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Research Article

Homological aspects of formal triangular matrix rings

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Abstract

Let $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ be a formal triangular matrix ring, where A and B are rings and U is a (B, A)-bimodule. We first give some computing formulas of projective, injective, flat and FP-injective dimensions of special left T-modules. Then we establish some formulas of (weak) global dimensions of T. It is proven that (1) If U_A is flat and ${}_BU$ is projective, $lD(A) \neq lD(B)$, then $lD(T) = \max\{lD(A), lD(B)\}$; (2) If U_A and ${}_BU$ are flat, $wD(A) \neq wD(B)$, then $wD(T) = \max\{wD(A), wD(B)\}$.

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1. Introduction

Formal triangular matrix rings play an important role in ring theory and the representation theory of algebras. This kind of rings are often used to construct examples and counterexamples [7, 13]. Homological properties on formal triangular matrix rings have also attracted more and more interest. For example, Fossum, Griffith and Reiten gave some estimations of global dimension of a formal triangular matrix ring in [6]. Asadollahi and Salarian studied the vanishing of the extension functor Ext over a formal triangular matrix ring and explicitly described the structure of modules of finite projective (resp. injective) dimension in [1]. Loustaunau and Shapiro obtained some bounds on global dimensions and weak global dimensions in a Morita context under certain assumptions [14] (The notion of Morita context is a generalization of formal triangular matrix rings). More generally, Psaroudakis provided bounds for global dimensions, finitistic dimensions and representation dimensions under recollement of abelian categories and then gave applications to formal triangular matrix rings [19]. Recently, the author also established some formulas of homological dimensions of special modules over a formal triangular matrix ring in [18]. In this note, we will continue to provide other computing formulas of homological dimensions of formal triangular matrix rings and modules over them.

Section 2 is devoted to some formulas of homological dimensions of special modules over a formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$, where A and B are rings and U is

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a (B,A)-bimodule. Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ be a left T-module. We prove that (1) If M_1 and M_2 are projective, then $pd(M) = 0$ or $pd(U \otimes_A M_1) + 1$; (2) If M_1 and M_2 are injective, then $id(M) = 0$ or $id(\mathrm{Hom}_B(U, M_2)) + 1$; (3) If M_1 and M_2 are flat, then $fd(M) = 0$ or $fd(U \otimes_A M_1) + 1$. Moreover, we establish the computing formulas of homological dimensions of simple left T-modules. On the other hand, let T be a left coherent ring and ${}_BU$ be finitely presented, $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ be a left T-module such that $\mathrm{Ext}^i_B(U, M_2) = 0$ for any $i \geq 1$, we prove that (1) If $\widetilde{\varphi}^M$ is an epimorphism, then $FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\ker(\widetilde{\varphi}^M))\}$; (2) If $\widetilde{\varphi}^M$ is a monomorphism, then $FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}$.

In Section 3, we give some computing formulas of global homological dimensions of a formal triangular matrix ring $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$. For example, we prove that (1) If U_A is flat and ${}_BU$ is projective, $lD(A) \neq lD(B)$, then $lD(T) = \max\{lD(A), lD(B)\}$; (2) If U_A and ${}_BU$ are flat, $wD(A) \neq wD(B)$, then $wD(T) = \max\{wD(A), wD(B)\}$. In addition, we give some estimations of other "global" dimensions of T such as $lIFD(T)$, $lIPD(T)$, $lPID(T)$ and $lFID(T)$.

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. For a module M, $pd(M)$, $id(M)$ and $fd(M)$ denote the projective, injective and flat dimensions of M, respectively, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+, $\mathrm{Gen}(M)$ is the class consisting of quotients of direct sums of copies of M and $\mathrm{Cogen}(M)$ is the class consisting of submodules of direct products of copies of M. $lD(R)$ and $wD(R)$ denote the left global dimension and weak global dimension of R, respectively. $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ always means a formal triangular matrix ring, where A and B are rings and U is a (B,A)-bimodule. By [9, Theorem 1.5], the category T-Mod of left T-modules is equivalent to the category Ω whose objects are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\varphi^M : U \otimes_A M_1 \to M_2$ is a B-morphism, and whose morphisms from $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}$ are pairs $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1), f_2 \in \mathrm{Hom}_B(M_2, N_2)$ and $\varphi^N(1 \otimes f_1) = f_2\varphi^M$. Given a triple $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ in Ω, we will denote by $\widetilde{\varphi^M}$ the A-morphism from M_1 to $\mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(x)(u) = \varphi^M(u \otimes x)$ for each $u \in U$ and $x \in M_1$. Analogously, the category Mod-T of right T-modules is equivalent to the category Γ whose objects are triples $M = (M_1, M_2)_{\varphi_M}$, where $M_1 \in$ Mod-A, $M_2 \in$ Mod-B and $\varphi_M : M_2 \otimes_B U \to M_1$ is an A-morphism, and whose morphisms from $(M_1, M_2)_{\varphi_M}$ to $(X_1, X_2)_{\varphi_X}$ are pairs (g_1, g_2) such that $g_1 \in \mathrm{Hom}_A(M_1, X_1), g_2 \in \mathrm{Hom}_B(M_2, X_2)$ and $\varphi_X(g_2 \otimes 1) = g_1\varphi_M$. In the paper, we will identify T-Mod (resp. Mod-T) with this category Ω (resp. Γ). Whenever there is no possible confusion, we will omit the morphism φ^M (resp. φ_M). For example, for the left T-module $\begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$, the B-morphism $U \otimes_A M_1 \to (U \otimes_A M_1) \oplus M_2$ is just the injection and for the left T-module $\begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$, the A-morphism $M_1 \oplus \mathrm{Hom}_B(U, M_2) \to \mathrm{Hom}_B(U, M_2)$ is just the projection.

