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Third Order Convergent Finite Difference Method for the Third Order Boundary Value Problem in ODEs

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Abstract. We propose a third order convergent finite-difference method for the approximate solution of the boundary value problems. We developed our numerical technique by employing Taylor series expansion and method of undetermined coefficients. The convergence property of the proposed finite difference method discussed. To demonstrate the computational accuracy and effectiveness of the proposed method numerical results presented.

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1. Introduction

In this article, we consider following third order boundary value problem which occurs in fluid dynamics, obstacle problems, moving boundary value problems and many other areas of studies;

$$u'''(x) = f(x, u(x), u'(x), u''(x)), \quad a < x < b \tag{1.1}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u'(a) = \beta \quad \text{and} \qquad u'(b) = \gamma,$$

where α, β and γ are real constants.

The solution of these problems is an important and interesting area of research. But it is not possible to find a solution for these problems for an arbitrary forcing function $f(x, u(x), u'(x), u''(x))$.

Thus, the existence and uniqueness of the solution to the problem (1.1) is assumed. However, the theory on the existence and uniqueness of the solution of higher order boundary value problems can be found in [1, 6] and for specific problem (1.1) in [4, 5, 9] and references there. Some solution method for problem (1.1) are finite difference method [2, 13], non polynomial spline method [7], quintic splines [8] and references therein.

The purpose of this article is to develop computationally efficient, inexpensive and third order accurate finite difference method to deal with the numerical solution of problems (1.1).

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In this article, we have presented our work in six sections including introduction. In Section 2, we have described finite difference method. In Section 3, we have outlined the derivation of the finite difference method. In Section 4, under proper condition, we have discussed and analysed the convergence of the proposed method. Numerical experiment and illustrative results presented in Section 5. A meaningful discussion on computational performance of the proposed method is presented in Section 6.

2. The Difference Method

We substitute domain $[a,b]$ by a discrete set of points and we wish to determine the numerical solution at these points. Thus, we define *N-1* numbers of $a \leq x_0 < x_1 < \cdots < x_N \leq b$ nodal points in the domain of $[a,b]$ using a uniform step length h such that $x_i = a + ih$, $i = 0, 1, \cdots, N$. We wish to determine the numerical approximation of the theoretical solution $u(x)$ of the problem (1.1) at the nodal points x_i, $i = 1, 2, \cdots, N-1$. We denote the numerical approximation of $u(x)$ at node $x = x_i$ as u_i, $i = 1, 2, \cdots, N-1$. Let us denote f_i as the approximation of the theoretical value of the source function $f(x, u(x), u'(x), u''(x))$ at node $x = x_i$, $i = 0, 1, 2, \cdots, N$. Thus, the finite difference method reduces the problem (1.1) to the following discrete problem at node $x = x_i$,

$$u_i''' = f_i, \quad a \leq x_i \leq b \tag{2.1}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u_0' = \beta \quad \text{and} \qquad u_N' = \gamma.$$

Let we define nodes $x_{i\pm\frac{1}{2}} = x_i \pm \frac{h}{2}$, $\quad i = 1, 2, ..., N-1$ and denote the solution of the problem (1.1) at these nodes as $u_{i\pm\frac{1}{2}}$. Following the idea in [10], we descretize problem (2.1) at these nodes in [a, b] as follows,

$$\overline{u}'_{i-\frac{5}{2}} = \frac{1}{12h}(-19u_{i-\frac{5}{2}} + 27u_{i-\frac{3}{2}} - 8u_{i-\frac{1}{2}} + hu_i'), \quad i = N, \tag{2.2}$$

$$\overline{u}'_{i-\frac{3}{2}} = \begin{cases} \frac{1}{30h}(15u_{i-\frac{3}{2}} + 20u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} - 32u_{i-2}), \quad i = 2 \\ \frac{1}{6h}(-2u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}}), \quad 3 \leq i \leq N-1 \\ \frac{1}{8h}(-3u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} - hu_i'), \quad i = N, \end{cases} \tag{2.3}$$

$$\overline{u}'_{i-\frac{1}{2}} = \begin{cases} \frac{1}{9h}(-28u_{i-1} + 27u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}} - 6hu'_{i-1}), \quad i = 1 \\ \frac{1}{60h}(-90u_{i-\frac{3}{2}} + 40u_{i-\frac{1}{2}} + 18u_{i+\frac{1}{2}} + 32u_{i-2}), \quad i = 2 \\ \frac{1}{6h}(u_{i-\frac{5}{2}} - 6u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} + 2u_{i+\frac{1}{2}}), \quad 3 \leq i \leq N-1 \\ \frac{1}{6h}(u_{i-\frac{5}{2}} - 6u_{i-\frac{3}{2}} + 5u_{i-\frac{1}{2}} + 2hu_i'), \quad i = N, \end{cases} \tag{2.4}$$

