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Research Article

QUASI-HARMONIC CONSTRAINTS FOR TORIC BEZIER SURFACES

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ABSTRACT

Toric Bezier patches generalize the classical tensor-product triangular and rectangular Bezier surfaces, extensively used in CAGD. The construction of toric Bezier surfaces corresponding to multi-sided convex hulls for known boundary mass-points with integer coordinates (in particular for trapezoidal and hexagonal convex hulls) is given. For these toric Bezier surfaces, we find approximate minimal surfaces obtained by extremizing the quasi-harmonic energy functional. We call these approximate minimal surfaces as the quasi-harmonic toric Bezier surfaces. This is achieved by imposing the vanishing condition of gradient of the quasi-harmonic functional and obtaining a set of linear constraints on the unknown inner mass-points of the toric Bezier patch for the above mentioned convex hull domains, under which they are quasi-harmonic toric Bezier patches. This gives us the solution of the Plateau toric Bezier problem for these illustrative instances for known convex hull domains.
Keywords: Harmonicity, minimal surfaces, toric bezier patches.

1. INTRODUCTION

The theory of minimal surfaces has its roots in the optimization problems of calculus of variations, based on the famous Euler- Lagrange equation which is a second order partial differential equation (*pde*). The solution of the Euler-Lagrange equation targets to find a function that extremizes a given functional and has many applications in the optimization theory. Many mathematicians have contributed to the subject of optimization theory and it has become a widely accepted discipline of Mathematics and Physics. A minimal surface is a surface which locally minimizes its area or equivalently a surface whose mean curvature vanishes everywhere on the surface. In the similar context, a problem known as the Plateau problem [1, 2] consists of finding the surface with least surface area bounded by a given boundary curve. It is named after Belgian physicist Joseph. A. Plateau [3] who experimentally demonstrated in 1849 that minimal surfaces can be associated to the soap films spanned by wire frames of different shapes. In the meantime, many mathematicians developed their interest in finding a minimal surface spanned by a fixed boundary curve such as Schwarz [4] (who studied the triply periodic surfaces namely the CLP (crossed layers of parallels), D (diamond), P (primitive), H (hexagonal) and T (tetragonal) surfaces, Weierstrass [1], Riemann [1] and R. Garnier [5] in the late 19th century. However, these

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were minimal surfaces for particular boundaries, until in 1931, American mathematician J. Douglas [6] and in 1933 Hungarian Tibor Rado [7] independently proved the existence of a minimal surface spanned by a closed curve by replacing the area functional by rather a simpler integral, now known as the Douglas-Dirichlet functional. The Douglus-Dirichlet functional does not have square root in its integrand as is the case with the area functional which makes it a suitable choice as an alternative to the area functional.

Exact mathematical solutions are known only for some specific boundaries. It is possible to find numerically the solution of a wide variety of problems giving rise to approximate minimal surfaces. Coppin and Greenspan [8] used a computer model of molecular structure and forces to approximate a minimal surface. K. Koohestani [9] also suggested the method involving non-linear force density to find minimal surfaces for membrane structures. Brakke [10] used the finite element method to approximate parameterized minimal surfaces. Level set method was proposed by Chopp [11] to cope with topological variations of a surface under linear convergence, whereas a variational approach to minimize the area of triply periodic surfaces was proposed by Jung et al. [12]. Ronquiust and Trasdahl [13] introduced an iterative scheme which involves parameterization of higher order polynomials to achieve a numerical approximation of a minimal surface with fixed boundaries. Similarly, Li et al. [14] numerically approximated the minimal surfaces with geodesic constraints over boundary curves. Kassabov [15] derived an equation of a canonical parameterized minimal surface and also pointed out its application. Xu et al. [16] proposed a parametric form of polynomial minimal surface with varying degrees which possess interesting properties helpful for geometric modeling in CAD.

Alternative energy functionals for minimization may be used to find an approximate minimal surface of a certain restricted class of surfaces. One of the widely used restriction is to find a minimal Bezier surface among all the Bezier surfaces

$$\mathbf{x}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,j}^{n,m}\left(u,v\right)\mathbf{P}_{ij}, \tag{1.1}$$

with

$$B_{i,j}^{n,m}\left(u,v\right)=B_i^n\left(u\right)B_j^m\left(v\right), \tag{1.2}$$

spanned by a given boundary in which $\mathbf{P}_{ij}$ represents a two dimensional control net over the domain $D=[0,1]\times[0,1]$ with u,v as the surface parameters, the bivariate functions $\{B_{i,j}^{n,m}\left(u,v\right):\mathbf{R}^2\to\mathbf{R}\}$ are the blending functions are the blending functions to specify the shape of the surface and

$$B_i^n\left(u\right)=\binom{n}{i}u^i\left(1-u\right)^{n-i} \tag{1.3}$$

are the Bernstein polynomials of degree *n* with $\binom{n}{i}=\frac{n!}{i!\left(n-1\right)!}$ as the binomial coeffcients.

An extremal of discrete version of Dirichlet functional giving minimal Bezier surfaces can be seen in the Monterde work [17]. X. D. Chen, G. Xu, and Y.Wang. [18] found approximate minimal surfaces as the solution of Plateau-Bezier problem using extended Dirichlet functional and the extended bending energy functional, the surfaces depend on the parameters λ and α (as they appear in eqs. (4) and (5) of the ref.[18]) for simple estimates of these parameters. Hao et al. [19] investigated the Plateau-quasi-Bezier problem, minimizing thereby the Dirichlet functional of surfaces for more generalized borders including the boundary curves like polynomial curves, catenaries and circular arcs. Another restriction could be to find a parametric polynomial minimal surface as has been proposed by Xu and Wang [20] to obtain a minimal surface for quintic parametric polynomial surface having the prescribed borders as polynomial curves. Ahmad and

Masud [21-23] gave an algorithm to find a quasi-minimal surface, variationally improving the non-minimal initial surface spanned by a fixed boundary composed of finite number of curves by minimizing its *rms* mean curvature functional instead of area functional which involves a square root in its integrand and applied this technique to a variety of surfaces. The idea may be extended to more generalized surfaces called toric Bezier surfaces to obtain a quasi-minimal surface by minimizing the quasi-harmonic functional as is done by Xu et al. [24] to find the quasi-harmonic surface as the solution of Plateau-Bezier problem. The related class of surfaces is called the harmonic mapping. The harmonic mappings find significant importance in the literature of minimal surfaces for the isothermal parameterization of the surfaces [1, 25]. This means that a positive definite metric in two dimensions

$$ds^2 = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2, \tag{1.4}$$

defined in the neighbourhood of a surface $\mathbf{x}(x,y)$ in local coordinates (x,y) takes the form

$$ds^2 = \lambda^2(x,y)(dx^2 + dy^2), \tag{1.5}$$

($i.e.$ $E(x,y) = G(x,y) = \lambda^2(x,y), F(x,y) = 0$) in the isothermal coordinates (x, y).