2. Homological dimensions of special modules over formal triangular matrix rings

Lemma 2.1. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *be a left* T*-module.*

(1) [11, Theorem 3.1] *M is a projective left T-module if and only if φ^M is a monomorphism, M_1 is a projective left A-module and* $\mathrm{coker}(\varphi^M)$ *is a projective left B-module.*

(2) [10, Proposition 5.1] and [1, p.956] *M is an injective left T-module if and only if $\widetilde{\varphi^M}$ is an epimorphism,* $\ker(\widetilde{\varphi^M})$ *is an injective left A-module and M_2 is an injective left B-module.*

(3) [6, Proposition 1.14] *M is a flat left T-module if and only if φ^M is a monomorphism, M_1 is a flat left A-module and* $\mathrm{coker}(\varphi^M)$ *is a flat left B-module.*

In [18], we establish some computing formulas of projective, injective and flat dimensions for those left T-modules $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ with φ^M (resp. $\widetilde{\varphi}^M$) a monomorphism or an epimorphism. Now we give some computing formulas of homological dimensions of other special left T-modules.

Proposition 2.2. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ *be a left* T*-module.*

(1) *If* $\mathrm{Tor}_i^A(U, M_1) = 0$ *for any* $i \geq 1$, $\mathrm{coker}(\varphi^M)$ *is a projective left B-module, then*
$$pd(M) = \max\{pd(M_1), pd(\ker(\varphi^M)) + 1\}.$$

(2) *If* $\mathrm{Ext}_B^i(U, M_2) = 0$ *for any* $i \geq 1$, $\ker(\widetilde{\varphi}^M)$ *is an injective left A-module, then*
$$id(M) = \max\{id(M_2), id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$

(3) *If* $\mathrm{Tor}_i^A(U, M_1) = 0$ *for any* $i \geq 1$ *and* $\mathrm{coker}(\varphi^M)$ *is a flat left B-module, then*
$$fd(M) = \max\{fd(M_1), fd(\ker(\varphi^M)) + 1\}.$$

Proof. (1) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} M_1 \\ \mathrm{im}(\varphi^M) \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} 0 \\ \mathrm{coker}(\varphi^M) \end{pmatrix} \to 0.$$

By Lemma 2.1(1), $\begin{pmatrix} 0 \\ \mathrm{coker}(\varphi^M) \end{pmatrix}$ is projective. So by [18, Theorem 2.4], we have

$$pd(M) = \max\{pd\begin{pmatrix} M_1 \\ \mathrm{im}(\varphi^M) \end{pmatrix}, pd\begin{pmatrix} 0 \\ \mathrm{coker}(\varphi^M) \end{pmatrix}\} = \max\{pd(M_1), pd(\ker(\varphi^M)) + 1\}.$$

(2) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} \ker(\widetilde{\varphi}^M) \\ 0 \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} \mathrm{im}(\widetilde{\varphi}^M) \\ M_2 \end{pmatrix} \to 0.$$

By Lemma 2.1(2), $\begin{pmatrix} \ker(\widetilde{\varphi}^M) \\ 0 \end{pmatrix}$ is injective. So by [18, Theorem 2.4], we have

$$id(M) = \max\{id\begin{pmatrix} \ker(\widetilde{\varphi}^M) \\ 0 \end{pmatrix}, id\begin{pmatrix} \mathrm{im}(\widetilde{\varphi}^M) \\ M_2 \end{pmatrix}\} = \max\{id(M_2), id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$

(3) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} M_1 \\ \mathrm{im}(\varphi^M) \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} 0 \\ \mathrm{coker}(\varphi^M) \end{pmatrix} \to 0.$$

By Lemma 2.1(3), $\begin{pmatrix} 0 \\ \mathrm{coker}(\varphi^M) \end{pmatrix}$ is flat. Therefore by [18, Theorem 2.4], we have

$$fd(M) = \max\{fd\begin{pmatrix} M_1 \\ \mathrm{im}(\varphi^M) \end{pmatrix}, fd\begin{pmatrix} 0 \\ \mathrm{coker}(\varphi^M) \end{pmatrix}\} = \max\{fd(M_1), fd(\ker(\varphi^M)) + 1\}.$$

□

Theorem 2.3. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ *be a left* T*-module.*

(1) *If* M_1 *and* M_2 *are projective, then* $pd(M) = 0$ *or* $pd(U \otimes_A M_1) + 1$.
(2) *If* M_1 *and* M_2 *are injective, then* $id(M) = 0$ *or* $id(\mathrm{Hom}_B(U, M_2)) + 1$.
(3) *If* M_1 *and* M_2 *are flat, then* $fd(M) = 0$ *or* $fd(U \otimes_A M_1) + 1$.

Proof. (1) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} 0 \\ U \otimes_A M_1 \end{pmatrix} \overset{\begin{pmatrix} 0 \\ f \end{pmatrix}}{\to} \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix} \overset{\begin{pmatrix} 1 \\ g \end{pmatrix}}{\to} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to 0,$$

where $f : U \otimes_A M_1 \to (U \otimes_A M_1) \oplus M_2$ is defined by $f(x) = (x, \varphi^M(x))$ for any $x \in U \otimes_A M_1$, $g : (U \otimes_A M_1) \oplus M_2 \to M_2$ is defined by $g(x, y) = \varphi^M(x) - y$ for any $x \in U \otimes_A M_1$ and $y \in M_2$. Since M_1 and M_2 are projective, $\begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$ is projective by Lemma 2.1(1).

For any left T-module $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$ and $i \geq 1$, by [15, Lemma 3.2], we have

$$\mathrm{Ext}_T^{i+1}(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}) \cong \mathrm{Ext}_T^i(\begin{pmatrix} 0 \\ U \otimes_A M_1 \end{pmatrix}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}) \cong \mathrm{Ext}_B^i(U \otimes_A M_1, X_2).$$

Thus $pd(M) = pd(U \otimes_A M_1) + 1$ if $pd(M) \neq 0$.

(2) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \overset{\begin{pmatrix} \alpha \\ 1 \end{pmatrix}}{\to} \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix} \overset{\begin{pmatrix} \beta \\ 0 \end{pmatrix}}{\to} \begin{pmatrix} \mathrm{Hom}_B(U, M_2) \\ 0 \end{pmatrix} \to 0,$$

where $\alpha : M_1 \to M_1 \oplus \mathrm{Hom}_B(U, M_2)$ is defined by $\alpha(x) = (x, \widetilde{\varphi^M}(x))$ for any $x \in M_1$, $\beta : M_1 \oplus \mathrm{Hom}_B(U, M_2) \to \mathrm{Hom}_B(U, M_2)$ is defined by $\beta(x, y) = \widetilde{\varphi^M}(x) - y$ for any $x \in M_1$ and $y \in \mathrm{Hom}_B(U, M_2)$. By Lemma 2.1(2), $\begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$ is injective since M_1 and M_2 are injective.