$$\overline{u}'_{i+\frac{1}{2}} = \begin{cases} \frac{1}{177h}(-20u_{i-1} - 243u_{i-\frac{1}{2}} + 263ui + \frac{1}{2} - 96hu'_{i-1}), \quad i = 1 \\ \frac{1}{30h}(75u_{i-\frac{3}{2}} - 100u_{i-\frac{1}{2}} + 57u_{i+\frac{1}{2}} - 32u_{i-2}), \quad i = 2, \end{cases} \tag{2.5}$$

$$\overline{u}''_{i-\frac{5}{2}} = \frac{1}{2h^2}(3u_{i-\frac{5}{2}} - 7u_{i-\frac{3}{2}} + 4u_{i-\frac{1}{2}} - hu_i'), \quad i = N.$$

$$\overline{u}''_{i-\frac{3}{2}} = \begin{cases} \frac{1}{5h^2}(-25u_{i-\frac{3}{2}} + 10u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} + 16u_{i-2}), \quad i = 2 \\ \frac{1}{h^2}(u_{i-\frac{5}{2}} - 2u_{i-\frac{3}{2}} + u_{i-\frac{1}{2}}), \quad 3 \leq i \leq N, \end{cases}$$

$$\overline{u}''_{i-1} = \frac{1}{9h^2}(-104u_{i-1} + 108u_{i-\frac{1}{2}} - 4u_{i+\frac{1}{2}} - 48hu'_{i-1}), \quad i = 1, \tag{2.6}$$

$$\overline{u}''_{i-\frac{1}{2}} = \begin{cases} \frac{1}{9h^2}(-8u_{i-1} + 8u_{i+\frac{1}{2}} - 12hu'_{i-1}), \quad i = 1 \\ \frac{1}{h^2}(u_{i-\frac{3}{2}} - 2u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}), \quad 2 \leq i \leq N-1 \\ \frac{1}{h^2}(u_{i-\frac{3}{2}} - u_{i-\frac{1}{2}} + hu_i'), \quad i = N, \end{cases} \tag{2.7}$$

$$\overline{u}''_{i+\frac{1}{2}} = \begin{cases} \frac{1}{9h^2}(184u_{i-1} - 216u_{i-1/2} + 32u_{i+1/2} + 60hu'_{i-1}), \quad i = 1 \\ \frac{1}{15h^2}(105u_{i-\frac{3}{2}} - 90u_{i-\frac{1}{2}} + 33u_{i+\frac{1}{2}} - 48u_{i-2}), \quad i = 2, \end{cases} \tag{2.8}$$