If a surface is parameterized using the isothermal parameterization [25], then such a parameterization is minimal if the coordinate functions are harmonic. In other words, a surface with isothermal parameterization is a minimal surface if and only if it is a harmonic surface. This is also useful in finding a minimal surface associated to a class of surfaces namely the Bezier surfaces. Monterde and Ugail [26] indicated that harmonic Bezier surfaces can only be specified by opposite boundary control points and thus making it impracticable to generate a harmonic Bezier surface from the prescribed four boundary Bezier curves. In order to overcome this dificulty, Xu et al. [24] proposed the quasi-harmonic surfaces which serve as the solution surfaces for Plateau-Bezier problem. They also showed that in particular cases when the corners of Bezier surface are almost isothermal, quasi-harmonic surfaces are better approximations when compared to surfaces generated by Dirichlet method.

Polynomial functions and splines are widely used in many structural design program softwares. The fundamental units of modeling a surface geometrically are the classical Bezier triangles and rectangular tensor product patches [27] in computer aided geometric designing (CAGD), however some applications require a more generalized form of multi-sided C1 patches rather than the classical Bezier surfaces. J. Warren [28] realized the usage of real toric surfaces in CAGD. His notable contribution is construction of a hexagonal patch from a rational Bezier triangle with zero weights and the corresponding control points located appropriately. The multi-sided patches bear more exibility and present interesting mathematical structures when dealt through Krasauskas's toric Bezier patches [29]. Toric Bezier patches are the generalization of the classical Bezier patches that deal only with triangular or rectangular patches. In 2002, Krasauskas and Goldman [30] presented the construction of toric Bezier patches of depth d by using the de Casteljau pyramid algorithm and blossoming algorithm for the associated patches. In recent work by Gang Xu, Tsz-Ho Kwok and Charlie C.L. Wang [31], a B-spline volumetric parameterization is constructed with semantic features for isogeometric analysis.

Further developments in toric Bezier surfaces include the work of Garcia-Puente et al. [32], they illustrated the geometrical importance of the structural system of toric Bezier patches, Sun and Zhu [33, 34] discussed the G^1 continuity of toric Bezier surfaces and found approximate minimal toric Bezier surfaces by minimizing the Dirichlet functional.

In this paper, we construct quasi-harmonic toric Bezier patches defined over multi-sided convex hulls with prescribed boundary mass-points by extremizing the quasi-harmonic functional to generate a system of linear equations for the unknown inner mass-points. This enables us to write down the parametric form of the solution of the Plateau-toric Bezier problem. The paper is organized as follows: In section 2, we give the preliminary introduction to toric Bezier patch of depth d in general and its construction consisting of indexing lattice polygon domains and the associated toric Bernstein polynomials. In the following sections 3 and 4, we utilize the quasi-harmonic energy functional as the objective functional to obtain the necessary and suficient conditions for a toric Bezier patch to be a quasi-harmonic toric Bezier patch which serves as the solution to the Plateau-toric Bezier problem. Finally, in section 5, we construct quasi-harmonic toric Bezier patches defined over trapezoidal convex hulls and hexagonal convex hull as illustrative applications. Constraints on mass-points of the toric Bezier patches defined over the above mentioned multi-sided domains are obtained by solving the respective systems of linear equations for the inner unknown mass-points. For the prescribed boundary mass points, quasi-harmonic toric Bezier patches, as illustrative applications, have also been obtained and shown that the inner mass-points satisfy the computed constraints.

2. TORIC BEZIER PATCHES AND RELATED TERMINOLOGY

In computer aided geometric designing (CAGD), three and four-sided patches namely the triangular and rectangular Bezier patches are commonly used for surface modeling but a multi-sided generalization of these Bezier schemes is required in order to fill *n*-sided holes. One of such schemes used to define multi-sided C1 patches is the Krasauskas's Toric Bezier patch as introduced in [29]. A scheme in section 4 is given to obtain quasi-harmonic toric Bezier surface by extremizing the quasi-harmonic functional introduced in the section 3. To comprehend the construction of these toric Bezier patches and then to extremize a given functional to find an approximate minimal surface, we give below the related terminology for the reader to get familiar with lattice polygons, Bernstein basis functions for these polygons, discrete convolution indexed by Minkowski sum and finally the construction of toric Bezier patches for given depth d.

Definition 2.1. (Lattice Polygons) The polygon formed by connecting the outer most sequence of points in the finite set $\sigma \in$ ¢^{2} in the plane is called the lattice polygon. The finite set σ is used as the index set for control points $\{P_{\sigma_i}\}_{\sigma_i \in \sigma}$ to form a polygonal array of control points.

The lattice polygons for the classical tensor-product Bezier patch and triangular Bezier patch are lattice rectangle and lattice triangle respectively which form the array of their corresponding control points. Other examples of multi-sides lattice polygons are given in fig 1.

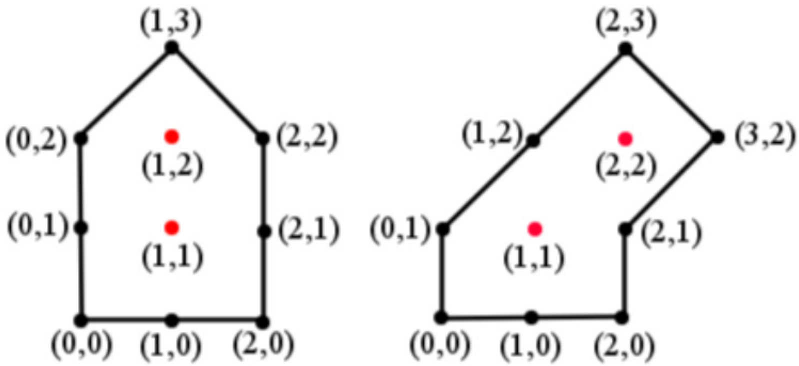

Figure 1. Multi-sided lattice polygons, a lattice pentagon (left) and a lattice hexagon (right) with inner lattice points (red dots).

Definition 2.2. (Bernstein Polynomial Functions for Lattice Polygons) Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \in Z^2$ be the set of finite integers in uv-plane. The lattice polygon I_σ denotes the convex hull of σ with corner points $v_1, v_2, ..., v_n$ and $\overline{L}_k(u,v) = \alpha_k u + \beta_k v + \gamma_k, k = 1, 2, ..n$, the kth edge of the convex hull I_σ. In addition, the direction of the normal vector (σ_k, β_k) to the line $\overline{L}_k(u,v)$ is in the convex hull I_σ and (σ_k, β_k) is the shortest normal vector with integer coordinates in that direction.