For any left T-module $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$ and $i \geq 1$, by [15, Lemma 3.2], we have

$$\mathrm{Ext}_T^{i+1}(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}, \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}) \cong \mathrm{Ext}_T^i(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}, \begin{pmatrix} \mathrm{Hom}_B(U, M_2) \\ 0 \end{pmatrix})$$
$$\cong \mathrm{Ext}_A^i(X_1, \mathrm{Hom}_B(U, M_2)).$$

Hence $id(M) = id(\mathrm{Hom}_B(U, M_2)) + 1$ if $id(M) \neq 0$.

(3) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} 0 \\ U \otimes_A M_1 \end{pmatrix} \overset{\begin{pmatrix} 0 \\ f \end{pmatrix}}{\to} \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix} \overset{\begin{pmatrix} 1 \\ g \end{pmatrix}}{\to} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to 0,$$

where $f : U \otimes_A M_1 \to (U \otimes_A M_1) \oplus M_2$ is defined by $f(x) = (x, \varphi^M(x))$ for any $x \in U \otimes_A M_1$, $g : (U \otimes_A M_1) \oplus M_2 \to M_2$ is defined by $g(x, y) = \varphi^M(x) - y$ for any $x \in U \otimes_A M_1$ and

$y \in M_2$. Since M_1 and M_2 are flat, $\begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}$ is a flat left T-module by Lemma 2.1(3).

For any right T-module $Y = (Y_1, Y_2)_{\varphi_Y}$ and $i \geq 1$, by [15, Lemma 3.5], we have

$$\mathrm{Tor}_{i+1}^T((Y_1,Y_2)_{\varphi_Y}, \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}) \cong \mathrm{Tor}_i^T((Y_1,Y_2)_{\varphi_Y}, \begin{pmatrix} 0 \\ U \otimes_A M_1 \end{pmatrix}) \cong \mathrm{Tor}_i^B(Y_2, U \otimes_A M_1).$$

So $fd(M) = fd(U \otimes_A M_1) + 1$ if $fd(M) \neq 0$. □

Proposition 2.4. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ *be a simple left* T*-module.*

(1) *If* $\mathrm{Tor}_i^A(U, M_1) = 0$ *for any* $i \geq 1$, *then*
$pd(M) = \max\{pd(M_1), pd(U \otimes_A M_1) + 1\}$ *or* $pd(M_2)$,
$fd(M) = \max\{fd(M_1), fd(U \otimes_A M_1) + 1\}$ *or* $fd(M_2)$.

(2) *If* $\mathrm{Ext}_B^i(U, M_2) = 0$ *for any* $i \geq 1$, *then*
$id(M) = \max\{id(M_2), id(\mathrm{Hom}_B(U, M_2)) + 1\}$ *or* $id(M_1)$.

Proof. By [12, Corollary 3.3.2], M_1 is simple and $M_2 = 0$, or $M_1 = 0$ and M_2 is simple.

(1) **Case (i)**: If M_1 is simple and $M_2 = 0$, then $pd(M) = \max\{pd(M_1), pd(U \otimes_A M_1) + 1\}$ and $fd(M) = \max\{fd(M_1), fd(U \otimes_A M_1) + 1\}$ by Proposition 2.2(1,3).

Case (ii): If $M_1 = 0$ and M_2 is simple, then $pd(M) = pd(M_2)$ and $fd(M) = fd(M_2)$ by [18, Theorem 2.4].

(2) **Case (i)**: If M_1 is simple and $M_2 = 0$, then $id(M) = id(M_1)$ by [18, Theorem 2.4].

Case (ii): If $M_1 = 0$ and M_2 is simple, then $id(M) = \max\{id(M_2), id(\mathrm{Hom}_B(U, M_2)) + 1\}$ by Proposition 2.2(2). □

Recall that R is a *left SF ring* if every simple left R-module is flat. R is called a *left* V*-ring* if every simple left R-module is injective.

As an immediate consequence of Proposition 2.4 and [12, Corollary 3.3.2], we have

Corollary 2.5. *The following assertions hold.*

(1) T *is a left SF ring if and only if* A *and* B *are left SF rings,* $U \otimes_A X = 0$ *for any simple left* A*-module* X.

(2) T *is a left* V*-ring if and only if* A *and* B *are left* V*-rings,* $\mathrm{Hom}_B(U, Y) = 0$ *for any simple left* B*-module* Y.

Given a left A-module X and a left B-module Y, there are two natural homomorphisms $\nu_Y : U \otimes_A \mathrm{Hom}_B(U, Y) \to Y$ defined by $\nu_Y(u \otimes f) = f(u)$ for any $f \in \mathrm{Hom}_B(U, Y)$ and $u \in U$, and $\eta_X : X \to \mathrm{Hom}_B(U, U \otimes_A X)$ defined by $\eta_X(x)(u) = u \otimes x$ for any $x \in X$ and $u \in U$.

Proposition 2.6. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ *be a left* T*-module.*

(1) *If* $\mathrm{Tor}_i^A(U, M_1) = 0$ *for any* $i \geq 1$, $M_2 \in Gen(U)$, $\widetilde{\varphi}^M$ *is an epimorphism, then*
$$pd(M) = \max\{pd(M_1), pd(\ker(\varphi^M)) + 1\},$$
$$fd(M) = \max\{fd(M_1), fd(\ker(\varphi^M)) + 1\}.$$

(2) *If* $\mathrm{Ext}_B^i(U, M_2) = 0$ *for any* $i \geq 1$, $M_1 \in Cogen(U^+)$, φ^M *is a monomorphism, then*
$$id(M) = \max\{id(M_2), id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$

Proof. (1) By [3, Lemma 2.1.2], $\nu_{M_2} : U \otimes_A \mathrm{Hom}_B(U, M_2) \to M_2$ is an epimorphism since $M_2 \in \mathrm{Gen}(U)$. So $\varphi^M = \nu_{M_2}(1 \otimes \widetilde{\varphi}^M) : U \otimes_A M_1 \to U \otimes_A \mathrm{Hom}_B(U, M_2) \to M_2$ is an epimorphism. By Proposition 2.2(1,3), $pd(M) = \max\{pd(M_1), pd(\ker(\varphi^M)) + 1\}$ and $fd(M) = \max\{fd(M_1), fd(\ker(\varphi^M)) + 1\}$.

