\in\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}\Leftrightarrow((t,v),\Psi(t)\times\Upsilon(v))\in\langle\Theta^{\cap(\alpha\times\beta)}\rangle_{\Re_1\times\Re_2}.$$

$(\Theta,\Re_1\times\Re_2)$ is a soft set over $\Re_1\times\Re_2$, where $\Theta:\Re_1\times\Re_2\to P(\Re_1\times\Re_2)$ is a set-valued function defined by $\Theta(t,v)=\Psi(t)\times\Upsilon(v)$. Hence, the proof is completed. □

Example 4. *Over the ring $\Re_1=(\mathbb{Z}_4,+,.)$, a soft set $(\Psi,\Re_1)$ given by $\Psi(0)=\{1,2,3\}$, $\Psi(1)=\{0\}$, $\Psi(2)=\{1,3\}$, $\Psi(3)=\{2\}$. For $\alpha=\{3\}$, $(\Psi,\Re_1)^{\cap\alpha}=\{0,2\}$ and $\langle\Psi^{\cap\alpha}\rangle_{\Re_1}=\{(0,\{1,2,3\}),(2,\{1,3\})\}\widetilde{<}\Re_1$. Given the ring $\Re_2=(\mathbb{Z}_6,+,.)$, a soft set $(\Upsilon,\Re_2)$ over $\Re_2$, defined by $\Upsilon(0)=\{0,1,2,5\}$, $\Upsilon(1)=\{3,4\}$, $\Upsilon(2)=\{4\}$, $\Upsilon(3)=\{0,2\}$, $\Upsilon(4)=\{3\}$ and $\Upsilon(5)=\{4\}$. For $\beta=\{2\}$, $(\Upsilon,\Re_2)^{\cap\beta}=\{0,3\}$ and $\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}=\{(0,\{0,1,2,5\}),(3,\{0,2\})\}\widetilde{<}\Re_2$. $\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\times\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}$ is a soft set given by*

$$\left\{\begin{array}{l}((0,0),\{(1,0),(1,1),(1,2),(1,5),(2,0),(2,1),(2,2),(2,5),(3,0),(3,1),(3,2),(3,5)\}),\\((2,0),\{(1,0),(1,1),(1,2),(1,5),(3,0),(3,1),(3,2),(3,5)\}),\\((0,3),\{(1,0),(1,2),(2,0),(2,2),(3,0),(3,2)\}),\\((2,3),\{(1,0),(1,2),(3,0),(3,2)\})\end{array}\right\}.$$

Now, let the soft set $(\Theta,\Re_1\times\Re_2)$ over $\Re_1\times\Re_2$, where $\Theta:\Re_1\times\Re_2\to P(\Re_1\times\Re_2)$ is a set-valued function defined by $\Theta(t,v)=\Psi(t)\times\Upsilon(v)$. Then, for $\alpha\times\beta=\{(3,2)\}$, it is easily seen that $\langle\Theta^{\cap(\alpha\times\beta)}\rangle_{\Re_1\times\Re_2}=\langle\Psi^{\cap\alpha}\rangle_{\Re_1}\times\langle\Upsilon^{\cap\beta}\rangle_{\Re_2}$.

5. Conclusions

In this paper, we are interested in the algebraic soft substructures of rings given in the article [29]. We introduced set-generated soft subrings of rings using non-empty subsets of rings. By theoretical directions, we applied some of the operations derived on soft sets to set-generated soft subrings. Moreover, we gave some relationships between the generators of soft subrings and studied their related various properties with assorted examples. To further this work, one could study the set-generated soft substructures of other algebraic structures such as fields, modules, vector spaces and algebras. Our future work will be based on the derivation of these algebraic structures and the investigation their application aspects.

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