$$
\begin{aligned}
\overline{f}_{i-\frac{5}{2}} &= f(x_{i-\frac{5}{2}}, u_{i-\frac{5}{2}}, \overline{u}'_{i-\frac{5}{2}}, \overline{u}''_{i-\frac{5}{2}}), \quad \overline{f}_{i-\frac{3}{2}} = f(x_{i-\frac{3}{2}}, u_{i-\frac{3}{2}}, \overline{u}'_{i-\frac{3}{2}}, \overline{u}''_{i-\frac{3}{2}}),\\
\overline{f}_{i-1} &= f(x_{i-1}, u_{i-1}, \overline{u}'_{i-1}, \overline{u}''_{i-1}), \quad \overline{f}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}, \overline{u}'_{i-\frac{1}{2}}, \overline{u}''_{i-\frac{1}{2}}),\\
\overline{f}_{i+\frac{1}{2}} &= f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}, \overline{u}'_{i+\frac{1}{2}}, \overline{u}''_{i+\frac{1}{2}}), \quad i = 1, 2, \cdots, N,
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
\overline{\overline{u}}''_{i-\frac{1}{2}} &= \overline{u}''_{i-\frac{1}{2}} - \begin{cases} \frac{58}{1003} h(\overline{f}_{i+\frac{1}{2}} - \overline{f}_{i-1}), & i = 1\\ \frac{59}{1296} h(\overline{f}_{i+\frac{1}{2}} - \overline{f}_{i-\frac{3}{2}}), & i = 2\\ \frac{1}{6} h(\overline{f}_{i-\frac{1}{2}} - \overline{f}_{i-\frac{3}{2}}), & 3 \le i \le N-1\\ \frac{13691}{77664} h(\overline{f}_{i-\frac{3}{2}} - \overline{f}_{i-\frac{5}{2}}), & i = N, \end{cases}\\
\overline{\overline{f}}_{i-\frac{1}{2}} &= f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}, \overline{u}'_{i-\frac{1}{2}}, \overline{\overline{u}}''_{i-\frac{1}{2}}), \quad i = 1, 2, \cdots, N,
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
9u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= 8u_{i-1} + 3hu'_{i-1} - \frac{3h^3}{160}(2\overline{f}_{i-1} + 17\overline{\overline{f}}_{i-\frac{1}{2}} + \overline{f}_{i+\frac{1}{2}}) + T_i, \quad i = 1\\
-15u_{i-\frac{3}{2}} + 10u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} &= -8u_{i-2} - \frac{h^3}{16}(14\overline{f}_{i-\frac{3}{2}} + 27\overline{\overline{f}}_{i-\frac{1}{2}} - \overline{f}_{i+\frac{1}{2}}) + T_i, \quad i = 2\\
u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= -\frac{h^3}{2}(\overline{f}_{i-\frac{3}{2}} + \overline{\overline{f}}_{i-\frac{1}{2}}) + T_i, \quad 3 \le i \le N-1\\
u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} &= hu'_i + \frac{h^3}{1920}(31\overline{f}_{i-\frac{5}{2}} - 1062\overline{f}_{i-\frac{3}{2}} - 809\overline{\overline{f}}_{i-\frac{1}{2}}) + T_i, \quad i = N,
\end{aligned} \tag{2.11}
$$

where $T_i, i = 1, 2, ..., N$ is truncation error.

Let truncate the terms T_i in (2.11). Thus, at nodes $x_{i-\frac{1}{2}}, i = 1, 2, ..., N$ we obtain the N system of equations in N unknown namely $u_{i-\frac{1}{2}}$. If source function $f(x, u, u', u'')$ is linear then system of equations will linear otherwise nonlinear. We obtain an approximate solution of the problem (1.1) by solving system of equations (2.11) by an appropriate method. However, we have applied an iterative method either Gauss Seidel or Newton-Raphson to solve a system of equations respectively for linear and nonlinear system of equations.

We computed numerical value of $u_i, i = 1, 2, ..., N$ using following third order approximation,

$$
u_i = \begin{cases} -3u_{i-1} + 4u_{i-\frac{1}{2}} - hu'_{i-1}, & i = 1\\ \frac{1}{8}(-u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}}), & i = 2, ..., N-1\\ \frac{1}{8}(-u_{i-\frac{3}{2}} + 9u_{i-\frac{1}{2}} + 3hu'_i), & i = N. \end{cases}
$$

3. Derivation of the Finite Difference Method

In this section, we will outline the derivation of proposed method (2.11). In detail we discuss equation for $i = 1$ here. It easy to verify that the approximations (2.2)-(2.5) provide $O(h^3)$ to respectively $u'_{i-\frac{5}{2}}, u'_{i-\frac{3}{2}}, u'_{i-\frac{1}{2}}$ and $u'_{i+\frac{1}{2}}$. Employing Taylor series expansion and from (2.6), (2.7) and (2.8) we will obtain

$$
\begin{aligned}
\overline{u}''_{i-1} &= u''_{i-1} - \frac{h^2}{16} u^{(4)}_{i-\frac{1}{2}} + O(h^3),\\
\overline{u}''_{i-\frac{1}{2}} &= u''_{i-\frac{1}{2}} + \frac{h^2}{16} u^{(4)}_{i-\frac{1}{2}} + O(h^3), && (3.1)\\
\overline{u}''_{i+\frac{1}{2}} &= u''_{i+\frac{1}{2}} - \frac{7h^2}{16} u^{(4)}_{i-\frac{1}{2}} + O(h^3). && (3.2)
\end{aligned}
$$

Let us define

$$
\overline{\overline{u}}''_{i-\frac{1}{2}} = \overline{u}'_{i-\frac{1}{2}} + h(a_0 \overline{f}_{i-1} + a_1 \overline{f}_{i+\frac{1}{2}}).
$$

