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DOMINATOR SEMI STRONG COLOR PARTITION IN GRAPHS

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Abstract. Let $G =(V, E)$ be a simple graph. A subset S is said to be Semi-Strong if for every vertex v in V, $|N(v) \cap S| \leq 1$, or no two vertices of S have the same neighbour in V, that is, no two vertices of S are joined by a path of length two in V. The minimum cardinality of a semi-strong partition of G is called the semi-strong chromatic number of G and is denoted by $\chi_s G$. A proper colour partition is called a dominator colour partition if every vertex dominates some colour class, that is , every vertex is adjacent with every element of some colour class. In this paper, instead of proper colour partition, semi-strong colour partition is considered and every vertex is adjacent to some class of the semi-strong colour partition.Several interesting results are obtained.

1. Introduction

Let $G = (V, E)$ be a finite, undirected graph. We follow standard definitions of graph theory [2, 8]. A proper vertex coloring of a graph is defined as coloring the vertices of a graph such that no two adjacent vertices are colored using same color. A subset S of a graph $G = (V, E)$ is said to be a dominting set if every vertex not in S is adjacent to at least one vertex of $V - S$. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G [9, 10]. S. M. Hedetniemi [11,12] introduced and discussed the concept of dominator coloring and dominator partition of graphs. S.Arumugam et.al. discussed further in dominator coloring in graphs [1]. The combination of the two most important fields in graph

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theory namely, Coloring and domination have a lot of research results. A dominator coloring of a graph G is a proper coloring, such that every vertex of G dominates at least one color class (possibly its own class). Gera et. al. [6] defined dominator colouring in a graph G as a proper colour partition in which every vertex dominates some color class. The dominator chromatic number of G, denoted by $\chi_d(G)$, is the minimum number of colors among all dominator colorings of G. Gera researched further in [7] on dominator coloring and safe clique partitions. Kazemi proposed the concept of total dominator coloring in graphs and studied its properties [15]. A proper coloring, such that each vertex of the graph is adjacent to every vertex of some (other) color class. For more results on the total dominator coloring, refer to [14,16]. M. Chellali and F. Maffray discussed Dominator colorings in some classes of graphs [4]. In 2015, Merouane et al. [17] proposed the dominated coloring which is defined as a proper coloring such that every color class is dominated by at least one vertex. The dominated chromatic number of G, denoted by $\chi_{dom}(G)$, is the minimum number of colors among all dominated colorings of G. For comprehensive results of coloring and domination in graphs, color class domination in graphs introudced and studied in detail. refer to [5, 20, 21]. As a generalization of strong set introduced by Claude Berge [3], E.Sampathkumar defined semi-strong sets [18] in a graph. In a simple graph G, a subset S of the vertex set $V(G)$ of G is called a semi-strong set of G if $|N[v] \cap S| \leq 1$ for v in $V(G)$. E.Sampathkumar also introduced Chromatic partition of a graph [19] and studied its properties. Also, G. Jothilakshmi et al studied (k,r) - Semi Strong Chromatic Number of a Graph [13]. Instead of proper color partition, semi-strong partition [18] of $V(G)$ is considered and domination property that every vertex dominates semi-strong color class is added. The minimum cardinality of such a partition is found for some classes of graphsand bounds are obtained. Interesting results in this new concepts are derived.

Definition 1. *A subset S of $V(G)$ is called a maximal semi-strong set of G if S is semi-strong and no proper super set of S is semi-strong. The maximum cardinality of a maximal semi-strong set of G is called semi-strong number of G and is denoted by $ss(G)$.*

Definition 2. *A **dominator coloring** of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class.*

Definition 3. *A semi-strong coloring of G is called a* **dominator semi-strong color partition** *of G if every vertex of G dominates an element of the partition. The minimum cardinality of such a partition is called the* **dominator semi-strong color partition number** *of G and is denoted by $\chi_s^d(G)$.*

Since the trivial partition is a semi-strong coloring of G, the existence of dominator semi-strong color partition is guaranteed in any graph.

2. $\chi_s^d(G)$ for Some Well-Known Graphs

Observation 1. *(i)* $\chi_s^d(K_n) = \chi_d(K_n) = n$.
(ia) $\chi_s^d(K_n - e) = n$ *(since $K_n - e$ has a full degree vertex).*
(ii) $\chi_s^d(K_{1,n}) = n + 1$, $\chi_s(K_{1,n}) + \gamma(K_{1,n}) = n + 1$.
(iii) $\chi_d(K_{m,n}) = 2 < \chi_s^d(K_{m,n})$ *if* $m \leq n$ *and* $n \geq 3$.