(2) By [3, Lemma 2.1.2], $\eta_{M_1} : M_1 \to \mathrm{Hom}_B(U, U \otimes_A M_1)$ is a monomorphism since $M_1 \in \mathrm{Cogen}(U^+)$. So $\widetilde{\varphi}^M = (\varphi^M)_* \eta_{M_1} : M_1 \to \mathrm{Hom}_B(U, U \otimes_A M_1) \to \mathrm{Hom}_B(U, M_2)$ is a monomorphism. By Proposition 2.2(2), $id(M) = \max\{id(M_2), id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}$. □

Corollary 2.7. *Let* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ *be a left* T*-module.*

(1) *If* $\mathrm{Tor}_i^A(U, M_1) = 0$ *for any* $i \geq 1$*,* $M_2 \in Gen(U)$ *and* M *is injective, then*

$$pd(M) = \max\{pd(M_1), pd(\ker(\varphi^M)) + 1\},$$

$$fd(M) = \max\{fd(M_1), fd(\ker(\varphi^M)) + 1\}.$$

(2) *If* $\mathrm{Ext}_B^i(U, M_2) = 0$ *for any* $i \geq 1$*,* $M_1 \in Cogen(U^+)$ *and* M *is flat, then*

$$id(M) = \max\{id(M_2), id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$

Proof. It follows from Lemma 2.1(2,3) and Proposition 2.6. □

Following [21], a left R-module X is called FP*-injective* if $\mathrm{Ext}_R^1(N, X) = 0$ for any finitely presented left R-module N. The FP*-injective dimension* of X, denoted by FP-$id(X)$, is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(N, X) = 0$ for every finitely presented left R-module N (if no such n exists, set FP-$id(X) = \infty$). If R is a left coherent ring, then FP-$id(X) = fd(X^+)$ by [5, Theorem 2.2].

Let ${}_BU$ be finitely presented, then $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ is an FP-injective left T-module if and only if $\widetilde{\varphi^M}$ is an epimorphism, $\ker(\widetilde{\varphi^M})$ is an FP-injective left A-module and M_2 is an FP-injective left B-module by [16, Theorem 3.3].

Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be a left T-module. Then $M^+ = (M_1^+, M_2^+)_{\varphi_{M^+}}$ is a character right T-module of M, where $\varphi_{M^+} : M_2^+ \otimes_B U \to M_1^+$ is defined by $\varphi_{M^+}(f \otimes u)(x) = f(\varphi^M(u \otimes x))$ for any $f \in M_2^+$, $u \in U$ and $x \in M_1$ (see [12, p.67]).

Next we give some computing formulas of FP-injective dimensions of special left T-modules.

Theorem 2.8. *Let* T *be a left coherent ring,* ${}_BU$ *be finitely presented,* $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ *be a left* T*-module such that* $\mathrm{Ext}_B^i(U, M_2) = 0$ *for any* $i \geq 1$*.*

(1) *If* $\widetilde{\varphi}^M$ *is an epimorphism, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\ker(\widetilde{\varphi}^M))\}.$$
(2) *If* $\widetilde{\varphi}^M$ *is a monomorphism, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$
(3) *If* φ^M *is a monomorphism and* $M_1 \in Cogen(U^+)$*, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$
(4) *If* $\ker(\widetilde{\varphi}^M)$ *is* FP*-injective, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}.$$
(5) *If* M_1 *and* M_2 *are* FP*-injective, then*
$$FP\text{-}id(M) = 0 \text{ or } FP\text{-}id(\mathrm{Hom}_B(U, M_2)) + 1.$$

Proof. By [17, Theorem 3.2], A and B are left coherent rings.

(1) Since $\widetilde{\varphi}^M$ is an epimorphism, we get the exact sequence

$$0 \to \ker(\widetilde{\varphi^M}) \to M_1 \overset{\widetilde{\varphi^M}}{\to} \mathrm{Hom}_B(U, M_2) \to 0,$$

which induces the exact sequence

$$0 \to \mathrm{Hom}_B(U, M_2)^+ \overset{(\widetilde{\varphi^M})^+}{\to} M_1^+ \to (\ker(\widetilde{\varphi^M}))^+ \to 0.$$

Since ${}_BU$ is finitely presented, $M_2^+ \otimes_B U \cong \mathrm{Hom}_B(U, M_2)^+$ by [20, Lemma 3.55]. So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_2^+ \otimes_B U & \xrightarrow{\varphi_{M^+}} & M_1^+ & \longrightarrow & \mathrm{coker}(\varphi_{M^+}) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \| & & \cong\downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_B(U, M_2)^+ & \xrightarrow{(\widetilde{\varphi^M})^+} & M_1^+ & \longrightarrow & (\ker(\widetilde{\varphi^M}))^+ & \longrightarrow & 0.
\end{array}$$

By [8, Lemma 1.2.11(d)], $\mathrm{Tor}_i^B(M_2^+, U) \cong \mathrm{Ext}_B^i(U, M_2)^+ = 0$ for any $i \geq 1$. By [18, Theorem 2.4], $FP\text{-}id(M) = fd(M^+) = fd(M_1^+, M_2^+)_{\varphi_{M^+}} = \max\{fd(M_2^+), fd(\mathrm{coker}(\varphi_{M^+})\} = \max\{fd(M_2^+), fd((\ker(\widetilde{\varphi}^M))^+)\} = \max\{FP\text{-}id(M_2), FP\text{-}id(\ker(\widetilde{\varphi}^M))\}$.