Hence, from (2.9), (2.10) and (3.1)-(3.2) we will obtain

$$2\overline{f}_{i-1} + 17\overline{\overline{f}}_{i-\frac{1}{2}} + \overline{f}_{i+\frac{1}{2}} = 2f_{i-1} + 17f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}} + h(a_0 + a_1)u^{(3)}_{i-\frac{1}{2}} + \frac{h^2}{2}(1 + 17(2a_1 - a_0))(u^{(4)}\frac{\partial f}{\partial u''})_{i-\frac{1}{2}} + O(h^3). \quad (3.3)$$

Thus, from (3.3) we conclude that, $2\overline{f}_{i-1} + 17\overline{\overline{f}}_{i-\frac{1}{2}} + \overline{f}_{i+\frac{1}{2}}$ will provide $O(h^3)$ approximation for $2f_{i-1} + 17f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}$ if

$$a_0 + a_1 = 0$$
$$1 + 17(2a_1 - a_0) = 0$$

Solving above system of equations, we will obtain $a_0 = \frac{1}{51} = -a_1$. Thus, for $i = 1$ in equation in (2.11) reduced to

$$\begin{aligned} 9u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= 8u_{i-1} + 3hu'_{i-1} - \frac{3h^3}{160}(2\overline{f}_{i-1} + 17\overline{\overline{f}}_{i-\frac{1}{2}} + \overline{f}_{i+\frac{1}{2}}) + T_i \\ &= 8u_{i-1} + 3hu'_{i-1} - \frac{3h^3}{160}(2f_{i-1} + 17f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}) + O(h^6) \end{aligned}$$

Following the above discussions, similarly, we can derive other equations in (2.11).

4. Convergence Analysis

We will consider following test equation for the purpose of convergence analysis of the proposed method (2.11);

$$u'''(x) = f(x, u(x), u'(x), u''(x)), \qquad a < x < b$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u'_0 = \beta \quad \text{and} \quad u'_N = \gamma.$$

Let

(i) $f(x, u(x), u'(x), u''(x))$ is continuous,

(ii) $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u'}$ and $\frac{\partial f}{\partial u''}$ exist and continuous,

(iii) $\frac{\partial f}{\partial u} > 0, |\frac{\partial f}{\partial u'}| < W_1$ and $|\frac{\partial f}{\partial u''}| < W_2, W_1, W_2 > 0$.

Let us define a source function $F_i = f(x_i, u_i, u'_i, u''_i)$ and $f_i = f(x_i, U_i, U'_i, U''_i)$. We can linearize source function F_i by the application of Taylor series expansion method, i.e.,

$$F_i - f_i = (u_i - U_i)(\frac{\partial f}{\partial U})_i + (u'_i - U'_i)(\frac{\partial f}{\partial U'})_i + (u''_i - U''_i)(\frac{\partial f}{\partial U''})_i.$$

Let $U_{i-\frac{1}{2}}$ and $u_{i-\frac{1}{2}}$, $i = 1, 2, \cdots, N$ are respectively approximate and exact solution of the system of equations (2.11). Let us define error in solution of the problem (1.1),

$$\epsilon_{i-\frac{1}{2}} = u_{i-\frac{1}{2}} - U_{i-\frac{1}{2}}, \quad i = 1, 2, \cdots, N.$$

Thus, we define N-dimensional vectors namely approximate solution $\mathbf{U}$, exact solution $\mathbf{u}$ and error $\mathbf{E}$. Also, we have truncation error vector $\mathbf{T} = (T_1, T_2, \cdots, T_N)$ associated with the proposed difference method (2.11).

Thus, we write a simplified form of error equation as matrix equation for the proposed method (2.11).

$$\mathbf{JE} = \mathbf{T} \quad (4.1)$$

where $\mathbf{J} = \mathbf{A} + \mathbf{B}$. These matrices $\mathbf{A}$, $\mathbf{B} = (b_{l,m})$ and $\mathbf{T} = (T_{l,1})$ are,

$$\mathbf{A} = \begin{pmatrix} 9 & -1 & & & & 0 \\ -15 & 10 & -3 & & & \\ 1 & -3 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -3 & 3 & -1 \\ 0 & & & 1 & -3 & 2 \end{pmatrix}_{N\times N},$$

$$b_{1,m} = \frac{3h^3}{160}\begin{cases} 17G_{0,1} + \frac{2928}{59h}G_{1,1} + \frac{20544}{531h^2}G_{2,1} - \frac{58h}{59}(G_{0,1} - \frac{81}{59h}G_{1,1} + \frac{154}{59h^2}G_{2,1})G_{2,1}, & m = 1\\ G_{0,1} + \frac{1792}{531h}G_{1,1} + \frac{4}{59h^2}G_{2,1} - \frac{58}{59}(\frac{263}{177}G_{1,1} + \frac{248}{177h}G_{2,1})G_{2,1}, & m = 2\\ 0, & \text{otherwise,}\end{cases}$$