Remark 1. *Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a dominator semi-strong color partition of G. A vertex $u \in V$ can dominate V_i if and only if $|V_i| = 1$.*

Theorem 1. *For any Path P_n, $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, $n \geq 2$.*

Proof. Let P_n be a path on n vertices.

Case 1: $n = 4k$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_4\}$, $V_{k+2} = \{v_8\}$, $\ldots$, $V_{2k+1} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k + 1 = \lceil \frac{n}{2} \rceil + 1$.

Let Π_1 be a χ_s^d-partition of P_n. The maximum cardinality of an element of Π_1 is at most $2k$. There are at least $2k$ singletons to dominate $4k$ elements, since no single element can dominate two elements of a set which are at a distance 2. Therefore $|\Pi| \geq 2k + 1$. Therefore $\chi_s^d(P_{4k}) = 2k + 1 = \lceil \frac{n}{2} \rceil + 1$.

Case 2: Let $n = 4k + 1$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_{4k+1}\}$, $V_{k+2} = \{v_4\}$, $V_{k+3} = \{v_8\}, \ldots, V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k + 2 = \lceil \frac{n}{2} \rceil + 1$.

Arguing as in case 1, $\chi_s^d(P_{4k+1}) \geq 2k+2 = \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, where $n = 4k + 1$.

Case 3: Let $n = 4k + 2$, $k \geq 0$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v_{4k+2}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_{4k+1}\}$, $V_{k+2} = \{v_4\}$, $V_{k+3} = \{v_8\}$, …, $V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k+2 = \lceil \frac{n}{2} \rceil + 1$.

Arguing as in case 1, $\chi_s^d(P_{4k+2}) \geq 2k+2 = \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, where $n = 4k + 2$.

Case 4: Let $n = 4k + 3$, $k \geq 0$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}, V_{2k+3}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v_{4k+2}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_{4k-3}\}$, $V_{k+2} = \{v_{4k+1}\}$, $V_{k+3} = \{v_4\}$, $V_{k+4} = \{v_8\}, \ldots, V_{2k+2} = \{v_{4k}\}$, $V_{2k+3} = \{v_{4k+3}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k + 3 = \lceil \frac{n}{2} \rceil + 1$.

Arguing as in case 1, $\chi_s^d(P_{4k+3}) \geq 2k+3 = \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, where $n = 4k + 3$. ∎

Theorem 2. $\chi_s^d(C_n) = \lceil \frac{n}{2} \rceil + 1$, $n \geq 3$.

Proof. Let C_n be a cycle on n vertices.

Case 1: $n = 4k$, $k \geq 1$

Let $V(C_n) = \{v_1, v_2, \ldots, v_{4k}\}$. Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}, V_2 = \{v_1\}, V_3 = \{v_5\}, \ldots, V_{k+1} = \{v_{4k-3}\}, V_{k+2} = \{v_4\}$, $V_{k+3} = \{v_8\}$, $\ldots$, $V_{2k+1} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of C_n. Therefore $\chi_s^d(C_n) \leq |\Pi| = 2k + 1 = \frac{4k}{2} + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k$ singletons and no single element can dominate a two element set which are at a distance 2. Therefore $\chi_s^d(C_{4k}) \geq \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(C_{4k}) = \lceil \frac{n}{2} \rceil + 1$.

Case 2: Let $n = 4k + 1$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_{k+1} = \{v_{4k-3}\}$, $V_{k+2} = \{v_{4k+1}\}$, $V_{k+3} = \{v_4\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(C_{4k+1}) \leq |\Pi| = 2k + 2 = \lceil \frac{4k+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k$ singletons and no single element can dominate a two element set which are at a distance 2. Therefore $\chi_s^d(C_{4k+1}) \geq \lceil \frac{n}{2} \rceil + 1$ and hence $\chi_s^d(C_{4k+1}) = \lceil \frac{n}{2} \rceil + 1$.

Case 3: Let $n = 4k + 2$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_{4k+1}, v_{4k+2}\}, V_3 = \{v_1\}, V_4 = \{v_5\}, \ldots, V_{k+2} = \{v_{4k-3}\}, V_{k+3} = \{v_4\}, V_{k+4} = \{v_8\}, \ldots, V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of C_n. Therefore $\chi_s^d(C_{4k+2}) \leq |\Pi| = 2k + 2 + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k$ singletons and no single element can dominate a two element set which are at a distance 2. Any doubleton must consist of consecutive vertices. Therefore $\chi_s^d(C_{4k+2}) \geq \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(C_{4k+2}) = \lceil \frac{n}{2} \rceil + 1$.

