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A Generalization of Two-Dimensional Bernstein-Stancu Operators

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Research Article

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Abstract
The aim in our study is giving a generalization of the two-dimensional (p,q)-Bernstein-Stancu operators in a particular domain. In addition, by creating some direct results of these operators, rate of convergence is studied by Lipschitz type functions and modulus of continuity.

Keywords: (p,q)-integers, Korovkin type approximation, (p,q)-Bernstein-Stancu Operators

İki Boyutlu Bernstein-Stancu Operatörlerinin Bir Genellemesi

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Öz
Çalışmamızın amacı, belirli bir aralıkta tanımlı iki boyutlu (p, q)-Bernstein-Stancu operatörlerinin bir genellemesini vermektir. Ayrıca, bu operatörlerin bazı direkt sonuçları oluşturularak, Lipschitz tipi fonksiyonlar ve süreklilik modülü ile yaklaşım hızı incelenmiştir.

Anahtar Kelimeler: (p,q)-tamsayı, Korovkin tipi yaklaşım, (p,q)-Bernstein-Stancu operatörleri

Introduction

The most researched operators in approximation theory is Bernstein operators and their modifications [1-15]. It has been an important field of study recently that the versions of the operators in the literature, which will be created using (p,q)-calculation, have better error estimation than the classical versions. In 2015, (p,q)-Bernstein operators defined and then various modifications of these operators have been studied by different authors in [2-10]. Thus, many well-known operators were transferred to post quantum calculus. In this study, based on the work of Karahan and Izgi [10], in which (p,q)-Bernstein operators are defined on a specific interval, important features of approximating to functions in a certain domain with bivariate (p,q)-Bernstein-Stancu type operators will be examined.

Our aim is to obtain a modification of the two-dimensional version of the (p,q)-Bernstein-Stancu operators on $E^2=\left[0,\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right]\times\left[0,\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right]$. Also, some important approximation theorems are proved using this operators and the rate of convergence is estimated.

Now we remember some main concepts of (p,q)-analysis. (p,q) integers are

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+pq^{n-2}+q^{n-1}=\begin{cases}\frac{p^n-q^n}{p-q}, & \text{if } p\neq q\neq 1;\\ np^{n-1}, & \text{if } p=q\neq 1\\ [n]_q, & \text{if } p=1;\\ n, & \text{if } p=q=1\end{cases} \tag{1}$$

for every $p,q>0$, here $[n]_q$ demonstrates q-integers for all $n\in\mathbb{N}\cup\{0\}$.

The (p,q)-factorial is described with

$$[n]_{p,q}!=\begin{cases}[n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}, & if\ n\geq 1,\\ 1, & if\ n=0.\end{cases} \tag{2}$$

Then (p,q)-binomial coefficient is characterized by

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}=\begin{bmatrix}n\\n-k\end{bmatrix}_{p,q} \tag{3}$$

for every $n,k\in\mathbb{N}$ and $n\geq k$. Also, following important equation are valid.

$$(ax+by)_{p,q}^n:=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}a^{n-k}b^kx^{n-k}y^k$$

Materials and Methods

In this section, the operators that we are working with is introduced and the status of the operators in the test functions is examined.

Let $i\in\{1,2\}$, $0<q_i<p_i\leq 1$ and $0<a<b$. Then, for all $0<\alpha<\beta$, we define a modification of two-dimensional version of the (p,q)-Bernstein-Stancu operators as follows.

$$\begin{aligned}\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)&=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}N_{n,k}(p_1,q_1;x)N_{m,j}(p_2,q_2;y)\\&\times f\left(\frac{\left([k]_{p_1,q_1}p_1^{n-k}+\alpha\right)[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}\left([n]_{p_1,q_1}+\beta\right)},\frac{\left([j]_{p_2,q_2}p_2^{m-j}+\alpha\right)[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}\left([m]_{p_2,q_2}+\beta\right)}\right)\end{aligned} \tag{5}$$