$$b_{2,m} = \frac{h^3}{16}\begin{cases} 14G_{0,2} - \frac{36}{h}G_{1,2} - \frac{50}{h^2}G_{2,2} + \frac{59h}{48}(G_{0,2} - \frac{2}{h}G_{1,2} - \frac{2}{h^2}G_{2,2})G_{2,2}, & m = 1\\ 27G_{0,2} + \frac{319}{6h}G_{1,2} - \frac{20}{h^2}G_{2,2} + \frac{59}{12}(G_{1,2} + \frac{2}{h}G_{2,2})G_{2,2}, & m = 2\\ -G_{0,2} + \frac{24}{5h}G_{1,2} + \frac{22}{h^2}G_{2,2} - \frac{59h}{48}(G_{0,2} + \frac{2}{h}G_{1,2} + \frac{12}{5h^2}G_{2,2})G_{2,2}, & m = 3\\ 0, & \text{otherwise,}\end{cases}$$

$$b_{l,m} = \frac{h^3}{2}\begin{cases} -\frac{1}{6h}G_{1,l} + \frac{1}{h^2}G_{2,l} - \frac{1}{6}(\frac{1}{2}G_{1,l} - \frac{1}{h}G_{2,l})G_{2,l}, & m = l-2\\ G_{0,l} - \frac{3}{2h}G_{1,l} - \frac{1}{h^2}G_{2,l} + \frac{h}{6}(G_{0,l} + \frac{1}{2h}G_{1,l} - \frac{3}{h^2}G_{2,l})G_{2,l}, & m = l-1\\ G_{0,l} + \frac{3}{2h}G_{1,l} - \frac{1}{h^2}G_{2,l} - \frac{h}{6}(G_{0,l} - \frac{1}{2h}G_{1,l} - \frac{3}{h^2}G_{2,l})G_{2,l}, & m = l\\ \frac{1}{3h}G_{1,l} + \frac{1}{h^2}G_{2,l} - \frac{1}{6}(\frac{1}{3}G_{1,l} + \frac{1}{h}G_{2,l})G_{2,l}, & m = l+1\\ 0, & \text{otherwise,}\end{cases}$$

and $3 \le l \le N-1$.

$$b_{N,m} = \frac{h^3}{1920}\begin{cases} -31G_{0,N} - \frac{643}{3h}G_{1,N} + \frac{2031}{2h^2}G_{2,N} + \frac{13691h}{96}(G_{0,N} - \frac{29}{24h}G_{1,N} + \frac{1}{h^2}G_{2,N})G_{2,N}, & m = N-2\\ 1062G_{0,N} - \frac{1277}{h}G_{1,N} - \frac{2413}{2h^2}G_{2,N} - \frac{13691h}{96}(G_{0,N} - \frac{21}{8h}G_{1,N} + \frac{3}{2h^2}G_{2,N})G_{2,N}, & m = N-1\\ 809G_{0,N} + \frac{2237}{3h}G_{1,N} - \frac{618}{h^2}G_{2,N} - \frac{13691}{96}(\frac{17}{12}G_{1,N} - \frac{1}{h}G_{2,N})G_{2,N}, & m = N\\ 0, & \text{otherwise,}\end{cases}$$

where $G_{0,l} = (\frac{\partial f}{\partial U})_{l-\frac{1}{2}}$, $G_{1,l} = (\frac{\partial f}{\partial U'})_{l-\frac{1}{2}}$, $G_{2,l} = (\frac{\partial f}{\partial U''})_{l-\frac{1}{2}}$ and

$$T_l = T_{l,1} = h^6\begin{cases} (\frac{1}{640}u^{(6)}_{l-\frac{1}{2}} + \cdots), & l = 1\\ (-\frac{1}{28}u^{(6)}_{l-\frac{1}{2}} + \cdots), & l = 2\\ o(h), & 3 \le l \le N-1\\ (\frac{159}{11520}u^{(6)}_{l-\frac{1}{2}} + \cdots), & l = N.\end{cases}$$