Case 4: Let $n = 4k + 3$, $k \geq 0$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}, V_{2k+3}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-1}, v_{4k+2}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_{k+1} = \{v_{4k-3}\}$, $V_{k+2} = \{v_{4k+1}\}$, $V_{k+3} = \{v_4\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$, $V_{2k+3} = \{v_{4k+3}\}$. Then Π is a dominator semi-strong color partition of C_n. Therefore $\chi_s^d(C_{4k+3}) \leq |\Pi| = 2k+3 = \lceil \frac{4k+3}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k + 1$ singletons and no single element can dominate a two element set which are at a distance 2. Any doubleton set must consist of consecutive vertices. Therefore $\chi_s^d(C_{4k+3}) \geq \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(C_{4k+3}) = \lceil \frac{n}{2} \rceil + 1$. □

Theorem 3. *For Complete bi-partite graph* $K_{m,n}, \chi_s^d(K_{m,n}) = max\{m, n\} + 1$.

Proof. Let V_1, V_2 be the partite sets of $K_{m,n}$.

Case 1: Let $m < n$.
Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$.
Let $\Pi = \{\{u_1, v_1\}, \ldots, \{u_{m-1}, v_{m-1}\}, \{u_m\}, \{v_m\}, \ldots, \{v_n\}\}$. Then each of $v_1, v_2, \ldots, v_n$ dominates $\{u_m\}$, and each of $u_1, u_2, \ldots, u_{m-1}$ dominates $\{v_n\}$. Therefore Π is a dominator semi-strong color partition of $K_{m,n}$.
Therefore $\chi_s^d(K_{m,n}) \leq |\Pi| = m + n - (m-1) = n + 1$.

No two elements of V_1 can belong to an element of Π. Also no two elements of V_2 can belong to an element of Π. Any element of V_1 dominates all elements of V_2. So is the case with V_2. Therefore Π must consist of at least one singleton from V_1 and one singletons from V_2. Therefore $\chi_s^d(K_{m,n}) \geq m - 1 + 2 + (n - m) = n + 1$.
Therefore $\chi_s^d(K_{m,n}) = n + 1 = max\{m, n\} + 1$.
Case 2: Let $m = n$

Let $\Pi = \{\{u_1, v_1\}, \ldots, \{u_{m-1}, v_{m-1}\}, \{u_m\}, \ldots, \{v_n\}\}$. Proceeding as in case 1, $\chi_s^d(K_{m,n}) = m + 1 = max\{m, n\} + 1$. □

Corollary 1. $\chi_s^d(K_{1,n}) = n + 1$.

Theorem 4. $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) = m + max\{a_1, a_2, \ldots, a_m\}$.

Proof. Let $a_1 \leq a_2 \leq \ldots \leq a_m$. Let $V(K_m(a_1, a_2, \ldots, a_m)) = \{u_1, u_2, \ldots, u_m, v_{1,1}, v_{1,2}, \ldots, v_{1,a_1}, \ldots, v_{m,1}, \ldots, v_{m,a_m}\}$. Let $\Pi = \{\{u_1\}, \ldots, \{u_m\}, \{v_{1,1}, v_{2,1}, \ldots, v_{m,1}\}, \ldots, \{v_{1,a_1}, v_{2,a_1}, \ldots, v_{m,a_1}\}, \{v_{2,a_2}, v_{3,a_2}, \ldots, v_{m,a_2}\}, \ldots, \{v_{m,a_m}\}\}$. Then $|\Pi| = m + a_m = m + max\{a_1, a_2, \ldots, a_m\}$.

Therefore $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) \leq m + max\{a_1, a_2, \ldots, a_m\}$. Any χ_s^d-partition must contain $u_1, u_2, \ldots, u_m$ as singletons for dominating the pendent vertices. Further no two pendent vertices at any u_i, $1 \leq i \leq m$ can belong to an element of the partition. Therefore $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) \geq m + max\{a_1, a_2, \ldots, a_m\}$. Therefore $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) = m + max\{a_1, a_2, \ldots, a_m\}$. □

Let G be the graph shown in Figure 1

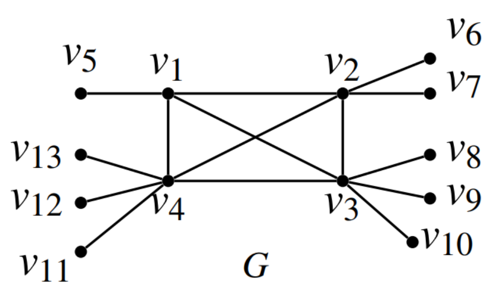

Figure 1. $G = K_4(1, 2, 3, 3)$ with $\chi_s^d(G) = 7$

Let $\Pi = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5, v_6, v_8, v_{11}\}, \{v_7, v_9, v_{12}\}, \{v_{10}, v_{13}\}\}$. Then Π is a χ_s^d-partition of G. Therefore $\chi_s^d(G) = |\Pi| = 4 + 3 = 7$.