The Bernstein polynomials $\beta_{\sigma i}(u, v)_{\sigma i \in \sigma}$ for (u, v) in the convex hull I_σ, for toric Bezier patch can be written as

$$\beta_{\sigma_i}(u,v) = c_{\sigma_i} \{\overline{L}_1(u,v)\}^{\overline{L}_1(\sigma_i)} \{\overline{L}_2(u,v)\}^{\overline{L}_2(\sigma_i)} ... \{\overline{L}_n(u,v)\}^{\overline{L}_n(\sigma_i)}, \tag{2.1}$$

where positive arbitrary normalizing constants $c_{\sigma i}$ are the coefficients of basis functions, chosen appropriately to get certain desired formulas. For toric Bezier patches, the Bernstein polynomials for lattice polygon $\{\beta_{\sigma_i}(u,v)\}_{(\sigma_i) \in I_\sigma}$ have the analogous properties as that of classical Bernstein polynomials (1.3) for which the classical bivariate functions (eq. (1.2) $\{B_{i,j}^{n,m}\}(u,v)$ are

$$B_{i,j}^{n,m}(u,v) = \binom{n}{i}\binom{m}{j} u^i (1-u)^{n-i} v^j (1-v)^{m-j}, \tag{2.2}$$

(for $i \in \{0, ..., n\}$; $j \in \{0, ..., m\}$) used to construct the triangular or rectangular Bezier patches. These Bernstein polynomials $\{\beta_{\sigma_i}(u,v)\}_{(\sigma_i) \in I_\sigma}$ (eq. (2.1)) indexed by the set σ with lattice polygon I_σ having corner points v_1, v_2, …, v_n satisfy the following properties: 1) $\beta_{\sigma_i}(u,v) > 0$ inside the lattice polygon I_σ, 2) $\beta_{\sigma_i}(u,v) = 0$ on the edge $v_k v_{k+1}$, if and only if $\sigma_i \notin v_k v_{k+1}$, 3) $\beta_{\sigma_i}(u,v) = 1$ if $\sigma_i = v_k$ and 4) $\{\beta_{\sigma_i}(u,v)\}$ are polynomial functions.

Definition 2.3. (Toric Bezier Patch) A toric Bezier patch is a rational surface $\mathcal{P}(u,v)$ in the real projective space RP^4 of dimension 4 with control structure consisting of mass-points $\{(\omega_{\sigma_i} P_{\sigma_i}, \omega_{\sigma_i})\}$ indexed by the lattice polygon I_σ. The mass-points $\{(\omega_{\sigma_i} P_{\sigma_i}, \omega_{\sigma_i})\}$ are four dimensional elements with $\omega_{\sigma i}$ as the scaler weights corresponding to control points $P_{\sigma i}$ in space. The Bernstein polynomials for lattice polygon $\beta_{\sigma i}(u, v)$ as given in eq. (2.1) are the blending functions which serve as the basis functions for toric Bezier patches defined over the domain lattice polygon I_σ and they are chosen to obtain the desired shape of the surface. The toric Bezier surface $\mathcal{P}(u,v)$ is defined by the expression

$$\mathcal{P}(u,v) = \sum_{\sigma_i \in I_\sigma} \beta_{\sigma_i}(u,v)\left(\omega_{\sigma_i} P_{\sigma_i}, \omega_{\sigma_i}\right), \ (u,v) \in I_\sigma, \tag{2.3}$$

where Bernstein polynomials $\{\beta_{\sigma_i}(u,v)\}_{(\sigma_i) \in I_\sigma}$ are given in eq. (2.1). A rational surface may be obtained by dividing the surface eq. (2.3) by $\sum_{\sigma_i \in I_\sigma} \beta_{\sigma_i}(u,v)$ provided that $\sum_{\sigma_i \in I_\sigma} \beta_{\sigma_i}(u,v) \neq 0$, throughout the domain. Krasauskas and Goldman [30] introduced the concept of depth for toric Bezier patches which is the analogue of degree used to define the classical higher order Bezier surfaces. It is based on the depth of lattice polygons defined with the help of repeated Minkowski sums.

Definition 2.4. (Minkowksi sum) Let A and B be any two sets of p-tuples. The Minkowski sum $A \oplus B$ of these two sets is the set with the sum of all elements from A and all elements of B given by,

$$A \oplus B = \{a + b \mid a \in A, b \in B\}.$$

Definition 2.5. (Discrete convolution indexed by Minkowski sum) Let $P = \{P_a \mid a \in A\}$ and $Q = \{Q_b \mid b \in B\}$ be two arrays. Then the discrete convolution $P \otimes Q$ indexed by the Minkowksi sum $A \oplus B$ i.e., $P \otimes Q = \{(P \otimes Q)_c \mid c \in A \oplus B\}$ is defined as

$$(P \otimes Q)_c = \sum_{a+b=c} P_a Q_b.$$

The indexing of discrete convolution indexed by Minkowski sum may be used to define toric Bezier patches with depth d, as given below. The depth d of toric Bezier patches as expressed by Krasasuskas and Goldman [30] is the analogue of degree used to define the classical higher order Bezier surfaces. It is based on the depth of lattice polygons defined with the help of repeated Minkowski sums as given above (definitions 2.4 and 2.5).

Definition 2.6. (Toric Bézier Patch with depth d) Let $\sigma^d = \overbrace{\sigma \oplus \sigma ... \oplus \sigma}^{d0fold}$ be the d -fold Minkowski sum of σ and I^d, the corresponding convex hull of σ^d. Then the toric Bernstein basis functions $\{\beta_\gamma^d(u,v)\}_{\gamma \in \sigma^d}$ on I^d are given by convolution of the Bernstein basis function $\{\beta_\sigma(u,v) = \beta_{\sigma_i}(u,v)\}_{\sigma_i \in \sigma}$ indexed by σ^d , $i.e.$,

$$\{\beta_\gamma^d(u,v)\}_{\gamma \in \sigma^d} = \overbrace{\beta_\sigma(u,v) \otimes \ \beta_\sigma(u,v) \otimes ... \otimes \beta_\sigma(u,v)}^{d-fold}. \tag{2.4}$$

A toric Bézier patch defined on lattice polygon of depth d and the corresponding convex hull I^d of σ^d in the projective space is a surface parameterized by the map $\mathcal{P} : I^d \to ¡\ \mathbb{P}^4$ (for $(u,v) \in I^d$) is defined as,

$$\mathcal{P}(u,v) = \sum_{\gamma \in \sigma^d} \beta_\gamma^d(u,v) \left(\omega_\gamma p_\gamma, \omega_\gamma \right), \tag{2.5}$$

the control structure consists of the mass-points $\left\{ \left(\omega_\gamma p_\gamma, \omega_\gamma \right) \right\}_{\gamma \in \sigma^d}$, where $\{p_\gamma\}_{\gamma \in \sigma^d}$ are the control points and $\{\omega_\gamma \geq 0\}_{\gamma \in \sigma^d}$ are the respective weights. $\mathcal{P}_\gamma^d(u,v)_{\gamma \in \sigma^d}$ are the blending functions, known as the *toric Bernstein basis functions* for I^d .