where

$$N_{n,k}(p_1,q_1;x)=\left(\frac{[n+b]_{p_1,q_1}}{[n+a]_{p_1,q_1}}\right)^n\begin{bmatrix}n\\k\end{bmatrix}_{p_1,q_1}p_1^{\frac{k(k-1)}{2}}x^k\prod_{s=0}^{n-k-1}\left(p_1^s\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}-q_1^sx\right), \tag{6}$$

$$N_{m,j}(p_2,q_2;y)=\left(\frac{[m+b]_{p_2,q_2}}{[m+a]_{p_2,q_2}}\right)^m p_2^{\frac{j(j-1)}{2}}\begin{bmatrix} m \\ j \end{bmatrix}_{p_2,q_2} y^j \prod_{r=0}^{m-j-1}\left(p_2^r\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}-q_2^r y\right). \quad (7)$$

Definition 1 Let $E^2=\left[0,\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right]\times\left[0,\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right]$, $0<q_i<p_i\le 1$, where $i\in\{1,2\}$ and

$0<a<b$. For $0<\alpha<\beta$, $f:E^2\to\mathbb{R}^2$ and $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$, we can make following demonstrations:

$${}^{x}\tilde{S}_n^{(p_1,q_1)}(f;x,y)=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n} f\left(\frac{[n+a]_{p_1,q_1}\left([k]_{p_1,q_1}p_1^{n-k}+\alpha\right)}{[n+b]_{p_1,q_1}\left([n]_{p_1,q_1}+\beta\right)},y\right)N_{n,k}(p_1,q_1;x) \quad (8)$$

and

$${}^{y}\tilde{S}_m^{(p_2,q_2)}(f;x,y)=\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{j=0}^{m} f\left(x,\frac{[m+a]_{p_2,q_2}\left([j]_{p_2,q_2}p_2^{m-j}+\alpha\right)}{[m+b]_{p_2,q_2}\left([m]_{p_2,q_2}+\beta\right)}\right)N_{m,j}(p_2,q_2;y). \quad (9)$$

Lemma 1 Let ${}^{x}\tilde{S}_n^{(p_1,q_1)}$, ${}^{y}\tilde{S}_m^{(p_2,q_2)}$ are defined on $C(E^2)$. Then the following results hold.

$$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)={}^{x}\tilde{S}_n^{(p_1,q_1)}(f;x,y)\,{}^{y}\tilde{S}_m^{(p_2,q_2)}(f;x,y)$$
$$={}^{y}\tilde{S}_m^{(p_2,q_2)}(f;x,y)\,{}^{x}\tilde{S}_n^{(p_1,q_1)}(f;x,y).$$

Lemma 2 Let $f:E^2\to\mathbb{R}^2$, $0<a<b$ and for $i\in\{1,2\}$, $0<q_i<p_i\le 1$. Then, for all $0<\alpha<\beta$ and $k\in\{0,1,2\}$, we have the next equalities for the functions $\sigma^k\varphi^k$;

i. $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y)=1,$ (10)

ii. $$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi;x,y)=\frac{[n]_{p_1,q_1}}{[n]_{p_1,q_1}+\beta}x+\frac{\alpha}{[n]_{p_1,q_1}+\beta}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right), \quad (11)$$

iii. $$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\sigma;x,y)=\frac{[m]_{p_2,q_2}}{[m]_{p_2,q_2}+\beta}y+\frac{\alpha}{[m]_{p_2,q_2}+\beta}\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right), \quad (12)$$

iv. $$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2;x,y)=\frac{[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}q_1x^2$$
$$+(p_1^{n-1}+2\alpha)\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\frac{[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x+\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}, \quad (13)$$

v. $$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\sigma^2;x,y)=\frac{[m]_{p_2,q_2}[m-1]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}q_2y^2$$
$$+(p_2^{m-1}+2\alpha)\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)\frac{[m]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}y+\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)^2\frac{\alpha^2}{\left([m]_{p_2,q_2}+\beta\right)^2}, \quad (14)$$

vi. $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2+\sigma^2;x,y)=\frac{q_1[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x^2+\frac{q_2[m]_{p_2,q_2}[m-1]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}y^2$

$$+(p_1^{n-1}+2\alpha)\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\frac{[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x$$

$$+(p_2^{m-1}+2\alpha)\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)\frac{[m]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}y$$

$$+\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2+\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)^2\frac{\alpha^2}{\left([m]_{p_2,q_2}+\beta\right)^2}\,. \tag{15}$$

