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Lower semi-continuity in a generalized metric space

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Abstract

In this paper, we verify the lower semi-continuity and Ekeland's variational principle for very recent results in a generalized metric space which introduced by Mohamed Jleli and Bessem Samet [2]. And in the sequel we obtain certain fixed point theorems and related topics.

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1. Preliminaries

Mohamed Jleli and Bessem Samet introduced very recent in [2] a new concept of generalized metric spaces for which they extended some well-known fixed point results including Banach contraction principle, Ćirić's fixed point theorem and so on; and new concept of generalized metric spaces recover various topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces. For more detail refer to [4, 5, 6, 7, 3].

Let X be a nonempty set and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, let us define the set

$$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0\}.$$

Definition 1.1 ([2]). *We say that D is a generalized metric on X if it satisfies the following conditions:*

$(D1)$ *for every $(x, y) \in X \times X$, we have $D(x, y) = 0 \Rightarrow x = y$;*
$(D2)$ *for every $(x, y) \in X \times X$, we have $D(x, y) = D(y, x)$;*
$(D3)$ *there exists $C > 0$ such that if $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$, then $D(x, y) \le C \limsup_{n\to\infty} D(x_n, y)$.*

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In this case, we say the pair (X, D) is a generalized metric space.

Definition 1.2 ([2]). *Let (X, D) be a generalized metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.*

1. *We say that $\{x_n\}$ D-converges to x if $\{x_n\} \in C(D, X, x)$.*
2. *We say that $\{x_n\}$ is a D-Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_{n+m}) = 0$.*
3. *It is said to be D-complete if every Cauchy sequence in X is convergent to some element in X.*

Proposition 1.3. *Let (X, D) be a generalized metric space. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ D-converges to x and $\{x_n\}$ D-converges to y, then $x = y$.*

2. Main result and fixed point theorems

Definition 2.1. *Let (X, D) be a complete generalized metric space and $\varphi : X \to \mathbb{R}^+$ be a given function. Then, φ is said to be a lower semi-continuous (l.s.c) function on X if*

$$\{x_n\} \in C(D, X, x) \Rightarrow \varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n),$$

for every $x \in X$.

Theorem 2.2. *Let (X, D) be a complete generalized metric space and $\varphi : X \to \mathbb{R}^+$ be a lower semi-continuous (l.s.c) function on X. Let $\varepsilon > 0$ and $x \in X$ be such that*

$$\varphi(x) \leq \inf_{t\in X} \varphi(t) + \varepsilon \quad \text{and} \quad \inf_{t\in X} D(x, t) < c', \tag{1}$$

where $c' = \min\left\{C, \frac{1}{C}\right\} \leq 1$. Then there exists some point $y \in X$ such that

$$\varphi(y) \leq \varphi(x), \quad D(x, y) \leq 1,$$

$$\forall z \in X, \quad z \neq y \quad \varphi(y) - \varphi(z) < \varepsilon c' D(y, z). \tag{2}$$

Proof. Let $x_1 := x$. Pick $\{x_n\}$ as follows $\varphi(x_n) \leq \varphi(x)$ and $D(x_n, x) \leq c'$. So we have two cases:

1. $\forall z \neq x_n \quad \varphi(x_n) - \varphi(z) < \varepsilon c' D(x_n, z)$.
2. $\exists z \neq x_n \quad \varphi(x_n) - \varphi(z) \geq \varepsilon c' D(x_n, z)$.