The toric Bezier patches are the rational surfaces lying in the affine or projective spaces. The derivative of a rational surface is not that straightforward in general but rather a little complicated. It is however advantageous to find the derivatives of the numerator and denominator parts of the rational surface first and then to apply the quotient rule of derivation to get the derivative of the quotient. Therefore, instead of derivative of the rational toric Bezier patch, the derivative of the corresponding toric Bezier surface in the space of mass-points is more useful. A detailed account of finding derivative of toric Bezier patch of depth *d w.r.t.* the surface parameters *u* and *v* can be seen in [30] (pages 82-84). The partial derivative *w.r.t. u* of Bernstein polynomials $\beta_\gamma^d(u,v)$ for lattice polygons of depth d is given by the following expression

$$\frac{\partial\ \beta_\gamma^d(u,v)}{\partial u} = d \sum_{\sigma_i \in \sigma} \frac{\partial\ \beta_{\sigma_i}(u,v)}{\partial u} \beta_{\gamma-\sigma_i}^{d-1}(u,v), \tag{2.6}$$

which leads to the first order partial differentiation *w.r.t.* *u* of the polynomial patch $\mathcal{P}(u,v)$ eq. (2.5) and is given by

$$\mathcal{P}_u(u,v) = d\sum_{\gamma\in\sigma^d}\left(\sum_{\sigma_i\in\sigma}\frac{\partial\beta_{\sigma_i}(u,v)}{\partial u}\beta_{\gamma-\sigma_i}^{d-1}(u,v)\right)\left(\omega_\gamma p_\gamma,\omega_\gamma\right). \tag{2.7}$$

The second order partial derivatives of toric Bezier patch *w.r.t.* its parameters *u* and *v* (later to be used in next section) can be computed and they are

$$\begin{aligned}\mathcal{P}_{uu}(u,v) = {} & d\sum_{\gamma\in\sigma^d}\left(\sum_{\sigma_i\in\sigma}\frac{\partial^2\beta_{\sigma_i}(u,v)}{\partial u^2}\beta_{\gamma-\sigma_i}^{d-1}(u,v)\right)\left(\omega_\gamma p_\gamma,\omega_\gamma\right)\\ & + d(d-1)\sum_{\gamma\in\sigma^d}\left(\sum_{\sigma_i\in\sigma}\frac{\partial\beta_{\sigma_i}(u,v)}{\partial u}\left(\sum_{\delta_i\in\sigma,\delta_i\neq\sigma_i}\frac{\partial\beta_{\delta_i}(u,v)}{\partial u}\beta_{\gamma-\sigma_i-\delta_i}^{d-2}(u,v)\right)\right)\left(\omega_\gamma p_\gamma,\omega_\gamma\right),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{P}_{vv}(u,v) = {} & d\sum_{\gamma\in\sigma^d}\left(\sum_{\sigma_i\in\sigma}\frac{\partial^2\beta_{\sigma_i}(u,v)}{\partial v^2}\beta_{\gamma-\sigma_i}^{d-1}(u,v)\right)\left(\omega_\gamma p_\gamma,\omega_\gamma\right)\\ & + d(d-1)\sum_{\gamma\in\sigma^d}\left(\sum_{\sigma_i\in\sigma}\frac{\partial\beta_{\sigma_i}(u,v)}{\partial v}\left(\sum_{\delta_i\in\sigma,\delta_i\neq\sigma_i}\frac{\partial\beta_{\delta_i}(u,v)}{\partial v}\beta_{\gamma-\sigma_i-\delta_i}^{d-2}(u,v)\right)\right)\left(\omega_\gamma p_\gamma,\omega_\gamma\right).\end{aligned} \tag{2.9}$$

Above partial derivatives of $\mathcal{P}(u,v)$ are helpful in the extremization of the quasi-harmonic functional used as objective function to obtain quasi-harmonic toric Bezier patch as the solution of Plateau toric Bezier problem, the task accomplished in section 4. The next section gives a brief description of the energy functionals that can be used as objective functions for extremization purpose to obtain an approximate minimal surface.

3. QUASI-HARMONIC FUNCTIONAL

To find an approximate minimal surface, several energy functionals have been used instead of area functional itself which involves a square root in its integrand. These functionals may be extremized to obtain quasi-minimal surfaces with prescribed boundary in general. Following section gives a brief description of different energy functionals which may be used as objective functions to trigger the extremization process for different surfaces along with the quasi-harmonic functional that is used in our next section to obtain a quasi-harmonic Bezier patch as an approximate solution to the Plateau-toric Bezier problem. In an optimization problem, one needs to minimize the area functional (eq. (3.1)) for any surface $\mathbf{x}(u,v)$. The area functional of the toric Bézier surface $\mathcal{P}(u,v)$ is

$$\mathcal{A}(\mathcal{P}(u,v)) = \int_{I^d}|\mathcal{P}(u,v)_u\times\mathcal{P}(u,v)_v\,|\,dudv, \tag{3.1}$$

where $I^d\subset ¢^2$ is the parametric domain over which the surface $\mathcal{P}(u,v)$ is defined as a map and $\mathcal{P}_u(u,v)$ and $\mathcal{P}_v(u,v)$ are the partial derivatives of $\mathcal{P}(u,v)$ with respect to parameters u and v. However, the non-linearity of this functional makes it difficult to find the solution of Plateau problem in general. Douglus [6] replaced the area functional for a surface $\mathbf{x}(u,v)$ with a relatively easy to manage Dirichlet functional

$$D\left(\mathbf{x}(u,v)\right) = \frac{1}{2}\int_R\left(\|\mathbf{x}_u\|^2+\|\mathbf{x}_v\|^2\right)dudv. \tag{3.2}$$

This functional was utilized by Monterde [17] to solve the Plateau-Bezier problem. Sun and Zhu [34] found the extremals of toric Bezier surfaces by minimizing the Dirichlet functional.

Monterde and Ugail [35], in 2006, introduced a general biquadratic functional

$$\mathcal{L}\left(\mathbf{x}(u,v)\right)=\frac{1}{2}\int_R\left(a\,\|\,\mathbf{x}_{uu}\,\|^2+b\langle\mathbf{x}_{uu},\mathbf{x}_{uv}\rangle+c\,|\,\mathbf{x}_{uv}\,\|^2+d\langle\mathbf{x}_{uv},\mathbf{x}_{vv}\rangle+e\,\|\,\mathbf{x}_{vv}\,\|^2\right)dudv, \tag{3.3}$$

with *a, b, c, d* and *e* being the real constants. By assigning different values to these constants, the functional coul d be reduced to other alternative functionals used for minimizing purposes such as Farin and Hansford functional [36], standard biharmonic functional introduced by Schneider and Kobbelt [37] or Bloor and Wilson's modified biharmonic functional [38]. The solution of the area problem for Bezier patches by extremizing the quasi-harmonic functional

$$\mathcal{H}\left(\mathbf{x}(u,v)\right)=\int_R\left(\mathbf{x}_{uu}+\mathbf{x}_{vv}\right)^2dudv. \tag{3.4}$$

for the surface x(u, v) is already known [24]. We choose this quasi-harmonic functional as an objective function to find the solution of Plateau's toric Bezier problem, as mentioned earlier that the toric Bezier patches generalize the classical rational triangular and tensor-product Bezier surfaces defined over multi-sided domains. It gives [24] better approximation of surfaces with lesser area and smaller mean curvature values at arbitrary points when compared to the Dirichlet functional for Bezier surfaces. The quasi-harmonic functional, taken as an objective function, for the toric Bézier patch $\mathcal{P}(u,v)$ (eq. (2.3)) is given by

$$\mathcal{H}(\mathcal{P}(u,v))=\int_{I^d}\left(\mathcal{P}_{uu}(u,v)+\mathcal{P}_{vv}(u,v)\right)^2dudv, \tag{3.5}$$

where $\mathcal{P}_{uu}(u,v)$ and $\mathcal{P}_{vv}(u,v)$ are given by eqs. (2.8) and (2.9). In the following section, necessary and sufficient condition for a toric Bezier patch to be a quasi-harmonic toric Bezier is computed by extremizing the above mentioned quasi-harmonic functional eq. (3.5).