Proof i. $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y)=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}N_{n,k}(p_1,q_1;x)\,N_{m,j}(p_2,q_2;y)=1$

is obtained.

ii. $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi;x,y)$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{[n+a]_{p_1,q_1}\left([k]_{p_1,q_1}p_1^{n-k}+\alpha\right)}{[n+b]_{p_1,q_1}\left([n]_{p_1,q_1}+\beta\right)},y\right)N_{n,k}(p_1,q_1;x)\,N_{m,j}(p_2,q_2;y)$$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\left(\frac{[n+b]_{p_1,q_1}}{[n+a]_{p_1,q_1}}\right)^{n-1}$$

$$\times\left[\frac{[n]_{p_1,q_1}x}{[n]_{p_1,q_1}+\beta}p_1^{\frac{n(n-1)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^{n-1}+\frac{\alpha}{[n]_{p_1,q_1}+\beta}p_1^{\frac{n(n-1)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^{n}\right]$$

$$=\frac{[n]_{p_1,q_1}}{[n]_{p_1,q_1}+\beta}x+\frac{\alpha}{[n]_{p_1,q_1}+\beta}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)$$

is completed.

iii. Similarly, $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\sigma;x,y)=\frac{[m]_{p_2,q_2}}{[m]_{p_2,q_2}+\beta}y+\frac{\alpha}{[m]_{p_2,q_2}+\beta}\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)$.

iv. $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2;x,y)$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\frac{1}{p_2^{\frac{m(m-1)}{2}}}\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{[n+a]_{p_1,q_1}^2\left([k]_{p_1,q_1}p_1^{n-k}+\alpha\right)^2}{[n+b]_{p_1,q_1}^2\left([n]_{p_1,q_1}+\beta\right)^2},y\right)N_{n,k}(p_1,q_1;x)N_{m,j}(p_2,q_2;y)$$

$$=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\left(\frac{[n+b]_{p_1,q_1}}{[n+a]_{p_1,q_1}}\right)^{n-2}$$

$$\times\left[\frac{p_1^{2n}[n]_{p_1,q_1}x}{\left([n]_{p_1,q_1}+\beta\right)^2}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p_1,q_1}[k+1]_{p_1,q_1}\,p_1^{\frac{(k+1)(k-4)}{2}}x^k\prod_{s=0}^{n-k-2}\left(p_1^s\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}-q_1^s x\right)\right.$$

$$+\frac{2\alpha p_1^n[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}\sum_{k=0}^{n-1} p_1^{\frac{(k-2)(k+1)}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p_1,q_1} x^{k+1}\prod_{s=0}^{n-k-2}\left(p_1^s\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}-q_1^s x\right)$$

$$\left.+\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2} p_1^{\frac{n(n-1)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^n\right]$$

here, using that $[1+k]_{p_1,q_1}=p_1^k+q_1[k]_{p_1,q_1}$, we can write

$$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2;x,y)=\frac{1}{p_1^{\frac{n(n-1)}{2}}}\left(\frac{[n+b]_{p_1,q_1}}{[n+a]_{p_1,q_1}}\right)^{n-2}$$

$$\times\left[\frac{[n]_{p_1,q_1}x}{\left([n]_{p_1,q_1}+\beta\right)^2} p_1^{\frac{(n-1)(n+2)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^{n-1}\right.$$

$$+\frac{p_1^{2n-3}q_1[n]_{p_1,q_1}[n-1]_{p_1,q_1}x^2}{\left([n]_{p_1,q_1}+\beta\right)^2} p_1^{\frac{(n-2)(n-3)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^{n-2}$$

$$\left.+\frac{2\alpha[n]_{p_1,q_1}x}{\left([n]_{p_1,q_1}+\beta\right)^2} p_1^{\frac{n(n-1)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^{n-1}+\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2} p_1^{\frac{n(n-1)}{2}}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^n\right]$$

$$=\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\frac{p_1^{n-1}[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x+\frac{q_1[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x^2$$

$$+\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\frac{2\alpha[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x+\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}$$

$$=\frac{q_1[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x^2+(p_1^{n-1}+2\alpha)\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\frac{[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x$$

$$+\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2.$$

v. $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\sigma^2;x,y)$ is obtained in a similar way.