We shall verify that the case (2) since by the case (1), assertion of theorem obtained by $y := x_n$. Put

$$S_n := \{z \in X : z \neq x_n \quad \varepsilon c' D(x_n, z) \leq \varphi(x_n) - \varphi(z)\}.$$

Choose $x_{n+1} \in S_n$, such that

$$\varphi(x_{n+1}) - \inf_{t\in S_n} \varphi(t) < \frac{1}{2}(\varphi(x_n) - \inf_{t\in S_n} \varphi(t)), \tag{3}$$

hence we have

$$\varepsilon c' D(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}).$$

$\{\varphi(x_n)\}$ is bounded bellow and non-increasing, so $\varphi(x_n) \to l$ for some l. Therefore

$$\varepsilon c' D(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}) \to 0,$$

also

$$\varepsilon c' D(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m) \to 0, \quad \text{as} \quad m, n \to \infty \tag{4}$$

so $\{x_n\}$ is Cauchy sequence and by completeness od X $x_n \to x^*$ in D for some $x^* \in X$. Thus

$$\varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) \leq \varphi(x),$$

and

$$D(x, x^*) \leq C \limsup_{n\to\infty} D(x, x_n) \leq Cc' \leq 1.$$

Now to prove (2) let it does not hold. So

$$\exists z \in X, \quad z \neq x^* \quad \varphi(x^*) - \varphi(z) \geq \varepsilon c' D(x^*, z). \tag{5}$$

So

$$\begin{aligned} \varphi(x^*) &\leq \liminf_{m\to\infty} \varphi(x_m) \\ &\leq \liminf_{m\to\infty} (\varphi(x_n) - \varepsilon c' D(x_n, x_m)) \quad \text{by (4)} \\ &\leq \varphi(x_n) - \varepsilon c' \limsup_{m\to\infty} D(x_n, x_m) \\ &\leq \varphi(x_n) - \varepsilon \frac{c'}{C} D(x_n, x^*). \end{aligned} \tag{6}$$

On the other hand by (6) and (5)

$$\begin{aligned} \varphi(z) &\leq \varphi(x^*) - \varepsilon c' D(x^*, z) \\ &\leq \varphi(x_n) - \varepsilon \frac{c'}{C} D(x_n, x^*) - \varepsilon c' D(x^*, z) \\ &\leq \varphi(x_n) - \varepsilon c' D(x_n, z), \end{aligned} \tag{7}$$

since we have

$$D(x_n, z) \leq \frac{1}{C} D(x_n, x^*) + D(x^*, z). \tag{8}$$

Because

$$\forall \varepsilon > 0 \ \exists N \ \forall n \ (n \geq N \Rightarrow D(x_n, z) \leq \limsup_{n\to\infty} D(x_n, z) + \varepsilon).$$

Thus

$$\begin{aligned} D(x_n, z) &\leq \limsup_{n\to\infty} D(x_n, z) + \varepsilon \\ &\leq \limsup_{n\to\infty} (\frac{1}{C} D(x_n, x^*) + D(x^*, z)) + \varepsilon, \\ &\leq D(x^*, z) + \varepsilon \\ &\leq \frac{1}{C} D(x_n, x^*) + D(x^*, z) + \varepsilon \end{aligned}$$

and since $\varepsilon > 0$ and arbitrary therefore

$$D(x_n, z) \leq \frac{1}{C} D(x_n, x^*) + D(x^*, z).$$

The (7) implies that $z \in S_n$. Now by (3)

$$2\varphi(x_{n+1}) - \varphi(x_n) \leq \inf_{t \in S_n} \varphi(t) < \varphi(z),$$

so when $\varphi(x_n) \to l$ hence $l \leq \varphi(z)$. By *l.s.c.* of φ we get $\varphi(x^*) \leq \liminf_{m\to\infty} \varphi(x_m) = l$. Thus $\varphi(x^*) \leq l \leq \varphi(z)$. But $z \neq x^*$ so from $D(x^*, z) > 0$ we have $\varphi(z) < \varphi(x^*)$, that is a contradiction. □

Theorem 2.3. *Let (X, D) be a complete generalized metric space and $\varphi : X \to \mathbb{R}^+$ be a lower semi-continuous (l.s.c) function on X. Let $\varepsilon > 0$ and $x \in X$ be such that Given $\varepsilon > 0$, then there exists $y \in X$ such that*

$$\varphi(y) \leq \inf_{t \in X} \varphi(t) + \varepsilon,$$

$$\forall z \in X, \quad \varphi(y) - \varphi(z) \leq \varepsilon D(y, z).$$

Theorem 2.4. *Let (X, D) be a complete generalized metric space and $\varphi : X \to \mathbb{R}^+$ be a lower semi-continuous (l.s.c) function on X. Then any mapping $T : X \to X$ satisfying*

$$D(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{9}$$

for each $x \in X$ has a fixed point in X.

