Araştırma Makalesi/Research Article

A Normed Projection Mapping on Unit Sphere

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Abstract: The unit vector of the position vector of all points at $IR^3 - \{0\}$ gives one point on unit sphere. In this paper, this mapping is called the normed projection and is shown Π_N. A normed projection of all regular curve not passing origin on unit sphere is obtained with Π_N. Every point of the projection curve defines a unit vector. Each of vector characterizes an orthogonal matrix with Cayley's theorem. Also, the Frenet vectors of spherical curve can be obtained using the Frenet vectors of original curve. When we take all of the curve, every regular curve not passing origin can be represented with a curve on orthogonal matrices set, $SO(3)$. This represent curve is called the action set of the first curve. In this paper, the obtaining of the action set and their results are studied.

Keywords: Action set, normed projection, orthogonal matrix, regular curve

Birim Küre Üzerinde Normlu İzdüşüm Dönüşümü

Özet: $IR^3 - \{0\}$ daki her noktanın yer vektörünün birim vektörü birim küre üzerinde bir nokta verir. Bu çalışmada, bunun dönüşüm hali normlu projeksiyon olarak adlandırıldı. Π_N ile gösterilen bu dönüşüm altında orijinden geçmeyen her regüler eğrinin birim küre üstünde bir normlu izdüşümü elde edilir. İzdüşüm eğrisinin her noktası bir birim vektör tanımlar. Herbir birim vektör Cayley teoremi gereğince bir ortogonal matris tanımlar. Ayrıca küresel eğrinin Frenet vektörleri, eğrinin Frenet vektörleri kullanılarak elde edilebilir. Eğrinin bütünü için ele aldığımızda orijinden geçmeyen her regüler eğri $SO(3)$ ortogonal matrisler cümlesi üstünde bir eğri ile temsil edilebilir. Bu temsil eğrisi ilk eğrinin etki cümlesi olarak isimlendirilir. Bu çalışmada etki cümlesinin elde edilişi ve sonuçları çalışıldı.

Anahtar kelimeler: Etki kümesi, normlu projeksiyon, ortogonal matris, regüler eğri

Introduction

The tangent indicatrix of a regular curve on IR^3 is known. The mapping

$$\alpha'(t) \to \frac{\alpha'(t)}{\|\alpha'(t)\|} \in S^2$$

defines the tangent indicatrix on S^2 of tangents of $\alpha(t)$ for $\alpha(t)$, regular curve. Similarly, the normal indicatrix and binormal indicatrix are also known (Struik, 1950).

In this paper we give another representation of a regular curve on unit sphere. For this, we define a normed projection on IR^3 and use this projection. One of the aims of this definition is to make up an action set with curve.

Firstly we define the normed projection mapping.

Definition 1 Π_N is defined as

$$\Pi_N : IR^3 - \{0\} \to S^2,$$

$$p \to \Pi_N(p) = q, \quad q = \frac{\overrightarrow{OP}}{\|\overrightarrow{OP}\|}$$

and called normed projection mapping on unit sphere.

We will restrict a normed projection mapping to a regular curve.

Let

$$\alpha : I \subseteq IR \to IR^3$$

be a regular curve not passing origin. Let

$$\Pi_N(\alpha(t)) = \beta(t).$$

$$\beta(t) = \left(\frac{\alpha_1(t)}{\|\alpha(t)\|}, \frac{\alpha_2(t)}{\|\alpha(t)\|}, \frac{\alpha_3(t)}{\|\alpha(t)\|}\right)$$

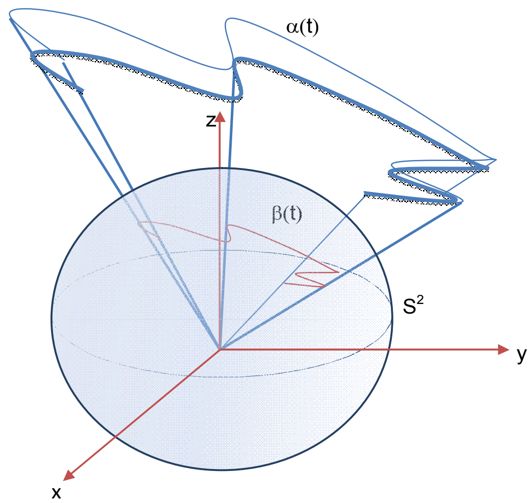

Figure 1: Spherical representation of regular curve not passing origin

Consequently, a spherical curve occurs on S^2 instead of all regular $\alpha(t)$ curve not passing origin (Figure 1). Some properties for $\alpha(t) \subset IR^3$ curve and the normed projection can be given as follows.