vi. On the other hand, for $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2+\sigma^2;x,y)$;

$$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2+\sigma^2;x,y)=\frac{[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}q_1x^2+q_2\frac{[m]_{p_2,q_2}[m-1]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}y^2$$

$$+(p_1^{n-1}+2\alpha)\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\frac{[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}x$$

$$+(p_2^{m-1}+2\alpha)\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)\frac{[m]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}y$$

$$+\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}+\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)^2\frac{\alpha^2}{\left([m]_{p_2,q_2}+\beta\right)^2}$$

can be found easily.

Remark 1 Let $0 < q_{1,n} < p_{1,n} \leq 1,\ 0 < q_{2,m} < p_{2,m} \leq 1$ and

$$\lim_{n\to\infty} p_{1,n} = \lim_{n\to\infty} q_{1,n} = 1\ ,\ \lim_{m\to\infty} p_{2,m} = \lim_{m\to\infty} q_{2,m} = 1. \tag{16}$$

In the next sections, proofs will be made using the following equations.

$$\lim_{n\to\infty} \frac{p_{1,n}^{n-1}}{[n]_{p_{1,n};q_{1,n}}} = \lim_{m\to\infty} \frac{p_{2,m}^{m-1}}{[m]_{p_{2,m};q_{2,m}}} = 0, \tag{17}$$

$$\lim_{n\to\infty} \frac{[n-1]_{p_{1,n};q_{1,n}}}{[n]_{p_{1,n};q_{1,n}}} q_{1,n} = \lim_{m\to\infty} \frac{[m-1]_{p_{2,m};q_{2,m}}}{[m]_{p_{2,m};q_{2,m}}} q_{2,m} = 1. \tag{18}$$

Results and Discussion

In this section, we will calculate moments by showing that our bivariate operators satisfy the approximation theorem.

Let $E_{nm}^2 = \left[0, \frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right] \times \left[0, \frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right]$.

Theorem 1 Let $0 < q_{1,n} < p_{1,n} \leq 1,\ 0 < q_{2,m} < p_{2,m} \leq 1, 0 < a < b$ and $\lim_{n\to\infty} p_{1,n} = \lim_{n\to\infty} q_{1,n} = 1$, $\lim_{m\to\infty} p_{2,m} = \lim_{m\to\infty} q_{2,m} = 1$. Then for every $f \in C(E_{nm}^2)$

$$\lim_{n,m\to\infty} \left\| \tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f; x, y) - f(x,y) \right\|_{C(E_{nm}^2)} = 0. \tag{19}$$

Proof In accordance to Volkov's theorem, since it is easy to show the cases i-iii in Lemma 2, it is sufficient only to show following equality.

$$\lim_{n,m\to\infty} \left\| \tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(\varphi^2 + \sigma^2; x, y) - (x^2 + y^2) \right\|_{C(E_{nm}^2)} = 0. \tag{20}$$

By definition of the norm, we get

$$\max_{(x,y)\in E_{nm}^2} \left| \tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(\varphi^2 + \sigma^2; x, y) - (x^2 + y^2) \right|$$

$$\leq \left| \left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}} \right)^2 \frac{[n]_{p_{1,n},q_{1,n}} \left(p_{1,n}^{n-1} + 2\alpha \right)}{\left([n]_{p_{1,n},q_{1,n}} + \beta \right)^2} \right|$$

$$+ \left| \left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}} \right)^2 \frac{[m]_{p_{2,m},q_{2,m}} \left(p_{2,m}^{m-1} + 2\alpha \right)}{\left([m]_{p_{2,m},q_{2,m}} + \beta \right)^2} \right|$$

$$+ \left| \left[\frac{q_{1,n} [n]_{p_{1,n},q_{1,n}} [n-1]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}} + \beta \right)^2} - 1 \right] \left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}} \right)^2 \right|$$