4. QUASI-HARMONIC TORIC BEZIER PATCHES FOR A GIVEN BOUNDARY

For the Plateau Toric Bézier problem, we minimize the quasi-harmonic functional to get a quasi-harmonic toric Bézier patch $\mathcal{P}(u,v)$. For this, we find the gradient of the $\mathcal{H}(\mathcal{P}(u,v))$ with respect to the inner unknown mass-points $\left(\omega_\lambda p_\lambda,\omega_\lambda\right)$ and equate it to zero to find the constraints as linear equations under which the $\mathcal{P}(u,v)$ is quasi-harmonic toric Bezier patch.

Theorem 4.1. *If the mass-points associated to the boundary lattice points of the convex hull I^d of the toric Bézier patch $\mathcal{P}(u,v)=\sum_{\gamma\in\sigma^d}\beta_\gamma^d(u,v)\left(\omega_\gamma p_\gamma,\omega_\gamma\right)$ are given, the patch $\mathcal{P}(u,v)$ is quasi-harmonic toric Bézier surface if and only if the inner unknown mass-points $\left(\omega_\lambda p_\lambda,\omega_\lambda\right)$ associated to the lattice points of the convex hull satisfy the following system of linear equations:*

$$\int_{I^d}\sum_{\gamma\in\sigma^d}\left((\xi^{\lambda,u}+(d-1)\eta^{\lambda,u})+(\xi^{\lambda,v}+(d-1)\eta^{\lambda,v})\right)\left((\xi^{\gamma,u}+(d-1)\eta^{\gamma,u})+(\xi^{\gamma,v}+(d-1)\eta^{\gamma,v})\right)(\omega_\gamma p_\gamma,\omega_\gamma)dudv=0, \tag{4.1}$$

where the coefficients $\xi^{\gamma,u}$ and $\eta^{\gamma,u}$ are,

$$\xi^{\gamma,u}=\sum_{\sigma_i\in\sigma}\frac{\partial^2\beta_{\sigma_i}(u,v)}{\partial u^2}\beta_{\gamma-\sigma_i}^{d-1}(u,v),$$
$$\eta^{\gamma,u}=\sum_{\sigma_i\in\sigma}\frac{\partial\beta_{\sigma_i}(u,v)}{\partial u}\left(\sum_{\delta_i\in\sigma,\delta_i\neq\sigma_i}\frac{\partial\beta_{\delta_i}(u,v)}{\partial u}\beta_{\gamma-\sigma_i-\delta_i}^{d-2}(u,v)\right). \tag{4.2}$$

Other coefficients $\xi^{\gamma,v}$, $\xi^{\lambda,u}$, $\xi^{\lambda,v}$, $\eta^{\gamma,v}$, $\eta^{\lambda,u}$ and $\eta^{\lambda,v}$ are obtained by replacing u by v and γ by λ in above eq. (4.2).

Proof: The quasi-harmonic functional (3.5) can be rewritten as

$$\mathcal{H}(\mathcal{P}(u,v)) = \int_{I^d} \langle \mathcal{P}_{uu}(u,v), \mathcal{P}_{uu}(u,v) \rangle + \langle \mathcal{P}_{vv}(u,v), \mathcal{P}_{vv}(u,v) \rangle + 2\langle \mathcal{P}_{uu}(u,v), \mathcal{P}_{vv}(u,v) \rangle dudv, \tag{4.3}$$

where the operator $\langle , \rangle$ denotes the inner product of the two functions. For an inner mass point $(\omega_\lambda p_\lambda, \omega_\lambda)$, $\lambda \in \sigma^d$ and $a \in \{1,2,3,4\}$, the gradient of the quasi-harmonic functional with respect to the coordinates of $(\omega_\lambda p_\lambda, \omega_\lambda)$ is given by

$$\begin{aligned}\frac{\partial \mathcal{H}(\mathcal{P}(u,v))}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a} = 2\int_{I^d} & \left\langle \frac{\partial \mathcal{P}_{uu}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a}, \mathcal{P}_{uu}(u,v) \right\rangle + \left\langle \frac{\partial \mathcal{P}_{vv}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a}, \mathcal{P}_{vv}(u,v) \right\rangle \\ & + \left\langle \frac{\partial \mathcal{P}_{uu}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a}, \mathcal{P}_{vv}(u,v) \right\rangle + \left\langle \frac{\partial \mathcal{P}_{vv}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a}, \mathcal{P}_{uu}(u,v) \right\rangle dudv.\end{aligned} \tag{4.4}$$

Differentiating partially $\mathcal{P}_{uu}(u,v)$ and $\mathcal{P}_{vv}(u,v)$, the 2nd order partial derivatives (eqs. (2.8) and (2.9) respectively) of the toric Bézier patch $\mathcal{P}(u,v)$ *w.r.t.* the inner mass-point coordinates $(\omega_\lambda p_\lambda, \omega_\lambda)$ gives us

$$\begin{aligned}&\frac{\partial \mathcal{P}_{uu}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a} = \\ & d\left(\sum_{\sigma_i \in \sigma} \frac{\partial^2 \beta_{\sigma_i}(u,v)}{\partial u^2} \beta^{d-1}_{\lambda-\sigma_i}(u,v) \right) e^a + d(d-1)\left(\sum_{\sigma_i \in \sigma} \frac{\partial \beta_{\sigma_i}(u,v)}{\partial u} \left(\sum_{\delta_i \in \sigma, \delta_i \neq \sigma_i} \frac{\partial \beta_{\delta_i}(u,v)}{\partial u} \beta^{d-2}_{\lambda-\sigma_i-\delta_i}(u,v) \right)\right) e^a,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\frac{\partial \mathcal{P}_{vv}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a} \\ &= d\left(\sum_{\sigma_i \in \sigma} \frac{\partial^2 \beta_{\sigma_i}(u,v)}{\partial v^2} \beta^{d-1}_{\lambda-\sigma_i}(u,v) \right) e^a + d(d-1)\left(\sum_{\sigma_i \in \sigma} \frac{\partial \beta_{\sigma_i}(u,v)}{\partial v} \left(\sum_{\delta_i \in \sigma, \delta_i \neq \sigma_i} \frac{\partial \beta_{\delta_i}(u,v)}{\partial v} \beta^{d-2}_{\lambda-\sigma_i-\delta_i}(u,v) \right)\right) e^a.\end{aligned} \tag{4.6}$$