Some Properties of a Curve in Special Choosings

Let $\alpha(t)$ be a curve on plane E not passing origin.

Property 2

If $\alpha(t)$ is a simple open curve then, $\beta(t)$, the normed projection, is a big circle arc.

Property 3

If $\alpha(t)$ is a simple closed curve, $\beta(t)$ is a big circle.

Let $C(IR^3)$ be shown all regular curves set not passed origin on IR^3.

Definition 4

Let $\sim$ relation defines as

$$\Pi_N(\alpha) = \Pi_N(\gamma) \Leftrightarrow \alpha \sim \gamma$$

on $C(IR^3)$. In this case, α and γ is called Π_N – related.

Proposition 5

Being a Π_N – related is an equivalence relation.

Reflection Property:

$\Pi_N(\alpha) = \Pi_N(\alpha)$

$\Rightarrow \alpha$ is Π_N – related.

Symmetry Property:

$\Pi_N(\alpha)=\Pi_N(\gamma)$

$\Rightarrow \alpha$ and γ are Π_N −related $\Rightarrow \gamma$ and α are Π_N −related $\Rightarrow$

$\Pi_N(\gamma)=\Pi_N(\alpha)$.

Transition Property:

$\Pi_N(\alpha)=\Pi_N(\gamma),\quad \Pi_N(\gamma)=\Pi_N(\xi)$

$\Rightarrow \alpha$ and γ are Π_N − related and γ and ξ are

Π_N − related $\Rightarrow \Pi_N(\alpha)=\Pi_N(\xi)$.

Hereafter, $\alpha(t)$ will be mean a curve not passing origin. If $\alpha(t)$ and $\gamma(t)$ are two curves, which their normed projections are the same $\beta(t)$ spherical curve, the separate property is the difference of their tangent vectors.

Namely, let

$$\beta(t)=\Pi_N(\alpha(t)) \tag{1}$$

and

$$\beta(t)=\Pi_N(\gamma(t)). \tag{2}$$

If (1) derives,

$$\beta(t)=\frac{\alpha(t)}{\|\alpha(t)\|}$$

$$\beta'(t)=\frac{\alpha'(t)\|\alpha(t)\|-\|\alpha(t)\|'\alpha(t)}{\|\alpha(t)\|^2}$$

$$\|\alpha(t)\|'=(\langle\alpha(t),\alpha(t)\rangle^{\frac{1}{2}})'=\frac{1}{2}(\langle\alpha(t),\alpha(t)\rangle)^{-\frac{1}{2}}(\langle\alpha'(t),\alpha(t)\rangle+\langle\alpha(t),\alpha'(t)\rangle)$$

$$=\frac{\langle\alpha'(t),\alpha(t)\rangle}{(\langle\alpha(t),\alpha(t)\rangle)^{\frac{1}{2}}}=\frac{\langle\alpha'(t),\alpha(t)\rangle}{\|\alpha(t)\|}$$

$$\beta'(t)=\frac{\alpha'(t)\|\alpha(t)\|^2-(\langle\alpha'(t),\alpha(t)\rangle)\alpha(t)}{\|\alpha(t)\|^3} \tag{3}$$

is obtained. If (2) derives,

$$\beta'(t)=\frac{\gamma'(t)\|\gamma(t)\|^2-(\langle\gamma'(t),\gamma(t)\rangle)\gamma(t)}{\|\gamma(t)\|^3} \tag{4}$$

is obtained. The norms of (3) and (4) are

$$\|\beta'_\alpha(t)\|=\left\|\frac{\alpha'(t)\|\alpha(t)\|^2-\langle\alpha'(t),\alpha(t)\rangle\alpha(t)}{\|\alpha(t)\|^3}\right\| \tag{5}$$

$$\|\beta'_\gamma(t)\|=\left\|\frac{\gamma'(t)\|\gamma(t)\|^2-\langle\gamma'(t),\gamma(t)\rangle\gamma(t)}{\|\gamma(t)\|^3}\right\|. \tag{6}$$

(3)

It is not required that (5) and (6) are equal for $\forall\ \alpha(t)$ and $\gamma(t)$.