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$$+\left|\left[\frac{q_{2,m}[m]_{p_{2,m},q_{2,m}}[m-1]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}-1\right]\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\right|$$

$$+\left|\frac{\alpha^2}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\right|+\left|\frac{\alpha^2}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\right|.$$

Here, using the equations

$$[n]_{p_{1,n},q_{1,n}}-p_{1,n}^{n-1}=[n-1]_{p_{1,n},q_{1,n}}q_{1,n} \tag{21}$$

and

$$[m]_{p_{2,m},q_{2,m}}-p_{2,m}^{m-1}=[m-1]_{p_{2,m},q_{2,m}}q_{2,m}\,, \tag{22}$$

we get

$$\lim_{n\to\infty}\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}=0,\ \lim_{n\to\infty}\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}=1,\ \lim_{n\to\infty}\frac{\alpha^2}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}=0, \tag{23}$$

$$\lim_{m\to\infty}\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}=0,\ \lim_{m\to\infty}\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}=1,\ \lim_{m\to\infty}\frac{\alpha^2}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}=0 \tag{24}$$

and

$$\lim_{n\to\infty}\left(1-\frac{[n]_{p_{1,n},q_{1,n}}[n-1]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}q_{1,n}\right)=0, \tag{25}$$

$$\lim_{m\to\infty}\left(1-\frac{[m]_{p_{2,m},q_{2,m}}[m-1]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}q_{2,m}\right)=0\,. \tag{26}$$

Taking into account the derivative for maximum of the above function; we get, $\lim_{n,m\to\infty}\left\|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n})(p_{2,m},q_{2,m})}(\varphi^2+\sigma^2;x,y)-(x^2+y^2)\right\|_{C(E_{nm}^2)}=0,$

then, using definition of sequences and the Volkov theorem, the desired result is obtained.

Lemma 3 For $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$ the following equations are true.

$$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\varphi-x)^2;x,y)=\left(\frac{[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}q_1-\frac{2[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)}+1\right)x^2$$
$$+\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\left(\frac{(p_1^{n-1}+2\alpha)[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}-\frac{2\alpha}{\left([n]_{p_1,q_1}+\beta\right)}\right)x$$
$$+\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2 \tag{27}$$

$$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\sigma-y)^2;x,y)=\left(\frac{[m]_{p_2,q_2}[m-1]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}q_2-\frac{2[m]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)}+1\right)y^2$$
$$+\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)\left(\frac{(p_2^{m-1}+2\alpha)[m]_{p_2,q_2}}{\left([m]_{p_2,q_2}+\beta\right)^2}-\frac{2\alpha}{\left([m]_{p_2,q_2}+\beta\right)}\right)y$$
$$+\left(\frac{[m+a]_{p_2,q_2}}{[m+b]_{p_2,q_2}}\right)^2\frac{\alpha^2}{\left([m]_{p_2,q_2}+\beta\right)^2}. \tag{28}$$

Proof From the definition of operators

$$\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\varphi-x)^2;x,y)=\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi^2;x,y)-2x\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(\varphi;x,y)$$
$$+x^2\,\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y)$$
$$=\left(\frac{q_1[n]_{p_1,q_1}[n-1]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}-\frac{2[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)}+1\right)x^2$$
$$+\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)\left(\frac{(p_1^{n-1}+2\alpha)[n]_{p_1,q_1}}{\left([n]_{p_1,q_1}+\beta\right)^2}-\frac{2\alpha}{\left([n]_{p_1,q_1}+\beta\right)}\right)x$$
$$+\frac{\alpha^2}{\left([n]_{p_1,q_1}+\beta\right)^2}\left(\frac{[n+a]_{p_1,q_1}}{[n+b]_{p_1,q_1}}\right)^2.$$

With a similar method, the desired equality for $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}((\sigma-y)^2;x,y)$ is obtained.

Rates of Convergences

Now, we give some convergence properties of $\tilde{S}_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$ by the following well known definitions of complete and first-second modulus of continuity.