It is to be noted that in above eqs. (4.5) and (4.6), the coefficients $\beta^{d-2}_{\lambda-\sigma_i-\delta_i}(u,v) = 0$ if $\lambda - \sigma_i - \delta_i \notin \sigma^{d-2}$, e^a denote the a^{th} vector of the standard basis, *i.e.* $e^1 = \{1,0,0,0\}, e^2 = \{0,1,0,0\}, e^3 = \{0,0,1,0\}$ and $e^4 = \{0,0,0,1\}$. Substituting the coefficients $\xi^{\gamma,u}$, $\xi^{\gamma,v}$, $\xi^{\lambda,u}$, $\xi^{\lambda,v}$ and $\eta^{\gamma,u}$, $\eta^{\gamma,v}$, $\eta^{\lambda,u}$, $\eta^{\lambda,v}$ (eqs. (4.2)) in eqs. (2.8)-(2.9), we get

$$\mathcal{P}_{uu}(u,v) = d\sum_{\gamma \in \sigma^d} \xi^{\gamma,u} (\omega_\gamma p_\gamma, \omega_\gamma) + d(d-1)\sum_{\gamma \in \sigma^d} \eta^{\gamma,u} (\omega_\gamma p_\gamma, \omega_\gamma), \tag{4.7}$$

$$\mathcal{P}_{vv}(u,v) = d\sum_{\gamma \in \sigma^d} \xi^{\gamma,v} (\omega_\gamma p_\gamma, \omega_\gamma) + d(d-1)\sum_{\gamma \in \sigma^d} \eta^{\gamma,v} (\omega_\gamma p_\gamma, \omega_\gamma). \tag{4.8}$$

so that the eqs. (4.5) and (4.6) reduce to

$$\frac{\partial \mathcal{P}_{uu}(u,v)}{\partial (\omega_\lambda p_\lambda, \omega_\lambda)^a} = d\xi^{\lambda,u} e^a + d(d-1)\eta^{\lambda,u} e^a, \tag{4.9}$$

and

$$\frac{\partial \mathcal{P}_{vv}(u,v)}{\partial\left(\omega_{\lambda} p_{\lambda}, \omega_{\lambda}\right)^{a}} = d\xi^{\lambda,v} e^{a} + d(d-1)\eta^{\lambda,v} e^{a}. \tag{4.10}$$

Now substitute eqs. (4.7) to (4.10) in eq. (4.4) to get

$$\frac{\partial \mathcal{H}(\mathcal{P}(u,v))}{\partial\left(\omega_{\lambda} p_{\lambda}, \omega_{\lambda}\right)^{a}} = 2d^{2} \int_{I^{d}} \sum_{\gamma\in\sigma^{d}} \left((\xi^{\lambda,u} + (d-1)\eta^{\lambda,u}) + (\xi^{\lambda,v} + (d-1)\eta^{\lambda,v})\right) \tag{4.11}$$

$$\left((\xi^{\gamma,u} + (d-1)\eta^{\gamma,u}) + (\xi^{\gamma,v} + (d-1)\eta^{\gamma,v})\right)(\omega_{\gamma} p_{\gamma}, \omega_{\gamma}) du dv.$$

We can now obtain the set of linear system of equations as stated in eq. (4.1) for which the toric Bezier patch is quasi-harmonic surface by setting $\frac{\partial \mathcal{H}(\mathcal{P}(u,v))}{\partial\left(\omega_{\lambda} p_{\lambda}, \omega_{\lambda}\right)^{a}} = 0$. □

5. QUASI-HARMONIC TORIC BEZIER PATCHES OVER MULTI-SIDED CONVEX HULLS

In this section, we construct toric Bezier patches over two different convex hulls namely 1) the trapezoidal convex hull and 2) hexagonal convex hull. We use the linear set of equations given in eq. (4.1) to compute the inner unknown mass-points of the toric Bezier patches spanned by the curves over these convex hulls in order to obtain the associated quasi-harmonic toric Bezier patch. In the former case we construct the quasi-harmonic toric Bezier patches for $n = 2$, $m = p = 1$ in which we find one condition on the unknown inner mass point and $n = 2$, $m = 3$, $p = 1$, we find three conditions on the three unknown inner mass points whereas in the latter case we construct the quasi-harmonic toric Bezier patch with depth $d = 2$, in this case there appear seven unknown inner points in terms of known boundary mass points. To simplify the calculations, the weights ω are all taken equal.

5.1. Quasi-harmonic toric Bezier patches over trapezoidal convex hulls

The general representation of toric Bezier patch $B(u,v)$ over a trapezoidal convex hull I_{σ} is defined as follows. Let n, $p \geq 1$ and $m \geq 0$ be integers and set

$$\sigma = \{(i,j) : 0 \leq j \leq n, 0 \leq i \leq m + pn + pj\} \tag{5.1}$$

$$\beta_{ij}(u,v) = c_{ij} u^{i} (m + pn - pv - u)^{m+pn-pj-i} v^{i} (n-v)^{n-j}. \tag{5.2}$$

Then the toric Bezier surface $B(u,v)$ defined over a general trapezoidal hull is expressed as

$$\mathcal{P}(u,v) = \sum_{(i,j)\in I} c_{ij} u^{i} (m + pn - pv - u)^{m+pn-pj-i} v^{i} (n-v)^{n-j} \left(\omega_{ij} P_{ij}, \omega_{ij}\right), \quad (u,v) \in I_{\sigma}. \tag{5.3}$$

Example 1. In particular, for $n = 2$, $m = p = 1$, the eq. (5.1) gives us the following set of integer lattice-points

$$\sigma = \{(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (1, 2), (2, 1), (1, 1)\},$$

with only one inner unknown mass-point p_{11} associated to $\sigma_{\text{i}} = (1, 1)$. The Bernstein polynomials

$$\beta_{ij}(u,v) = c_{ij} u^{i} (2-v)^{2-j} v^{j} (3-u-v)^{3-i-j}, \tag{5.4}$$

for the respective lattice-points come out to be

$$\beta_{00}(u,v)=\frac{1}{108}(2-v)^2(-u-v+3)^3,\quad \beta_{10}(u,v)=\frac{1}{16}u(2-v)^2(-u-v+3)^2,$$
$$\beta_{20}(u,v)=\frac{1}{16}u^2(2-v)^2(-u-v+3),\quad \beta_{30}(u,v)=\frac{1}{108}u^3(2-v)^2,$$
$$\beta_{01}(u,v)=\frac{1}{4}(2-v)v(-u-v+3)^2,\quad \beta_{02}(u,v)=\frac{1}{4}v^2(-u-v+3),\qquad (5.5)$$
$$\beta_{12}(u,v)=\frac{uv^2}{4},\quad \beta_{21}(u,v)=\frac{1}{4}u^2(2-v)v,$$
$$\beta_{11}(u,v)=u(2-v)v(-u-v+3),$$

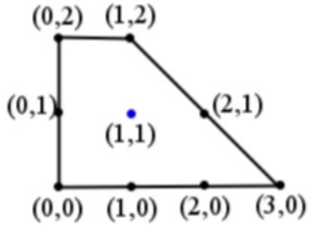

Figure 2. Trapezoidal domain with 1 inner lattice point, shown as the blue dot, indexing the corresponding unknown mass-point

in which c_{ij} have been chosen appropriately. The toric Bezier patch over the given trapezoidal convex hull I_σ, as shown in fig.2 with corresponding Bernstein polynomials defined over lattice points is expressed as

$$\mathcal{P}(u,v)=\sum_{(i,j)\in I}\beta_{ij}(u,v)(\omega_{ij}P_{ij},\omega_{ij}),\qquad (5.6)$$

where $(u, v) \in I_\sigma$. We find the constraints for the toric Bezier patch with unknown inner mass-points to be quasi-harmonic by substituting the second order partial derivative and their gradient with respect to the inner unknown mass-point p11 in eq. (4.1). The toric Bezier patch is quasi-harmonic if and only if the mass-points of the patch satisfy the following constraint equation

$$p_{11}=0.0904\,p_{00}-0.1973\,p_{01}+0.01430\,p_{02}+0.1970\,p_{10}+0.1006\,p_{12}+0.09269\,p_{20}-0.1390\,p_{21}+0.0438\,p_{30}.\qquad (5.7)$$