(2) Since $\widetilde{\varphi}^M$ is a monomorphism, we get the exact sequence

$$0 \to M_1 \overset{\widetilde{\varphi^M}}{\to} \mathrm{Hom}_B(U, M_2) \to \mathrm{coker}(\widetilde{\varphi^M}) \to 0,$$

which induces the exact sequence

$$0 \to (\mathrm{coker}(\widetilde{\varphi^M}))^+ \to \mathrm{Hom}_B(U, M_2)^+ \overset{(\widetilde{\varphi^M})^+}{\to} M_1^+ \to 0.$$

So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\varphi_{M^+}) & \longrightarrow & M_2^+ \otimes_B U & \xrightarrow{\varphi_{M^+}} & M_1^+ & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \| & & \\
0 & \longrightarrow & (\mathrm{coker}(\widetilde{\varphi^M}))^+ & \longrightarrow & \mathrm{Hom}_B(U, M_2)^+ & \xrightarrow{(\widetilde{\varphi^M})^+} & M_1^+ & \longrightarrow & 0.
\end{array}$$

By [18, Theorem 2.4], $FP\text{-}id(M) = fd(M^+) = \max\{fd(M_2^+), fd(\ker(\varphi_{M^+})) + 1\} = \max\{fd(M_2^+), fd((\mathrm{coker}(\widetilde{\varphi}^M))^+) + 1\} = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}$.

(3) By [3, Lemma 2.1.2], $\eta_{M_1} : M_1 \to \mathrm{Hom}_B(U, U \otimes_A M_1)$ is a monomorphism since $M_1 \in \mathrm{Cogen}(U^+)$. So $\widetilde{\varphi}^M = (\varphi^M)_* \eta_{M_1} : M_1 \to \mathrm{Hom}_B(U, U \otimes_A M_1) \to \mathrm{Hom}_B(U, M_2)$ is a monomorphism. By (2), $FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}$.

(4) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} \ker(\widetilde{\varphi}^M) \\ 0 \end{pmatrix} \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} \mathrm{im}(\widetilde{\varphi}^M) \\ M_2 \end{pmatrix} \to 0.$$

Since $\begin{pmatrix} \ker(\widetilde{\varphi}^M) \\ 0 \end{pmatrix}$ is FP-injective by [16, Theorem 3.3], we have $FP\text{-}id(M) = \max\{FP\text{-}id\begin{pmatrix} \ker(\widetilde{\varphi}^M) \\ 0 \end{pmatrix}, FP\text{-}id\begin{pmatrix} \mathrm{im}(\widetilde{\varphi}^M) \\ M_2 \end{pmatrix}\} = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\widetilde{\varphi}^M)) + 1\}$ by (2).

(5) There exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \to \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix} \to \begin{pmatrix} \mathrm{Hom}_B(U, M_2) \\ 0 \end{pmatrix} \to 0.$$

Since M_1 and M_2 are FP-injective, we have $\begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}$ is FP-injective by [16, Theorem 3.3]. Therefore, for any finitely presented left T-module $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}$, $\mathrm{Ext}_T^i(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}, \begin{pmatrix} M_1 \oplus \mathrm{Hom}_B(U, M_2) \\ M_2 \end{pmatrix}) = 0$ for any $i \geq 1$ by [21, Lemma 3.1]. So

$$\mathrm{Ext}_T^{i+1}(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}, \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}) \cong \mathrm{Ext}_T^i(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_{\varphi^X}, \begin{pmatrix} \mathrm{Hom}_B(U, M_2) \\ 0 \end{pmatrix})$$

$$\cong \mathrm{Ext}^i_A(X_1, \mathrm{Hom}_B(U, M_2)).$$

Note that X_1 is finitely presented. Thus $FP\text{-}id(M) = FP\text{-}id(\mathrm{Hom}_B(U, M_2)) + 1$ if $FP\text{-}id(M) \neq 0$. □

Corollary 2.9. *Let R be a left coherent ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$, $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \neq 0$ be a left $T(R)$-module.*

(1) *If φ^M is an epimorphism, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\ker(\varphi^M))\}.$$
(2) *If φ^M is a monomorphism, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\varphi^M)) + 1\}.$$
(3) *If $\ker(\varphi^M)$ is FP-injective, then*
$$FP\text{-}id(M) = \max\{FP\text{-}id(M_2), FP\text{-}id(\mathrm{coker}(\varphi^M)) + 1\}.$$

Proof. It is an immediate consequence of Theorem 2.8 since $T(R)$ is a left coherent ring by [16, Corollary 3.7]. □

3. Global dimensions of formal triangular matrix rings

Theorem 3.1. *Let U_A be flat. Then the following assertions hold.*

(1) *If ${}_BU$ is projective and $lD(A) \neq lD(B)$, then $lD(T) = \max\{lD(A), lD(B)\}$.*
(2) *If ${}_BU$ is flat and $wD(A) \neq wD(B)$, then $wD(T) = \max\{wD(A), wD(B)\}$.*

Proof. (1) We first note that $\max\{lD(A), lD(B)\} \leq lD(T) \leq \max\{lD(A)+1, lD(B)\}$ by [15, Corollary 3.3].

Next we prove that $lD(T) \leq \max\{lD(A), lD(B) + 1\}$. For any left T-module $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N} \neq 0$, there exists an exact sequence in T-Mod

$$0 \to \begin{pmatrix} 0 \\ N_2 \end{pmatrix} \to \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N} \to \begin{pmatrix} N_1 \\ 0 \end{pmatrix} \to 0.$$

By [18, Theorem 2.4], $pd(N) \leq \max\{pd\begin{pmatrix} N_1 \\ 0 \end{pmatrix}, pd\begin{pmatrix} 0 \\ N_2 \end{pmatrix}\} = \max\{\max\{pd(N_1), pd(U \otimes_A N_1) + 1\}, pd(N_2)\} \leq \max\{\max\{lD(A), lD(B) + 1\}, lD(B)\} = \max\{lD(A), lD(B) + 1\}$, which means that $lD(T) \leq \max\{lD(A), lD(B) + 1\}$.

Case (i): $lD(A) = \infty$ or $lD(B) = \infty$.

Since $\max\{lD(A), lD(B)\} \leq lD(T)$, $lD(T) = \infty$. So $lD(T) = \max\{lD(A), lD(B)\}$.