For every $f\in C(E_{nm}^2)$ and $(\varphi,\sigma),(x,y)\in E_{nm}^2$ complete and partial modulus of continuity are defined as following respectively (see for example in [16]).

$$\omega\left(f,\delta_{n,m}\right)=\sup\left\{|f(\varphi,\sigma)-f(x,y)|:\sqrt{(\varphi-x)^2+(\sigma-y)^2}\le\delta_{n,m}\right\} \tag{29}$$

$$\omega^1(f;\delta)=\sup\{|f(x_1,y)-f(x_2,y)|:y\in E_{nm}\ ve\ |x_1-x_2|\le\delta\} \tag{30}$$

$$\omega^2(f;\delta)=\sup\{|f(x,y_1)-f(x,y_2)|:x\in E_{nm}\ ve\ |y_1-y_2|\le\delta\} \tag{31}$$

Theorem 2 For sufficiently large n,m and every $f\in C(E_{nm}^2)$, rate of convergence of operators is examined with following inequality using the modulus of continuity

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right|\le 2\omega\left(f;\delta_{n,m}\right)$$

where

$$\delta_{n,m}=\left[\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left[\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right]\right.$$

$$+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left[\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]\Bigg]^{1/2}. \quad (32)$$

Proof By the definition of complete modulus of continuity, we have

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right|$$

$$\leq \omega(f;\delta_{n,m})\left\{1+\frac{1}{\delta_{n,m}}\left[\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left(\frac{q_{1,n}[n]_{p_{1,n},q_{1,n}}[n-1]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}-\frac{2[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}+1\right)\right.\right.$$

$$+\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left(\frac{\left(p_{1,n}^{n-1}+2\alpha\right)[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}-\frac{2\alpha}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}+\frac{\alpha^2}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right)$$

$$+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left(\frac{q_{2,m}[m]_{p_{2,m},q_{2,m}}[m-1]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}-\frac{2[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}+1\right)$$

$$\left.\left.+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left(\frac{\left(p_{2,m}^{m-1}+2\alpha\right)[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}-\frac{2\alpha}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}+\frac{\alpha^2}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right)\right]^{1/2}\right\}$$

$$\leq\left\{1+\frac{1}{\delta_{n,m}}\left[\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left(\frac{[n]_{p_{1,n},q_{1,n}}\left([n]_{p_{1,n},q_{1,n}}-p_{1,n}^{n-1}\right)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}-\frac{2[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}+1\right)\right.\right.$$

$$+\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left(\frac{\left(p_{1,n}^{n-1}+2\alpha\right)[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}-\frac{2\alpha}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}+\frac{\alpha^2}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right)$$

$$+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left(\frac{[m]_{p_{2,m},q_{2,m}}\left([m]_{p_{2,m},q_{2,m}}-p_{2,m}^{m-1}\right)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}-\frac{2[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}+1\right)$$

$$\left.\left.+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left(\frac{\left(p_{2,m}^{m-1}+2\alpha\right)[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}-\frac{2\alpha}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}+\frac{\alpha^2}{\left([m]_{p_2,q_2}+\beta\right)^2}\right)\right]^{1/2}\right\}$$

$$\times\,\omega(f;\delta_{n,m})$$

$$\leq \omega(f;\delta_{n,m})\left\{1+\frac{1}{\delta_{n,m}}\left[\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2\right.\right.$$

$$+\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2$$

$$\left.\left.+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]^{1/2}\right\}$$

$$\leq \omega\left(f;\delta_{n,m}\right)\left\{1+\frac{1}{\delta_{n,m}}\left[\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left[\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right]\right.\right.$$

$$\left.\left.+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left[\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]\right]^{1/2}\right\}.$$

Using Remark 1 and choosing

$$\delta_{n,m}=\left[\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left[\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right]\right.$$

$$\left.+\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left[\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]\right]^{1/2}$$

we get our desired result.