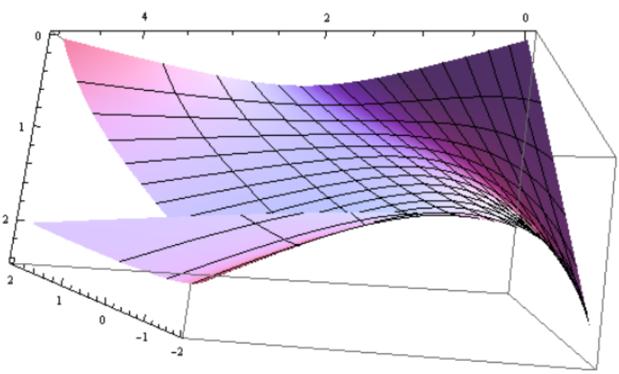

Figure 3. A quasi-harmonic toric Bezier patch with 1 inner lattice point indexing the unknown mass-point which is computed by using the eq. (5.7)

A particular example of a toric Bezier patch over trapezoidal convex hull with 1 unknown inner mass-point is given in figure 3 by taking known mass-points on the boundary of the convex hull. The unknown inner mass-point p_{11} is computed by using the result as stated in eq. (5.7).

Example 2. For $n = 2$, $m = 3$, $p = 1$, the set of integer lattice points is given as

$$\sigma = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (0, 1), (0, 2), (1, 2), (2, 2), (3, 2), (2, 1), (3, 1), (4, 1), (1, 1)\}$$

with 3 inner unknown mass-point p_{11}, p_{21} and p_{31}. The Bernstein polynomials

$$\beta_{ij}(u,v) = c_{ij} u^i (2-v)^{2-j} v^j (5-u-v)^{5-i-j}, \tag{5.8}$$

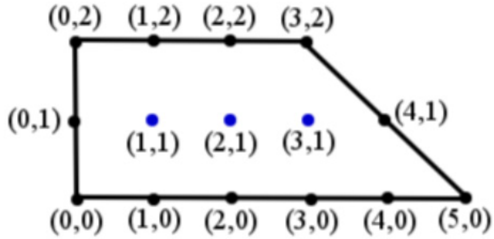

Figure 4. Trapezoidal domain with 3 inner lattice point, marked as blue dots, associated to the corresponding unknown mass-points

for the respective lattice points are

$$\begin{aligned}
&\beta_{00}(u,v) = \frac{1}{12500}(2-v)^2(5-u-v)^5, && \beta_{10}(u,v) = \frac{1}{2500}u(2-v)^2(5-u-v)^4,\\
&\beta_{20}(u,v) = \frac{1}{1250}u^2(2-v)^2(5-u-v)^3, && \beta_{30}(u,v) = \frac{1}{1250}u^3(2-v)^2(5-u-v)^2,\\
&\beta_{40}(u,v) = \frac{1}{2500}u^4(2-v)^2(5-u-v), && \beta_{50}(u,v) = \frac{1}{12500}u^5(2-v)^2,\\
&\beta_{01}(u,v) = \frac{1}{512}(2-v)(5-u-v)^4 v, && \beta_{02}(u,v) = \frac{1}{108}(5-u-v)^3 v^2,\\
&\beta_{11}(u,v) = \frac{1}{128}u(2-v)(5-u-v)^3 v, && \beta_{21}(u,v) = \frac{3}{256}u^2(2-v)(5-u-v)^2 v,\\
&\beta_{31}(u,v) = \frac{1}{128}u^3(2-v)(5-u-v)v, && \beta_{41}(u,v) = \frac{1}{512}u^4(2-v)v,\\
&\beta_{12}(u,v) = \frac{1}{36}u(5-u-v)^2 v^2, && \beta_{22}(u,v) = \frac{1}{36}u^2(5-u-v)v^2,\\
&\beta_{32}(u,v) = \frac{1}{108}(u^3 v^2).
\end{aligned} \tag{5.9}$$

for an appropriate choice of c_{ij}. The toric Bezier patch over the given convex hull I_σ is defined as

$$\mathcal{P}(u,v) = \sum_{(i,j)\in I} \beta_{ij}(u,v)(\omega_{ij} p_{ij}, \omega_{ij}), \tag{5.10}$$

where $(u, v) \in I_\sigma$. We can find the constraints for the toric Bezier patch with unknown inner mass-points to be quasi-harmonic by substituting the second order partial derivatives and their gradient with respect to each unknown inner mass points, namely p_{11}, p_{21} and p_{31} in eq. (4.1). The toric Bezier patch over the given trapezoidal convex hull is quasi-harmonic if and only if the mass-points of the patch satisfy the following system of equations

$$\begin{aligned} p_{11} &= 0.8157p_{00} - 1.027p_{01} + 0.3170p_{02} + 0.1815p_{10} + 0.2146p_{12} + 0.1106p_{20} - 0.0981p_{22} + 0.06520p_{30} \\ &+ 0.1122p_{32} + 0.0210p_{40} - 0.07081p_{41} + 0.02990p_{50}, \\ p_{21} &= -0.4561p_{00} + 0.5450p_{01} - 0.1718p_{02} + 0.2450p_{10} + 0.0246p_{12} + 0.2527p_{20} + 0.7647p_{22} + 0.03338p_{30} \\ &- 0.3465p_{32} + 0.02031p_{40} + 0.2384p_{41} - 0.09169p_{50}, \\ p_{31} &= 0.0871p_{00} - 0.0955p_{01} + 0.0341p_{02} - 0.0211p_{10} - 0.0221p_{12} + 0.1053p_{20} - 0.3099p_{22} + 0.3448p_{30} \\ &+ 0.6595p_{32} + 0.1629p_{40} - 0.4250p_{41} + 0.2278p_{50}. \end{aligned} \tag{5.11}$$

5.2. Quasi-harmonic toric Bezier patches of depth 2

Consider a toric Bezier patch defined over hexagonal convex hull, shown by the dotted line in fig. 6 with lattice-points,

$$\sigma = \{(0, 0), (1, 0), (0, 1), (1, 1), (1, 2), (2, 1), (2, 2)\},$$

where the edges of the hexagonal convex hull I_σ are

$$\overline{L}_1(u,v) = v; \overline{L}_2(u,v) = -v+2; \overline{L}_3(u,v) = -u+2; \overline{L}_4(u,v) = u; \overline{L}_5(u,v) = v-u+1; \overline{L}_6(u,v) = -v+u+1.$$