$$\|\beta'_\alpha(t)\|^2=(\langle\alpha'(t)\|\alpha(t)\|^2-\langle\alpha'(t),\alpha(t)\rangle\alpha(t),\alpha'(t)\|\alpha(t)\|^2-\langle\alpha'(t),\alpha(t)\rangle\alpha(t)\rangle)\frac{1}{\|\alpha(t)\|^6}$$

$$=(\langle\alpha'(t),\alpha'(t)\rangle\|\alpha(t)\|^4-\langle\alpha'(t),\alpha(t)\rangle\langle\alpha(t),\alpha'(t)\rangle\|\alpha(t)\|^2-$$

$$\langle\alpha'(t),\alpha(t)\rangle\langle\alpha'(t),\alpha(t)\rangle\|\alpha(t)\|^2+\langle\alpha'(t),\alpha(t)\rangle^2\langle\alpha(t),\alpha(t)\rangle)\frac{1}{\|\alpha(t)\|^6}$$

and norm of $\beta'_\alpha(t)$ is obtained as

$$\left\|\beta_{\alpha}'(t)\right\| = (\left\|\alpha'(t)\right\|^2 \left\|\alpha(t)\right\|^4 - 2\left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2 + \left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2)^{\frac{1}{2}} \frac{1}{\left\|\alpha(t)\right\|^3}.$$

However, $\beta_{\alpha}(t)$ and $\beta_{\gamma}(t)$ are the different parameter curves for $\beta(t)$. $\left\|\overrightarrow{\beta'(t)}\right\| = 1$ parameter can be choosen, because $\beta(t)$ will be a regular curve. So we can find Frenet elements of $\beta(t)$ .We write

$$(\left\|\alpha'(t)\right\|^2 \left\|\alpha(t)\right\|^4 - 2\left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2 + \left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2)^{\frac{1}{2}} \frac{1}{\left\|\alpha(t)\right\|^3} = 1$$

$$(\left\|\alpha'(t)\right\|^2 \left\|\alpha(t)\right\|^4 - 2\left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2 + \left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2) \frac{1}{\left\|\alpha(t)\right\|^6} =$$

$$\left\|\alpha'(t)\right\|^2 \left\|\alpha(t)\right\|^4 - 2\left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2 + \left\langle\alpha'(t),\alpha(t)\right\rangle^2 \left\|\alpha(t)\right\|^2 = \left\|\alpha(t)\right\|^6$$

for $\alpha(t)$. Let $\beta(t)$ gives arc length parameter. Let $\left\|\overrightarrow{\beta_{\alpha}'(t)}\right\| = 1$. The tangent vector of spherical represent is obtained as

$$\beta_{\alpha}'(t) = \frac{\alpha'(t)\left\|\alpha(t)\right\|^2 - (\left\langle\alpha'(t),\alpha(t)\right\rangle)\alpha(t)}{\left\|\alpha(t)\right\|^3} = \overrightarrow{T_{\beta}(t)}.$$

$$\overrightarrow{N_{\beta}(t)} = \frac{\beta''(t)}{\left\|\beta''(t)\right\|}$$

$$= \Big(\frac{(\alpha'(t)\left\|\alpha(t)\right\|^2 - \left\langle\alpha'(t),\alpha(t)\right\rangle\alpha(t))'\left\|\alpha(t)\right\| - \left\|\alpha(t)\right\|'(\alpha'(t)\left\|\alpha(t)\right\|^2 - \left\langle\alpha'(t),\alpha(t)\right\rangle\alpha(t))}{\left\|\alpha(t)\right\|^4}$$

$$- \frac{2\left\langle\alpha'(t),\alpha(t)\right\rangle\alpha'(t)\left\|\alpha(t)\right\|^2 - 2\left\langle\alpha'(t),\alpha(t)\right\rangle^2\alpha(t)}{\left\|\alpha(t)\right\|^5}\Big)\frac{1}{\left\|\beta''(t)\right\|}$$

and we have binormal vector as

$$\overrightarrow{B_{\beta}(t)} = \overrightarrow{T_{\beta}(t)} \times \overrightarrow{N_{\beta}(t)}$$

Finally, the Frenet vectors of spherical curve can be expressed using Frenet vectors of original curve.

Orthogonal Representation

In this section, the obtaining of the action set will be calculate for a regular $\alpha(t)$.Let

$$\alpha : I \to IR^3$$

be regular curve not passing origin.

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

($\alpha(t) \neq 0$). If $\beta(t)$ is the normed projection of $\alpha(t)$, then

$$\Pi_N(\alpha(t)) = \beta(t), \left\|\overrightarrow{O\beta(t)}\right\| = 1$$

$$\beta(t) = (b_1(t), b_2(t), b_3(t)),$$

$$\forall i, b_i(t) = \frac{\alpha_i(t)}{\|\alpha(t)\|}.$$

When a unit vector is given, the rotation is known with this vector. According to this, for $\forall\ \overrightarrow{O\beta(t)}$ provided that θ is known. $R_\theta(\beta(t))$ is the rotating matrix about $\overrightarrow{O\beta(t)}$ axis with rotation angle θ. It is apparent as follows.