Case (ii): $lD(A) = m < \infty$ and $lD(B) = n < \infty$.

Since $m \neq n$, we have $\max\{m, n\} \leq lD(T) \leq \min\{\max\{m + 1, n\}, \max\{m, n + 1\}\} = \max\{m, n\}$. So $lD(T) = \max\{m, n\}$.

It follows that $lD(T) = \max\{lD(A), lD(B)\}$.

(2) We first note that $\max\{wD(A), wD(B)\} \leq wD(T) \leq \max\{wD(A) + 1, wD(B)\}$ by [15, Corollary 3.6].

Next we prove that $wD(T) \leq \max\{wD(A), wD(B) + 1\}$. For any left T-module $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N} \neq 0$, we have $fd(N) \leq \max\{fd\begin{pmatrix} N_1 \\ 0 \end{pmatrix}, fd\begin{pmatrix} 0 \\ N_2 \end{pmatrix}\} = \max\{\max\{fd(N_1), fd(U \otimes_A N_1) + 1)\}, fd(N_2)\} \leq \max\{\max\{fd(A), fd(B) + 1\}, fd(B)\} \leq \max\{fd(A), fd(B) + 1\}$ by [18, Theorem 2.4]. So $wD(T) \leq \max\{wD(A), wD(B) + 1\}$.

Case (i): $wD(A) = \infty$ or $wD(B) = \infty$.

Since $\max\{wD(A), wD(B)\} \leq wD(T)$, we have $wD(T) = \infty$. Therefore $wD(T) = \max\{wD(A), wD(B)\}$.

Case (ii): $wD(A) = m < \infty$ and $wD(B) = n < \infty$.

Since $m \neq n$, we have $\max\{m,n\} \leq wD(T) \leq \min\{\max\{m+1,n\}, \max\{m,n+1\}\} = \max\{m,n\}$. So $wD(T) = \max\{m,n\}$.

Consequently $wD(T) = \max\{wD(A), wD(B)\}$. □

It is well known that if $U = 0$, then $lD(T) = \max\{lD(A), lD(B)\}$ and $wD(T) = \max\{wD(A), wD(B)\}$. However, the conditions "$lD(A) \neq lD(B)$" and "$wD(A) \neq wD(B)$" in Theorem 3.1 is not superfluous.

Example 3.2. Let R be a ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$, then $lD(T(R)) = lD(R)+1 \neq lD(R)$ and $wD(T(R)) = wD(R) + 1 \neq wD(R)$ by [15, Corollaries 3.4 and 3.7].

Example 3.3. Let S be a commutative von Neumann regular ring which is not semisimple Artinian. Then there is an ideal I such that I is not a direct summand of S. Let $R = S/I$ and $T = \begin{pmatrix} S & 0 \\ R & R \end{pmatrix}$. Then $wD(R) = wD(S) = 0$. But $wD(T) = 1 \neq \max\{wD(S), wD(R)\}$ (see [13, 2.34, p.47]).

The condition that "${}_BU$ is projective" in Theorem 3.1 is not superfluous.

Example 3.4. Let $T = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{pmatrix}$. Note that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module but is not a projective $\mathbb{Z}$-module, $1 = wD(\mathbb{Z}) = lD(\mathbb{Z}) \neq lD(\mathbb{Q}) = wD(\mathbb{Q}) = 0$. Then we have $wD(T) = \max\{wD(\mathbb{Q}), wD(\mathbb{Z})\} = 1$ but $lD(T) \neq \max\{wD(\mathbb{Q}), wD(\mathbb{Z})\} = 1$ (see [7, Exercises 11, p.113]).

By taking the supremums of one of projective, injective or flat dimensions of specified R-modules, one obtains various "global" dimensions of R. We write

$lIFD(R) = \sup\{fd(E) : E$ is an injective left R-module$\}$ (see [4]);
$lIPD(R) = \sup\{pd(E) : E$ is an injective left R-module$\}$;
$lPID(R) = \sup\{id(P) : P$ is a projective left R-module$\}$;
$lFID(R) = \sup\{id(F) : F$ is a flat left R-module$\}$.

The following theorem gives an estimation of these "global" dimensions of a formal triangular matrix ring T.

Theorem 3.5. *Let U_A be flat. Then the following assertions hold.*

(1) *If ${}_BU$ is flat, then*
$\max\{lIFD(A), lIFD(B)\} \leq lIFD(T) \leq \max\{lIFD(A)+1, lIFD(B)\}$.
(2) *If ${}_BU$ is projective, then*
$\max\{lIPD(A), lIPD(B)\} \leq lIPD(T) \leq \max\{lIPD(A)+1, lIPD(B)\}$.
(3) *If ${}_BU$ is projective, then*
$\max\{lPID(A), lPID(B)\} \leq lPID(T) \leq \max\{lPID(A), lPID(B)+1\}$.
(4) *If ${}_BU$ is projective, then*
$\max\{lFID(A), lFID(B)\} \leq lFID(T) \leq \max\{lFID(A), lFID(B)+1\}$.

Proof. (1) Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}$ be an injective left T-module. By Lemma 2.1(2), we get the exact sequence

$$0 \to \ker(\widetilde{\varphi^M}) \to M_1 \overset{\widetilde{\varphi^M}}{\to} \mathrm{Hom}_B(U, M_2) \to 0$$

with $\ker(\widetilde{\varphi^M})$ and M_2 injective. Since U_A is flat, $\mathrm{Hom}_B(U, M_2)$ is injective and so M_1 is injective. By [15, Corollary 3.6], $fd(M) \leq \max\{fd(M_1)+1, fd(M_2)\} \leq \max\{lIFD(A)+1, lIFD(B)\}$. So $lIFD(T) \leq \max\{lIFD(A)+1, lIFD(B)\}$.

Let N be an injective left A-module. Then $\begin{pmatrix} N \\ 0 \end{pmatrix}$ is injective by Lemma 2.1(2). So $fd(N) \leq fd\begin{pmatrix} N \\ 0 \end{pmatrix} \leq lIFD(T)$ by [15, Corollary 3.6]. Let G be an injective left B-module.