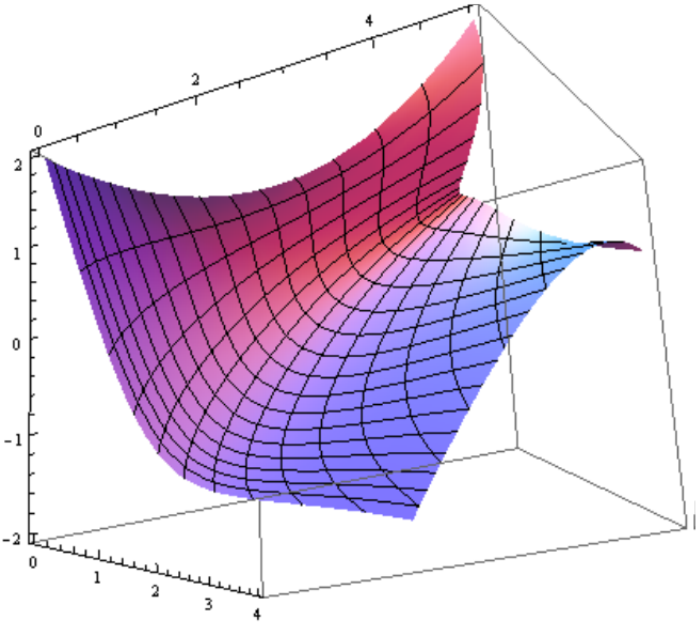

Figure 5. A quasi-harmonic toric patch defined over trapezoidal convex hull with 3 inner lattice points indexing the unknown mass-points which are computed by using system of eqs. (5.11)

The toric Bernstein polynomials for each lattice-point $\sigma_i \in \sigma$ can be defined using the following relation

$$\beta_{\sigma_i} = c_{\sigma_i} \overline{L}_1(u,v)^{\overline{L}_1(\sigma_i)} \overline{L}_2(u,v)^{\overline{L}_2(\sigma_i)} \overline{L}_3(u,v)^{\overline{L}_3(\sigma_i)} \overline{L}_4(u,v)^{\overline{L}_4(\sigma_i)} \overline{L}_5(u,v)^{\overline{L}_5(\sigma_i)} \overline{L}_6(u,v)^{\overline{L}_6(\sigma_i)}. \tag{5.12}$$

Whereas, the Bernstein polynomials $\{\beta_\gamma^d\}_{\gamma \in I^d}$ for the toric Bezier patch of depth $d = 2$ can be computed by convolving the Bernstein polynomials $\beta_{\sigma i}$ (u, v) as stated above in eq. (5.12) indexed by the Minkowski sum $\sigma \oplus \sigma = \sigma^2$. The toric Bezier patch of depth 2 over the hexagonal convex hull (as shown as solid line in fig. 6) I^d, with corresponding Bernstein polynomials is defined as

$$\mathcal{P}(u,v) = \sum_{\gamma \in \sigma^d} \beta_\gamma^d(u,v)\left(\omega_\gamma p_\gamma, \omega_\gamma\right), \tag{5.13}$$

where $(u, v) \in I^d$. Similarly, as we already have shown for the toric Bezier patches over trapezoidal convex hull, the constraints on the mass-points for this patch can also be computed by using eq. (4.1). The toric Bezier patch of depth 2 over the hexagonal convex hull with 7 unknown inner-mass points is quasi-harmonic if and only if these inner- mass points of the patch satisfy the following linear system of constraints

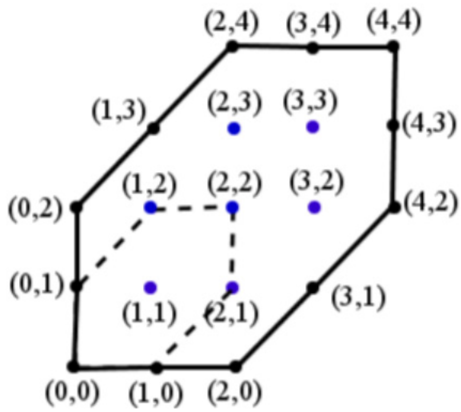

Figure 6. A hexagonal convex hull (solid line) of depth $d = 2$ with 19 lattice points indexed by the set I_σ^2 with 7 inner lattice points, marked as blue dots corresponding to the unknown mass-points. The dotted lines represent the hexagonal hull of I_σ for toric Bezier patch of depth $d = 1$

$$\begin{aligned}
p_{11} &= -0.1986p_{00} - 0.0264p_{01} - 0.8067p_{02} + 0.1294p_{10} - 5.9296p_{13} - 0.4091p_{20} - 19.2105p_{24} - 5.8759p_{31} \\
&\quad -38.9767p_{34} - 41.2599p_{42} + 23.8580p_{44}, \\
p_{21} &= -0.0018p_{00} + 0.0033p_{01} - 0.0582p_{02} + 0.0045p_{10} - 1.9341p_{13} - 0.0698p_{20} - 0.9901p_{24} - 0.2803p_{31} \\
&\quad +10.1815p_{34} - 1.1566p_{42} + 2.804p_{44}, \\
p_{12} &= 0.1143p_{00} - 0.5075p_{01} + 5.7024p_{02} - 0.3095p_{10} + 25.0159p_{13} + 6.2165p_{20} + 110.1030p_{24} + 25.0099p_{31} \\
&\quad -3.9603p_{34} + 104.0531p_{42} - 142.3752p_{44}, \\
p_{22} &= 0.0004p_{01} - 0.0434p_{02} + 0.0034p_{10} - 0.1725p_{13} - 0.0285p_{20} - 0.3504p_{24} - 0.1228p_{31} - 0.41809p_{34} \\
&\quad -0.4812p_{42} - 0.0297p_{44}, \\
p_{32} &= -0.0010p_{00} + 0.0020p_{01} + 0.4605p_{02} - 0.0834p_{10} + 2.6567p_{13} + 0.1535p_{20} + 8.5203p_{24} + 1.1316p_{31} \\
&\quad -0.0712p_{34} + 6.7355p_{42} + 4.4061p_{44}, \\
p_{23} &= -0.0168p_{00} + 0.0555p_{01} - 0.9415p_{02} + 0.0297p_{10} - 4.0360p_{13} - 0.8251p_{20} - 19.3765p_{24} - 3.0281p_{31} \\
&\quad -2.3478p_{34} - 11.6685p_{42} + 10.8462p_{44}, \\
p_{33} &= 0.0005p_{00} - 0.0007p_{01} - 0.0393p_{02} + 0.0060p_{10} - 0.2230p_{13} - 0.0299p_{20} - 0.9192p_{24} - 0.1673p_{31} \\
&\quad +0.2163p_{34} - 0.9267p_{42} - 1.0000p_{43} - 2.9665p_{44}.
\end{aligned} \tag{5.14}$$

Toric Bezier patches defined over any polygonal convex hull of domains with prescribed boundary mass-points can be approximated to quasi-harmonic toric Bezier patch using the result stated in eq. (4.1).

6. CONCLUSION

In this paper, we considered the quasi-Plateau problem which consists of finding the quasi-minimal surface with prescribed entire or partial border. In particular, we find a solution to the Plateau-toric Bezier problem for toric Bezier surface, which is the generalization of classical rational triangular and tensor-product Bezier surfaces defined over multi-sided domains. Quasi harmonic functional is used as the objective functional which is extremized to obtain a toric Bezier patch among all the possible patches with prescribed boundary, which we termed as quasi-harmonic toric Bezier patch. This patch serves as the solution to quasi Plateau-toric Bezier problem. The vanishing condition for gradient of the quasi-harmonic functional yields the constraints on mass-points of the toric Bezier patch as system of linear equations under which it is a quasi-harmonic toric Bezier patch. This scheme is applied to toric Bezier patches for different prescribed borders defined over the multi-sided convex hulls to illustrate its effectiveness and exibility.

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