Theorem 5. *$\chi_s^d(K_{a_1,a_2,\ldots,a_m}) = a_1 + a_2 + \ldots + a_m$ if $m \geq 3$.*

Proof. Let $m \geq 3$. Then any vertex of $K_{a_1,a_2,\ldots,a_m}$ is a common vertex of two vertices. Hence no two vertices can be included in an element of a χ_s^d-partition. Hence $\chi_s^d(K_{a_1,a_2,\ldots,a_m}) = a_1 + a_2 + \ldots + a_m$ if $m \geq 3$. □

Theorem 6. *$\chi_s^d(P) = 7$ where P is the Petersen graph.*

Proof. Consider the graph in Figure 2. Let $V(P) = \{v_1, v_2, \ldots, v_{10}\}$.
Let $\Pi = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}, \{v_6, v_9\}, \{v_7\}, \{v_8\}, \{v_{10}\}\}$. Then Π is a dominator semi-strong color partition of P. Therefore $\chi_s^d(P) \leq 7$.

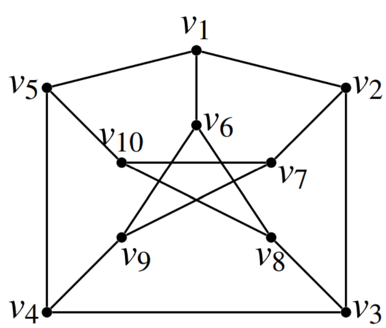

FIGURE 2. Petersen Graph

In any χ_s^d-partition of P, no three-element set can appear. Since for any three element set of P, there exists a vertex which is adjacent to two of the element of that set. Any three 2 element sets must have three singletons for domination. Hence the remaining one element must appear as a singleton. Therefore $\chi_s^d(P) \geq 7$. Therefore $\chi_s^d(P) = 7$. □

Remark 2. *(i) $1 \leq \chi_s^d(G) \leq n$.*
(ii) $\chi_s^d(G) = 1$ if and only if $G = K_1$.

Observation 2. *Let G be a graph with full degree vertex. Then $\chi_s^d(G) = |V(G)|$.*

Proof. Let Π be a χ_s^d-partition of G. Let $V_1 \in \Pi$. If $|V_1| \geq 2$, then any two points of V_1 are adjacent with full degree vertex, a contradiction. Therefore $|V_1| = 1$. Therefore $\chi_s^d(G) = |V(G)|$. □

Corollary 2. *$\chi_s^d(W_n) = n$.*

Corollary 3. *$\chi_s^d(F_n) = n$.*

3. Main Results

Theorem 7. *$max\{\chi_s(G), \gamma(G)\} \leq \chi_s^d(G) \leq \chi_s(G) + \gamma(G)$.*

Proof. Since any χ_s^d-partition of G is a χ_s-partition of G, $\chi_s(G) \leq \chi_s^d(G)$. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ where $k = \chi_s^d(G)$ be a χ_s^d-partition of G. Let $x_i \in V_i$, $1 \leq i \leq k$. Let $S = \{x_1, x_2, \ldots, x_k\}$. Let $v \in V - S$. Then v dominates some color class, say V_i. Therefore v is adjacent with x_i. Therefore $\{x_1, x_2, \ldots, x_k\}$ dominates G. That is, S is a dominating set of G. That is, $\gamma(G) \leq |S| = k = \chi_s^d(G)$. Therefore $max\{\chi_s(G), \gamma(G)\} \leq \chi_s^d(G)$.

Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a χ_s-coloring of G. Assign colors $\chi_s(G)+1, \ldots, \chi_s(G) + \gamma(G)$ to the vertices of a minimum dominating set of G, leaving the rest of the vertices colored as before. Then the resulting partition is a dominator semi-strong color partition of G. Therefore, $\chi_s^d(G) \leq |\Pi| + \gamma(G) = \chi_s(G) + \gamma(G)$. □

Remark 3. *The set S need not be a minimum dominating set. For example, when $G = P_6$, $\chi_s^d(G) = 4$. But $\gamma(P_6) = 2$.*

Theorem 8. *Let a, b be positive integers with $a \leq b$. Then there exists a graph G such that $\chi_d(G) = a$ and $\chi_s^d(G) = b$.*

Proof. When $a = b$, $\chi_d(K_a) = \chi_s^d(K_a) = a$. Let $a < b$. Let $G = K_{a_1, a_2, \ldots, a_k}$ where $k = a$. Then $\chi_d(G) = a$. Choose $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \ldots + a_k = b$. Then $\chi_s^d(G) = b$. □

Theorem 9. *$\chi_s^d(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $\chi_s^d(G) = 2$. Suppose $\chi_s^d(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be a χ_s^d-partition of G. Suppose $|V_1| \geq 2$. Then any vertex of V_2 dominates V_1 unless $|V_2| = 1$. If $|V_2| > 1$, then it is a contradiction. Therefore $|V_2| = 1$. Similarly, $|V_1| = 1$. Therefore $G = K_2$. □

Corollary 4. *Suppose T is a tree of order $n \geq 2$. Then $\chi(T) = 2$. $\chi_s^d(T) = \chi(T)$ if and only if $\chi_s^d(T) = 2$. That is if and only if $G = K_2$.*

Theorem 10. *Let G be a connected unicyclic graph. Then $\chi_s^d(G) = \chi(G)$ if and only if $G = C_3$.*

Proof. If G is a cycle, then $\chi_s^d(G) = \chi(G)$ if and only if $G = C_3$. Suppose G contains C_{2n}. Then $\chi(G) = 2$, but $\chi_s^d(G) \geq 3$, a contradiction. Therefore G contains an odd cycle C_{2n+1}. Then $\chi(G) = 3$. If there exists a path attached with a vertex of C_{2n+1}, then $\chi_s^d(G) \geq 4$, a contradiction. Therefore G is a cycle. Since $\chi_s^d(G) = \chi(G)$, $G = C_3$. □

Theorem 11. *Let G be a connected graph. Then $\chi_s^d(G) = n$ if and only if either G has a full degree vertex or $N(G) = K_n$.*

Proof. Let $\chi_s^d(G) = n$. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Then $\Pi = \{\{u_1\}, \{u_2\}, \ldots, \{u_n\}\}$ is a χ_s^d-partition of G. Let $diam(G) = k \geq 3$. Let u and v be the end vertices of a diametrical path. Let $u = u_1, u_2, \ldots, u_{k+1} = v$. Then u and v have no common adjacent vertex. Therefore $\Pi_1 = \{\{u, v\}, \ldots, \{u_n\}\}$. Then u dominates $\{u_2\}$ and v dominates $\{u_k\}$. Also $\{u, v\}$ is dominated by a single vertex. Therefore Π_1 is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq n - 1$, a contradiction. Therefore $diam(G) \leq 2$.

Suppose u_1 and u_2 are adjacent and u_1u_2 is not the edge of a triangle. Then $\{u_1, u_2\}$ can be taken as an element of a dominator semi-strong color partition of G with all other vertices as singletons. If u_1 is adjacent with some u_i, $i \geq 3$ and u_2 is adjacent with some u_j, $j \neq \{1, 2\}$, then $\chi_s^d(G) \leq n - 1$, a contradiction. Therefore if $|V(G)| \geq 4$ and $diam(G) \leq 2$ and u_1u_2 is an edge such that u_1 and u_2 have separate adjacent vertices, then u_1u_2 is the edge of a triangle. In such case, $N(G) = K_n$. Suppose u_1 is adjacent with some vertex u_3 and u_2 is not adjacent with any vertex of G other than u_1. Suppose u_3 is adjacent with some vertex u_4. If u_1 is not adjacent with u_4, then $\Pi_2 = \{\{u_1, u_3\}, \{u_2\}, \{u_4\}, \ldots, \{u_n\}\}$ is a dominator semi-strong color partition of G, a contradiction. If u_3 is adjacent with u_1, then u_4 is also adjacent with u_1. Therefore G is a connected graph with a full degree vertex.