Then $\begin{pmatrix}\mathrm{Hom}_B(U,G)\\ G\end{pmatrix}$ is injective by Lemma 2.1(2). So $fd(G) \le fd\begin{pmatrix}\mathrm{Hom}_B(U,G)\\ G\end{pmatrix} \le lIFD(T)$ by [15, Corollary 3.6]. Thus $\max\{lIFD(A), lIFD(B)\} \le lIFD(T)$.

(2) Let $M = \begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$ be an injective left T-module. Then M_1 and M_2 are injective. By [15, Corollary 3.3], $pd(M) \le \max\{pd(M_1)+1, pd(M_2)\} \le \max\{lIPD(A)+1, lIPD(B)\}$. So $lIPD(T) \le \max\{lIPD(A)+1, lIPD(B)\}$.

Let N be an injective left A-module. Then $pd(N) \le pd\begin{pmatrix}N\\ 0\end{pmatrix} \le lIPD(T)$ by [15, Corollary 3.3]. Let G be an injective left B-module. Then $pd(G) \le pd\begin{pmatrix}\mathrm{Hom}_B(U,G)\\ G\end{pmatrix} \le lIPD(T)$ by [15, Corollary 3.3]. So $\max\{lIPD(A), lIPD(B)\} \le lIPD(T)$.

(3) Let $M = \begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$ be a projective left T-module. By Lemma 2.1(1), we get the exact sequence

$$0 \to U \otimes_A M_1 \overset{\varphi^M}{\to} M_2 \to \mathrm{coker}(\varphi^M) \to 0$$

with M_1 and $\mathrm{coker}(\varphi^M)$ projective. Since ${}_BU$ is projective, $U \otimes_A M_1$ is projective and so M_2 is projective. By [15, Corollary 3.3], we have

$$id(M) \le \max\{id(M_1), id(M_2)+1\} \le \max\{lPID(A), lPID(B)+1\}.$$

So $lPID(T) \le \max\{lPID(A), lPID(B)+1\}$.

Let N be a projective left A-module. Then $\begin{pmatrix}N\\ U\otimes_A N\end{pmatrix}$ is a projective left T-module by Lemma 2.1(1). So $id(N) \le id\begin{pmatrix}N\\ U\otimes_A N\end{pmatrix} \le lPID(T)$ by [15, Corollary 3.3]. Let G be a projective left B-module. Then $\begin{pmatrix}0\\ G\end{pmatrix}$ is a projective left T-module by Lemma 2.1(1). So $id(G) \le id\begin{pmatrix}0\\ G\end{pmatrix} \le lPID(T)$ by [15, Corollary 3.3]. Thus $\max\{lPID(A), lPID(B)\} \le lPID(T)$.

(4) Let $M = \begin{pmatrix}M_1\\ M_2\end{pmatrix}_{\varphi^M}$ be a flat left T-module. By Lemma 2.1(3), there exists the exact sequence

$$0 \to U \otimes_A M_1 \overset{\varphi^M}{\to} M_2 \to \mathrm{coker}(\varphi^M) \to 0$$

with M_1 and $\mathrm{coker}(\varphi^M)$ flat. Since ${}_BU$ is projective, $U \otimes_A M_1$ is flat and so M_2 is flat. By [15, Corollary 3.3], $id(M) \le \max\{id(M_1), id(M_2)+1\} \le \max\{lFID(A), lFID(B)+1\}$. So $lFID(T) \le \max\{lFID(A), lFID(B)+1\}$.

Let N be a flat left A-module. Then $\begin{pmatrix}N\\ U\otimes_A N\end{pmatrix}$ is a flat left T-module by Lemma 2.1(3). So $id(N) \le id\begin{pmatrix}N\\ U\otimes_A N\end{pmatrix} \le lFID(T)$ by [15, Corollary 3.3]. Let G be a flat left B-module. Then $\begin{pmatrix}0\\ G\end{pmatrix}$ is a flat left T-module by Lemma 2.1(3). So $id(G) \le id\begin{pmatrix}0\\ G\end{pmatrix} \le lFID(T)$ by [15, Corollary 3.3]. Thus $\max\{lFID(A), lFID(B)\} \le lFID(T)$. □

Remark 3.6. It is easy to verify that if $U = 0$, then

$$lIFD(T) = \max\{lIFD(A), lIFD(B)\},$$
$$lIPD(T) = \max\{lIPD(A), lIPD(B)\},$$
$$lPID(T) = \max\{lPID(A), lPID(B)\},$$
$$lFID(T) = \max\{lFID(A), lFID(B)\}.$$

It is known that R is a quasi-Frobenius ring if and only if every injective left R-module is projective if and only if every projective (flat) left R-module is injective.

Recall that R is a *left IF ring* [2] if every injective left R-module is flat.

Proposition 3.7. *Let R be a ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. Then*

(1) $lIFD(T(R)) = lIFD(R) + 1$.
(2) $lIPD(T(R)) = lIPD(R) + 1$.
(3) $lPID(T(R)) = lPID(R) + 1$.
(4) $lFID(T(R)) = lFID(R) + 1$.

Consequently, R is a left IF ring if and only if $lIFD(T(R)) = 1$; R is a quasi-Frobenius ring if and only if $lIPD(T(R)) = 1$ if and only if $lPID(T(R)) = 1$ if and only if $lFID(T(R)) = 1$.

Proof. (1) Let $lIFD(R) = n < \infty$.

Case (i): If $n = 0$, then $lIFD(T(R)) \leq 1$ by Theorem 3.5. Since $\begin{pmatrix} R^+ \\ 0 \end{pmatrix}$ is an injective left $T(R)$-module but not a flat left $T(R)$-module by Lemma 2.1(2,3), $lIFD(T(R)) \geq fd\begin{pmatrix} R^+ \\ 0 \end{pmatrix} \geq 1$. So $lIFD(T(R)) = 1$.