T, verifying (9), is called a Caristi mapping on (X, m).

Proof. Put $\varepsilon := \frac{1}{2}$ in the Theorem 2.3 for φ in (9).

$$\exists y \in X \quad \text{such that} \quad \varphi(y) - \varphi(z) \leq \frac{1}{2} D(y, z) \quad \forall z \in X.$$

So for $z = Ty$, we get

$$\varphi(y) - \varphi(Ty) \leq \frac{1}{2} D(y, Ty).$$

Therefore by (9), one can find

$$D(y, Ty) \leq \varphi(y) - \varphi(Ty).$$

Thus

$$D(y, Ty) \leq \frac{1}{2} D(y, Ty),$$

which implies that $D(y, Ty) = 0$, so $Ty = y$, that is, T has a fixed point. □

The following Corollaries hold for every p-metric by [2, Proposition 2.8].

Corollary 2.5 ([1])**.** *Let (X, p) be a complete p-metric space and $\varphi : X \to \mathbb{R}^+$ be a l.s.c. function on X. Let $\varepsilon > 0$ and $x \in X$ be such that*

$$\varphi(x) \leq \inf_{t \in X} \varphi(t) + \varepsilon \quad \text{and} \quad \inf_{t \in X} p(x, t) < 1.$$

Then there exists some point $y \in X$ such that

$$\varphi(y) \leq \varphi(x), \quad p(x, y) \leq 1,$$

$$\forall z \in X, \quad z \neq y \quad \varphi(y) - \varphi(z) < \varepsilon p(y, z).$$

Corollary 2.6 ([1])**.** *Let (X, p) be a complete p-metric space and $\varphi : X \to \mathbb{R}^+$ be a l.s.c. function on X. Given $\varepsilon > 0$, then there exists $y \in X$ such that*

$$\varphi(y) \leq \inf_{t \in X} \varphi(t) + \varepsilon,$$

$$\forall z \in X, \quad \varphi(y) - \varphi(z) \leq \varepsilon p(y, z).$$

Corollary 2.7 ([1])**.** *Let (X, p) be a complete p-metric space and $\varphi : X \to \mathbb{R}^+$ be a l.s.c. function on X. Then any mapping $T : X \to X$ satisfying:*

$$p(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

for each $x \in X$ has a fixed point in X.

References

[1] M. Asadi, On Ekeland's Variational Principle in M-metric spaces, Journal of Convex Analysis, 17(6) (2016) 1151-1158.

[2] M. Jleli and B. Samet, A generalized metric space and related fixed point theorems, Fixed Point Theory and Applications (2015) 2015:61, DOI 10.1186/s13663-015-0312-7

[3] D. Khantwal, S. Aneja, U.C. Gairola, A fixed point theorem for (ϕ, φ)–convex contraction in metric spaces, 5(2) (2021) 240–245, Article Id: 2021:21.

[4] F. Khojasteh, E. Karapınar, H. Khandani, Some applications of Caristi's fixed point theorem in metric spaces, Fixed Point Theory and Applications, 2016:16.

[5] F. Khojasteh, A. Razani, S. Moradi, A fixed point of generalized TF-contraction mappings in cone metric spaces, Fixed Point Theory and Applications 2011:14.

[6] M. Omidvari, S.M. Vaezpour, R. Saadati and S.J. Lee, Best proximity point theorems with Suzuki distances, Journal of Inequalities and Applications volume 2015:27.

[7] A. Tas, On Double Controlled Metric–Like Spaces and Related Fixed Point Theorems, 5(2) (2021) 167-172, Article Id: 2021:13.