Theorem 3 For all $f\in C(E_{nm}^2)$, the following inequality holds

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right|\leq 2\left(\omega^1(f;\delta_n)+\omega^2(f;\delta_m)\right) \tag{33}$$

where

$$\delta_n=\sqrt{\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left[\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right]}, \tag{34}$$

$$\delta_m=\sqrt{\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left[\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]}. \tag{35}$$

Proof Using Cauchy-Schwartz inequality, we have

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right|$$

$$\leq \omega^1(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2\left[\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right]\right)^{1/2}\right]$$

$$+\omega^2(f;\delta_m)\left[1+\frac{1}{\delta_m}\left(\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2\left[\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2+\frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]\right)^{1/2}\right]$$

By choosing

$$\delta_n = \sqrt{\left(\frac{[n+a]_{p_{1,n},q_{1,n}}}{[n+b]_{p_{1,n},q_{1,n}}}\right)^2 \left[\left(1-\frac{[n]_{p_{1,n},q_{1,n}}}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)}\right)^2 + \frac{\alpha(\alpha-2\beta)}{\left([n]_{p_{1,n},q_{1,n}}+\beta\right)^2}\right]}$$

$$\delta_m = \sqrt{\left(\frac{[m+a]_{p_{2,m},q_{2,m}}}{[m+b]_{p_{2,m},q_{2,m}}}\right)^2 \left[\left(1-\frac{[m]_{p_{2,m},q_{2,m}}}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)}\right)^2 + \frac{\alpha(\alpha-2\beta)}{\left([m]_{p_{2,m},q_{2,m}}+\beta\right)^2}\right]}$$

the proof is completed.

We recall Lipschitz class $Lip_M(\alpha_1,\alpha_2)$ for the bivariate functions as follows:

Let $\alpha_1,\alpha_2 \in (0,1]$ also $(\varphi,\sigma),(x,y) \in E_{nm}^2$. There exists $M>0$:

$$|f(\varphi,\sigma)-f(x,y)| \le M|\varphi-x|^{\alpha_1}|\sigma-y|^{\alpha_2}, \tag{36}$$

then f is called Lipschitz continuous function. The set of Lipschitz continuous functions is denoted by $Lip_M(\alpha_1,\alpha_2)$.

Theorem 4 Let $(x,y) \in E_{nm}^2$ and $f \in Lip_M(\alpha_1,\alpha_2)$. (δ_n) and (δ_m) are the sequences defined in (34) and (35), then we have next inequalities for operators,

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right| \le M\left(\sqrt{\delta_n}\right)^{\alpha_1}\left(\sqrt{\delta_m}\right)^{\alpha_2}. \tag{37}$$

Proof Let $f \in Lip_M(\alpha_1,\alpha_2)$. Using linearity and positivity of operators

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right|$$
$$\le M\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(|\varphi-x|^{\alpha_1};x)\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(|\sigma-y|^{\alpha_2};y).$$

Taking

$p'=\frac{2}{\alpha_1}$, $q'=\frac{2}{2-\alpha_1}$ and $p''=\frac{2}{\alpha_2}$, $q''=\frac{2}{2-\alpha_2}$

also applying Hölder's inequality, we write

$$\left|\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(f;x,y)-f(x,y)\right|$$
$$\le M\left\{\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(|\varphi-x|^2;x)\right\}^{\alpha_1/2}\left\{\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(1;x)\right\}^{\alpha_1/2}$$
$$\times\left\{\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(|\sigma-y|^2;y)\right\}^{\alpha_2/2}\left\{\tilde{S}_{n,m}^{(p_{1,n},q_{1,n}),(p_{2,m},q_{2,m})}(1;y)\right\}^{\alpha_2/2}$$
$$\le M\left(\sqrt{\delta_n}\right)^{\alpha_1}\left(\sqrt{\delta_m}\right)^{\alpha_2}.$$

So, Theorem is proved.

Conclusions

In this paper, a generalization of (p, q)-Bernstein Stancu operators on certain domain was given. Then, the approximation properties of our operators; for bivariate functions, rate of convergence and using the properties of the Lipschitz class was investigated. Furthermore, important results were obtained with regard Bernstein Stancu type operators means of (p, q)-integers.

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