Suppose G has no full degree vertex. Then the case that only one of u_1, u_2 which are adjacent, has some other adjacent vertex does not hold. Therefore both u_1 and u_2 have different adjacent vertices. Therefore u_1u_2 is the edge of a triangle. Therefore $diam(G) \leq 2$ and when u_1u_2 is an edge, then u_1u_2 is the edge of a triangle. Therefore $N(G) = K_n$. The converse is obvious. □

Remark 4. *Let G be the graph given in Figure 3.*

Then $G = N(G)$, $N(G)$ is not complete and G has no full degree vertex. Therefore $\chi_s^d(G) = 4$ and $\chi_s(G) = 3$.

Remark 5. *Let G be the graph shown in Figure 4.*

Then $N(G) = K_5 - \{e\}$. G has a full degree vertex and hence $\chi_s^d(G) = 5$ eventhough $N(G)$ is not complete. Hence $\chi_s(G) = 4$ and $\chi_s^d(G) = 5$.

Remark 6. *Let G be a complete multipartite graph $K_{a_1,a_2,\ldots,a_n}$, $n \geq 3$. Then G has no full degree vertex. $\chi_s^d(G) = n$ and hence $N(G) = K_n$.*

Observation 3. *Let G be a cycle C_n with pendent vertex attached with exactly one vertex of C_n. Then* $\chi_s^d(G) = \begin{cases} \chi_s^d(C_n) + 1 & \text{if } n \not\equiv 1 \ (mod\ 4) \\ \chi_s^d(C_n) & \text{otherwise} \end{cases}$

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Let u_{n+1} be a pendent vertex attached with u_1.

Case 1: Let $n = 4k$.

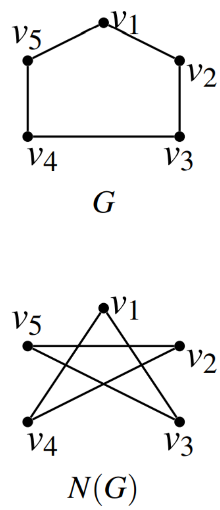

FIGURE 3. $G = N(G) = C_5$

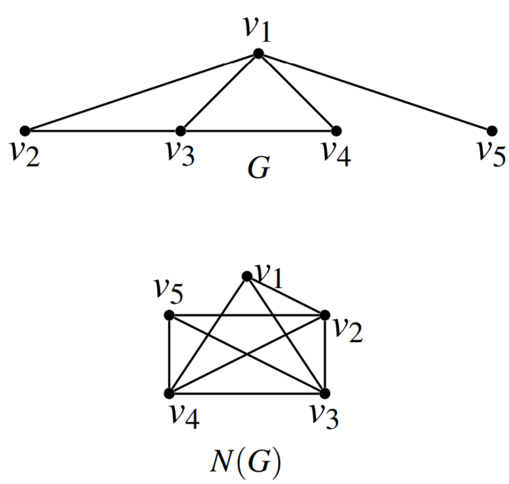

FIGURE 4. G and $N(G)$

Let $\Pi = \{\{u_{4k+1}, u_3, u_4, u_7, u_8, \ldots, u_{4k-5}, u_{4k-4}, u_{4k-1}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k-3}\}, \{u_{4k-2}\}, \{u_{4k}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq 1 + 2k + 1 = 2k + 2 = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

There are at least $2k$ singletons and no single element can dominate a 2 element set whose elements are at distance 2. Also for the pendent vertex either it appears as a singleton or its support appears as a singleton. Therefore $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil + 2$. Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Case 2: Let $n = 4k + 1$.

Let $\Pi = \{\{u_{4k+2}, u_3, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k+1}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq 1 + k + 1 + k = 2k + 2 = \lceil \frac{n}{2} \rceil + 1 = \chi_s^d(C_n)$.

If $\chi_s^d(G) < \lceil \frac{n}{2} \rceil + 1$, then removing the pendent vertex we get that $\chi_s^d(C_n) < \lceil \frac{n}{2} \rceil + 1$, a contradiction. Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 1 = \chi_s^d(C_n)$.

Case 3: Let $n = 4k + 2$.

Let $\Pi = \{\{u_{4k+3}, u_3, u_4, u_7, u_8, \ldots, u_{4k-5}, u_{4k-4}, u_{4k-1}, u_{4k}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k-3}\}, \{u_{4k-2}\}, \{u_{4k+1}\}, \{u_{4k+2}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Arguing as in case 1, we get that $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil + 2$.

Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Case 4: Let $n = 4k + 3$.

Let $\Pi = \{\{u_{4k+4}, u_3, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k+1}\}, \{u_{4k+2}\}, \{u_{4k+3}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq |\Pi| = 1 + k + 1 + k + 2 = 2k + 4 = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Arguing as in case 1, we get that $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil + 2$.

Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$. □

Proposition 1. *If $diam(G) \leq 2$, then $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil$, where $|V(G)| = n$.*

Proof. Let G be a connected graph and $diam(G) \leq 2$. If $diam(G) = 1$, then $G = K_n$ and $\chi_s^d(G) = n \geq \lceil \frac{n}{2} \rceil$. Suppose $diam(G) = 2$. Then $\chi_s^d(G) \geq \chi_s(G) \geq \lceil \frac{n}{2} \rceil$ [?]. □

Remark 7. *The converse of the above proposition need not be true.*
For: $\chi_s^d(C_n) = \lceil \frac{n}{2} \rceil + 1 > \lceil \frac{n}{2} \rceil$ for all $n \geq 3$. When $n \geq 6$, $diam(C_n) \geq 3$.

Definition 4. *$C_m(a_1, a_2, \ldots, a_m)$ is the graph obtained from the cycle C_m by attaching a_i (≥ 1) pendent vertices at the vertex u_i of C_m, $1 \leq i \leq m$.*

Proposition 2. *$\chi_s^d(C_m(a_1, a_2, \ldots, a_m)) = m + max\{a_1, a_2, \ldots, a_m\}$.*

Proof. The proof follows on the same line as the proof of the Theorem 4. □

Theorem 12. *Let G be a connected graph. Then $\chi_s^d(G) = n - 1$, where $|V(G)| = n$ if and only if $n \geq 4$. When $n = 4$, $G = P_4$ or C_4. When $n = 5$, G is one of the ten graphs P_5, C_5, $D_{1,2}$ or G_i, $(1 \leq i \leq 7)$ given in Figure 5. When $n \geq 6$, there exist two vertices say u_1, u_2 such that u_1 and u_2 may be either adjacent or independent and there exist u_i, $(3 \leq i \leq n)$ adjacent with u_1 and not with u_2, there exist u_j, $(j \neq i)$, $(3 \leq k \leq n)$ such that u_r and u_s are adjacent with u_k and u_1 may*

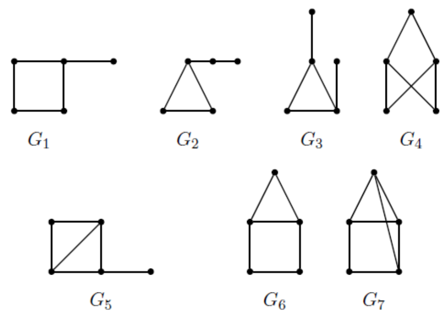

FIGURE 5. A set of graphs $G_1, G_2, G_3, G_4, G_5, G_5, G_7$ with $n = 5$ and $\chi_s^d(G) = n - 1$

be adjacent with any u_k, $(k \neq j)$, u_2 *may be adjacent with any* u_k, $(k \neq i)$ *but* u_1 *and* u_2 *are not together adjacent with any* u_k.

Proof. Let G be a connected graph. Let $\chi_s^d(G) = n - 1$. Let $\Pi = \{\{u_1, u_2\}, \{u_3\}, \{u_4\}, \ldots, \{u_n\}\}$ be a χ_s^d-partition of G.

Case 1: u_1 and u_2 are adjacent.

Let u_i, $3 \leq i \leq n$, be such that u_i is not adjacent with both u_1 and u_2. That is, either u_i is adjacent with u_1 and not with u_2 or u_i is adjacent with u_2 and not with u_1 or u_i is not adjacent with both u_1 and u_2. Since Π is a χ_s^d-partition, there exist some u_i, $3 \leq i \leq n$ adjacent with u_1 and some u_j, $j \neq i$, $3 \leq j \leq n$, adjacent with u_2. Then u_i, u_2 have a common vertex u_1 and u_j, u_1 have a common vertex u_2. Any two of the vertices $u_3, \ldots, u_n$ have a common vertex that is, $d(u_r, u_s) \leq 2$. Let $n \geq 6$. Suppose u_r and u_s are adjacent, $r \neq s$, $r, s \notin \{1, 2\}$, $3 \leq r, s \leq n$. Then there exist u_k, $3 \leq k \leq n$, $k \neq \{r, s\}$ such that u_i, u_j, u_k form a triangle. If u_r and u_s are independent, then there exist u_k, $3 \leq k \leq n$, $k \neq \{i, j\}$ such that u_r, u_s, u_k form a path of length 2. If $n = 5$, then only one vertex is left other than u_1, u_2, u_i, u_j, and the graph is either P_5 or $D_{1,2}$ or C_5, a contradiction.