Let order of square matrix $\mathbf{S}$ and identity matrix $\mathbf{I}$ are same. If $\|\mathbf{S}\| < 1$ then matrix $(\mathbf{I} + \mathbf{S})$ is invertible [3, 12] and

$$\|(\mathbf{I} + \mathbf{S})^{-1}\| < \frac{1}{1 - \|\mathbf{S}\|},$$

where $\mathbf{I}$ is an identity matrix and same order of $\mathbf{S}$. Let us assume

$$\|\mathbf{A}^{-1}\|\|\mathbf{B}\| < 1.$$

Thus, from (4.1), we have

$$\|\mathbf{E}\| < \frac{1}{1 - \|\mathbf{A}^{-1}\|\|\mathbf{B}\|}\|\mathbf{A}^{-1}\|\|\mathbf{T}\|. \tag{4.2}$$

Let

$$M = \max_{x\in[a,b]} |u^{(6)}(x)|, \quad D = \max_{x\in[a,b]} \frac{\partial f}{\partial U},$$
$$D_1 = \max_{x\in[a,b]} \frac{\partial f}{\partial U'}, \quad \text{and} \quad D_2 = \max_{x\in[a,b]} \frac{\partial f}{\partial U''}, \quad D, D_1, D_2 > 0,$$
$$D_1 \le W_1 \quad \text{and} \quad D_2 \le W_2.$$

Thus, it is easy to compute $\|\mathbf{B}\|$ and we have $\|\mathbf{A}^{-1}\| < \frac{(b-a)^3}{12h^3}$ in [10]. Thus, from (4.2) we obtained,

$$\|\mathbf{E}\| < \frac{53(b-a)^3 M h^3}{320(12 - \|\mathbf{B}\|(b-a)^3)}. \tag{4.3}$$

It follows from equation (4.3) that, $\|\mathbf{E}\| \to 0$ as $h \to 0$. This establishes that our proposed method (2.11) is convergent, and the order of convergence of the method is at least $O(h^3)$.

5. Numerical Results

To test the computational efficiency and validity of the theoretical development of a proposed method, we have considered linear and nonlinear model problems. In each model problem, we took uniform step size h. In Table 1 - Table 2, we have shown MAU the maximum absolute error in the solution $u(x)$ of the problem (1.1) for different values of N. We have used the following formula in computation of MAU,

$$MAU = \max_{1\leq i\leq N} |u(x_i) - u_i|.$$

We have used an iterative method to solve system of equations arise from equation (2.11). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compilers (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iterations reached 10^3.

Problem 1. The model linear problem in [11] and given by

$$u'''(x) = -2u''(x) + 4u'(x) - u(x) + \frac{1}{4}x(8-x) + \frac{9\exp(-2x) + 2x - 9}{4(1-\exp(-2))}, \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 1 \quad , \quad u'(0) = 1 \qquad \text{and} \quad u'(1) = 1$$

The analytical solution of the problem is

$$u(x) = C_1\exp(x) + C_2\exp(-\frac{3-\sqrt{13}}{2}x) + C_3\exp(-\frac{3+\sqrt{13}}{2}x) + \frac{1}{4}(4-x^2) + \frac{\exp(-2x) + 2x - 1}{4(1-\exp(-2))},$$

where constants C_1, C_2 and C_3 are to be determined so that boundary conditions satisfied exactly. The MAU computed by a method (2.11) for different values of N are presented in Table 1.

Table 1. Maximum absolute error (Problem 1).

	Maximum absolute error			
	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
MAU	.10943413(-2)	.26059151(-3)	.23603439(-4)	.33378601(-5)

Problem 2. The nonlinear model problem given by

$$u'''(x) = x^2(u''(x) - u'(x)) + u^2(x) + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u'(0) = -1 \qquad \text{and} \quad u'(1) = \sin(1),$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = (x-1)\sin(x)$. The MAU computed by a method (2.11) for different values of N are presented in Table 2.

Table 2. Maximum absolute error (Problem 2).

	Maximum absolute error			
	$N = 16$	$N = 32$	$N = 64$	$N = 128$
MAU	.12617186(-3)	.34049153(-4)	.62119216(-6)	.81956387(-7)

We have tested our numerical method for numerical solution of linear and nonlinear model problems. The numerical result of model problems for different values of N are presented in table 1-2. Numerical result approves the convergence of the proposed method and consistent with the theoretical development.

6. Conclusion

We have developed third order finite difference method for the numerical solution of third order boundary value problem. We discretized the problem (1.1) at a discrete set of points. Thus, we have obtained $N \times N$ a system of algebraic equations (2.11). Our proposed method (2.11) produced a good numerical solution for the considered model problems. Thus, we arrived at the conclusion that our method is computationally efficient and accurate.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Author worked on the results and he has read and agreed to the published version of the manuscript.

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