$$R_\theta(\beta(t)) = \begin{bmatrix} b_1^2(1-\cos\theta)+\cos\theta & b_1b_2(1-\cos\theta)-b_3\sin\theta & b_1b_3(1-\cos\theta)+b_2\sin\theta \\ b_1b_2(1-\cos\theta)+b_3\sin\theta & b_2^2(1-\cos\theta)+\cos\theta & b_2b_3(1-\cos\theta)-b_1\sin\theta \\ b_1b_3(1-\cos\theta)-b_2\sin\theta & b_2b_3(1-\cos\theta)+b_1\sin\theta & b_3^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

is a rotation matrix for $\forall t \in I$, where $\beta(t)$ defines rotation about an fixed axis with rotation angle θ (Fu,1987; Parkin,1997). In other words,

$$R_\theta(\beta(t)) = \cos\theta.I_3 + (1-\cos\theta)[b_ib_j] + \sin\theta.S$$

;where, $S = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$.

Thus, when we take this process along the curve $\alpha(t)$,

$$R_\alpha : I \to SO(3), t \to R_\alpha(t)$$

is the representing curve on the set of the orthogonal matrix of α regular curve.

The rotating matrices set , $R_\theta(\beta(t))$, is the rotating matrices set which their origin is $\alpha(t)$. Thereby, the mapping

$$C(IR^3) \to SO(3)$$
$$\alpha(t) \to R_\alpha(\beta(t))$$

can be given. $R_\alpha(\beta(t))$ is called the action set of $\alpha(t)$. It is denoted with R_α.

The orbit of $X=(x,y,z) \in IR^3$ under R_α is obvious as

$$R_\alpha(X) = (\cos\theta.I_3 + (1-\cos\theta)[b_ib_j] + \sin\theta.S)X.$$

The point of X rotates about the rotating axis, $\overrightarrow{\beta(t)}$, with θ for $\forall t \in I$ (Figure 2).

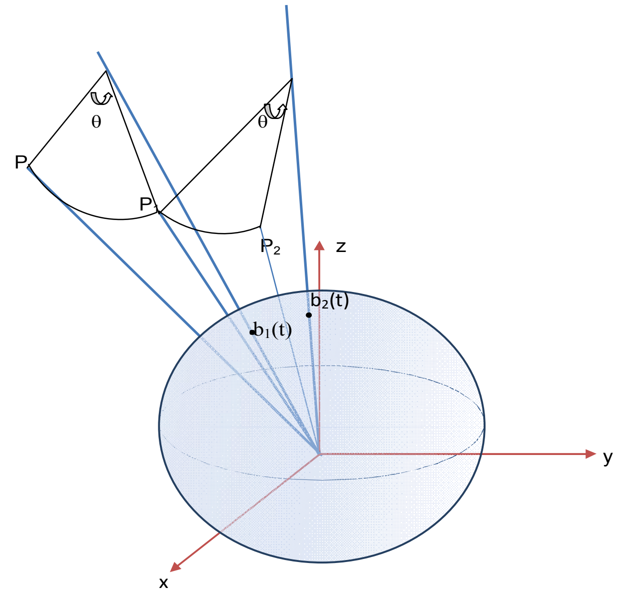

Figure 2: The successive rotations of $X \in IR^3$

The tangent vector of the orbit of X is $R_\alpha^{'}(X) = (1-\cos\theta)M + \sin\theta.N)X$,where

$$M = \frac{1}{\|\alpha\|^2}[\alpha_i^{'}.\alpha_j^{'}] - \frac{\langle \alpha,\alpha^{'}\rangle^2}{\|\alpha\|^2}[\alpha_i^{'}.\alpha_j + \alpha_i.\alpha_j^{'}] + \frac{\langle \alpha,\alpha^{'}\rangle^2}{\|\alpha\|^6}[\alpha_i.\alpha_j]$$

and

$$N = \frac{1}{\|\alpha\|}(\|\alpha\|S)^{'} - \frac{\langle \alpha,\alpha^{'}\rangle}{\|\alpha\|^2}S.$$

Property 6

The same R_α orthogonal representation for all of the cone surface which its vertex is $O=(0,0,0)$ and receives α as the base curve is obtained.

Property 7

All of the curves which have the same base curve of cone surface are Π_N − related.

Property 8

Let α *and* γ be two curves which can be the base curve for K cone surface. If $\alpha(t)=A$, $\beta(t)=B$ and A and B are on the same generated line, then we have $\Pi_N(A)=\Pi_N(B)=C \in \beta$.

Conclusion

In this paper we give another representation of a regular curve on unit sphere. For this, we define a normed projection on IR^3 and use this projection. One of the aims of this definition is to make up an action set with a curve. Additionally, the Frenet vectors of spherical curve can be obtained using the Frenet vectors of original curve.

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