Subcase 1: $n = 3$

Then $G = P_3$ or K_3. Then $\chi_s^d(G) = 3$, a contradiction. Therefore $n \geq 4$.

Subcase 2: $n = 4$

Then $G = P_4$, C_4, K_4, $K_{1,3}$, $K_4 - \{e\}$. When $G = K_4, K_{1,3}, K_4 - \{e\}$, G has a full degree vertex. Therefore $\chi_s^d(G) = 4$, a contradiction. Hence $G = P_4$ or C_4.

Subcase 3: $n = 5$

Then $G = P_5$, C_5, K_5, $K_{1,4}$, $K_5 - \{e\}$, $K_5 - \{e_1, e_2\}$ or one of the following graphs shown in Figure 6:

Therefore $\chi_s^d(G) = 4$ if $G = P_5, C_5, D_{1,2}$ or one of the following graphs shown in Figure 7:

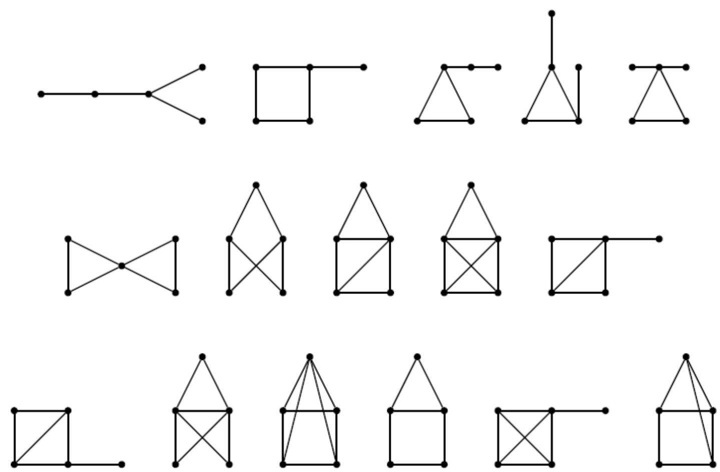

FIGURE 6. A set of graphs $G_1, G_2, G_3, G_4, G_5, G_5, G_7$ with $n = 5$

Case 2: u_i and u_j are independent.

Let u_i, $3 \leq i \leq n$, be not adjacent with both u_1 and u_2. That is, either u_i is adjacent with u_1 and not with u_2 or u_i is adjacent with u_2 and not with u_1 or u_i is not adjacent with both u_1 and u_2. Then Π is a χ_s^d-partition, there exist some u_i, $3 \leq i \leq n$ adjacent with u_1 and some u_j, $j \neq i$, $3 \leq j \leq n$, adjacent with u_2. Then u_i, u_2 have a common vertex u_1 and u_j, u_1 have a common vertex u_2. Any two of the vertices $u_3, \ldots, u_n$ have a common vertex that is, $d(u_r, u_s) \leq 2$. Let $n \geq 6$. Suppose u_r and u_s are adjacent, $r \neq s$, $r, s \notin \{1, 2\}$, $3 \leq r, s \leq n$. Then there exist u_k, $3 \leq k \leq n$, $k \neq \{r, s\}$ such that u_r, u_s, u_k form a triangle. If u_r and u_s are independent, then there exist u_k, $3 \leq k \leq n$, $k \neq \{r, s\}$ such that u_r, u_s, u_k form a path of length 2. If $n = 5$, then only one vertex is left other than u_1, u_2, u_i, u_j, and the graph is either P_5 or a contraction.

□

4. Conclusion

In this paper, a study of dominator semi-strong partition and the parameter $\chi_s^d(G)$ is initiated. Further study can be made on the complexity of the parameter and Nordhaus-Gaddum type results for $\chi_s^d(G)$.

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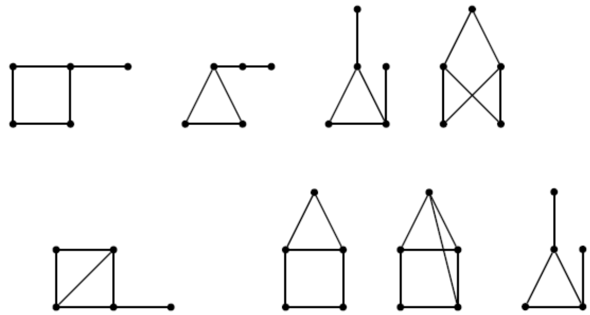

FIGURE 7. A set of graphs $G_1, G_2, G_3, G_4, G_5, G_5, G_7$ with $n = 5$

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