Case (ii): If $n \geq 1$, then there is an injective left R-module G such that $fd(G) = n$. So there is a right R-module X such that $\mathrm{Tor}_n^R(X, G) \neq 0$. By [15, Lemma 3.5], $\mathrm{Tor}_n^{T(R)}((0, X), \begin{pmatrix} 0 \\ G \end{pmatrix}) \cong \mathrm{Tor}_n^R(X, G) \neq 0$ and $\mathrm{Tor}_n^{T(R)}((0, X), \begin{pmatrix} G \\ G \end{pmatrix}) \cong \mathrm{Tor}_n^R(0, G) = 0$. The exact sequence $0 \to \begin{pmatrix} 0 \\ G \end{pmatrix} \to \begin{pmatrix} G \\ G \end{pmatrix} \to \begin{pmatrix} G \\ 0 \end{pmatrix} \to 0$ induces the exact sequence

$$\mathrm{Tor}_{n+1}^{T(R)}((0, X), \begin{pmatrix} G \\ 0 \end{pmatrix}) \to \mathrm{Tor}_n^{T(R)}((0, X), \begin{pmatrix} 0 \\ G \end{pmatrix}) \to \mathrm{Tor}_n^{T(R)}((0, X), \begin{pmatrix} G \\ G \end{pmatrix}) = 0.$$

So $\mathrm{Tor}_{n+1}^{T(R)}((0, X), \begin{pmatrix} G \\ 0 \end{pmatrix}) \neq 0$. Since $fd\begin{pmatrix} G \\ 0 \end{pmatrix} \leq fd(G) + 1 = n + 1$ by [15, Corollary 3.6], $fd\begin{pmatrix} G \\ 0 \end{pmatrix} = n + 1$. Also $\begin{pmatrix} G \\ 0 \end{pmatrix}$ is injective, hence $lIFD(T(R)) \geq fd\begin{pmatrix} G \\ 0 \end{pmatrix} = n + 1$. But $lIFD(T(R)) \leq n + 1$ by Theorem 3.5. So $lIFD(T(R)) = n + 1$.

(2) Let $lIPD(R) = m < \infty$.

Case (i): If $m = 0$, then $lIPD(T(R)) \leq 1$ by Theorem 3.5. Since $\begin{pmatrix} R^+ \\ 0 \end{pmatrix}$ is an injective left $T(R)$-module but not a projective left $T(R)$-module by Lemma 2.1(1,2), $lIPD(T(R)) \geq pd\begin{pmatrix} R^+ \\ 0 \end{pmatrix} \geq 1$. So $lIPD(T(R)) = 1$.

Case (ii): If $m \geq 1$, then there exists an injective left R-module E such that $pd(E) = m$. So there exists a left R-module Y such that $\mathrm{Ext}_R^m(E, Y) \neq 0$. By [15, Lemma 3.2], $\mathrm{Ext}_{T(R)}^m(\begin{pmatrix} E \\ E \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix}) \cong \mathrm{Ext}_R^m(E, 0) = 0$ and $\mathrm{Ext}_{T(R)}^m(\begin{pmatrix} 0 \\ E \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix}) \cong \mathrm{Ext}_R^m(E, Y) \neq 0$. The exact sequence $0 \to \begin{pmatrix} 0 \\ E \end{pmatrix} \to \begin{pmatrix} E \\ E \end{pmatrix} \to \begin{pmatrix} E \\ 0 \end{pmatrix} \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}_{T(R)}^m(\begin{pmatrix} E \\ E \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix}) \to \mathrm{Ext}_{T(R)}^m(\begin{pmatrix} 0 \\ E \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix}) \to \mathrm{Ext}_{T(R)}^{m+1}(\begin{pmatrix} E \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix}).$$

Therefore $\mathrm{Ext}^{m+1}_{T(R)}(\begin{pmatrix}E\\0\end{pmatrix},\begin{pmatrix}0\\Y\end{pmatrix}) \neq 0$. But $pd\begin{pmatrix}E\\0\end{pmatrix} \leq m+1$ by [15, Corollary 3.4]. So $pd\begin{pmatrix}E\\0\end{pmatrix} = m+1$. Hence $lIP(T(R)) \geq pd\begin{pmatrix}E\\0\end{pmatrix} = m+1$. Also $lIPD(T(R)) \leq m+1$ by Theorem 3.5. Thus $lIP(T(R)) = m+1$.

(3) Let $lPID(R) = k < \infty$.

Case (i): If $k=0$, then $lPID(T(R)) \leq 1$ by Theorem 3.5. Since $\begin{pmatrix}0\\R\end{pmatrix}$ is a projective left $T(R)$-module but not an injective left $T(R)$-module by Lemma 2.1(1,2), $lPID(T(R)) \geq id\begin{pmatrix}0\\R\end{pmatrix} \geq 1$. So $lPID(T(R)) = 1$.

Case (ii): If $k \geq 1$, then there exists a projective left R-module P such that $id(P) = k$. So there exists a left R-module H such that $\mathrm{Ext}^k_R(H,P) \neq 0$. By [15, Lemma 3.2], $\mathrm{Ext}^k_{T(R)}(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}P\\P\end{pmatrix}) \cong \mathrm{Ext}^k_R(0,P) = 0$ and $\mathrm{Ext}^k_{T(R)}(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}P\\0\end{pmatrix}) \cong \mathrm{Ext}^k_R(H,P) \neq 0$. The exact sequence $0 \to \begin{pmatrix}0\\P\end{pmatrix} \to \begin{pmatrix}P\\P\end{pmatrix} \to \begin{pmatrix}P\\0\end{pmatrix} \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}^k_{T(R)}(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}P\\P\end{pmatrix}) \to \mathrm{Ext}^k_{T(R)}(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}P\\0\end{pmatrix}) \to \mathrm{Ext}^{k+1}_{T(R)}(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\P\end{pmatrix}).$$

Whence $\mathrm{Ext}^{k+1}_{T(R)}(\begin{pmatrix}H\\0\end{pmatrix},\begin{pmatrix}0\\P\end{pmatrix}) \neq 0$. Since $id\begin{pmatrix}0\\P\end{pmatrix} \leq k+1$ by [15, Lemma 3.2], $id\begin{pmatrix}0\\P\end{pmatrix} = k+1$. Hence $lPI(T(R)) \geq pd\begin{pmatrix}0\\P\end{pmatrix} = k+1$. But $lPID(T(R)) \leq k+1$ by Theorem 3.5. Thus $lPI(T(R)) = k+1$.

The proof of (4) is similar to that of (3). □

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