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# Frenet Frames of Trigonometric Bézier Curves

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Abstract. The geometry of curves and surfaces plays a very important role in computer-aided geometric design (CAGD). The goal of this paper is to construct the Frenet frames of trigonometric Bézier curves in Euclidean 2 and 3−space. Especially, the curvatures of these curves are investigated at the beginning and the ending points.

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**Keywords:** Computer aided geometric design (CAGD), curvature, Frenet frame field, trigonometric Bézier curve, trigonometric polynomial.

## 1. Introduction

Studying the geometry of curves and surfaces is very essential because it has many important applications in numerous different areas. Therefore, various curves and surfaces have been studied by a lot of authors for many years. Recently, due to its different structure and property, Bézier curves have attracted the attention of several researchers. Bézier curves are introduced firstly by Pierre Bézier in 1968. Bézier curves are the most important mathematical representations of curves which are applied to computer graphics and related areas.

In recent years, trigonometric Bézier curves (T-Bézier curve) with different degrees have been studied in the literature. The quadratic trigonometric Bézier curve with single shape parameter which provides more control on the shape of the curve was handled by Bashir et al. [2]. The cubic trigonometric Bézier curves with a shape parameter is analysized by [6]. The cubic trigonometric Bézier curve with two shape parameters was investigated by Xi-An Han et al. [5]. Sharma et al. studied quartic trigonometric Bézier curves and surfaces with shape parameter [3, 9]. Misro et al. examined the quintic trigonometric Bézier curve with two shape parameters [7]. Also, Sun and Ji gave a novel parametric method to design a kitchen product in the residential environment, a kitchen cabinet, by using cubic T-Bézier curves with constraints of geometric continuities [10]. Then, Ammad and Misro get a class of quintic trigonometric Bézier like basis functions, which is an extension of a traditional Bernstein basis and constructed three different types of shape adjustable surfaces such as general surface, swept surface and swung surface according to these basis functions [1]. In our work, we deal with the geometric structure of trigonometric Bézier curves with aid of Frenet frame. The Frenet

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frame of a curve in Euclidean 2 or 3-space has been the standard tool for characterization and analysing geometry of the curve.

The rest part of the paper is given as follows: Section 2 gives some basic notations and definitions for needed throughout the study. In section 3, we present and examine the Serret-Frenet frame and curvature of a cubic planar T-Bézier curve at the end points. Then, in section 4, we characterize Frenet apparatus of a cubic spatial T-Bézier curve at the beginning and the ending points. In the final section, we give some results and talk about our future works.

## 2. Preliminaries

A cubic trigonometric Bézier curve with two shape parameters, i.e., the T-Bézier curve for short with control points $P_j$ is defined as [5]

$$r(t) = \sum_{j=0}^{3} P_j b_j(t), t \in R, \lambda, \mu \in [-2, 1]\,, \tag{2.1}$$

where

$$\begin{aligned}
b_0(t) &= (1 - \sin\frac{\pi}{2}t)^2(1 - \lambda\sin\frac{\pi}{2}t),\\
b_1(t) &= \sin\frac{\pi}{2}t(1 - \sin\frac{\pi}{2}t)(2 + \lambda - \lambda\sin\frac{\pi}{2}t),\\
b_2(t) &= \cos\frac{\pi}{2}t(1 - \cos\frac{\pi}{2}t)(2 + \mu - \mu\cos\frac{\pi}{2}t),\\
b_3(t) &= (1 - \cos\frac{\pi}{2}t)^2(1 - \mu\cos\frac{\pi}{2}t).
\end{aligned}$$

Shape parameters allow us more control on the shape of the curve compared to the ordinary Bézier curve. So, this parameters are crucial tools in constructing curves and surfaces. The presence of shape parameters allow the curve to be more flexible without changing its control points. Furthermore, even if the value of the shape parameters changes, the curve still preserves its geometrical features [7].

Any space curve is studied by assigning at each point a specific frame in differential geometry. In the 3-dimensional spaces a frame field is a set of three unit vector fields and the ratio of these vectors along the curve is usually expressed with regard to the vectors themselves by the famous Frenet formulas [7]. The curvature and the torsion functions of arc length in the Frenet formulas has attention because a curve in 3-dimensional spaces is completely determined (up to Euclidean motions) by these functions. In the 2-dimensional case, we need the following operator for constructing the Frenet frame field.

**Definition 2.1** ( [4])**.** $J : E^2 \to E^2$ is a linear transformation which is defined by the following equation:

$$J(P_1, P_2) = (-P_2, P_1).$$

By using Definition 2.1, the Frenet frame field of a planar curve can be given in Euclidean 2-space.

**Definition 2.2** ( [4])**.** Let $\beta : I \to E^2$ be a non-unit speed planar curve. The Serret-Frenet frame $\{T(t), N(t)\}$ and curvature $\kappa(t)$ of $\beta(t)$ for $\forall t \in I$ are defined by the following equations:

$$\begin{aligned}
T(t) &= \frac{\beta'(t)}{\|\beta'(t)\|},\\
N(t) &= \frac{J\beta'(t)}{\|\beta'(t)\|},\\
\kappa(t) &= \frac{<\beta''(t), J\beta'(t)>}{\|\beta'(t)\|^3}.
\end{aligned} \tag{2.2}$$

In the following definition, we give the Frenet frame field along a space curve $\beta$.

**Definition 2.3** ( [8]). Let $\beta : I \to E^3$ be a non-unit speed planar curve. The Serret-Frenet frame $\{T(t), N(t), B(t)\}$ and curvature $\kappa(t)$ and torsion $\tau$ of $\beta(t)$ for $\forall t \in I$ are defined by the following equations:

$$\begin{aligned} T(t) &= \frac{\beta'(t)}{\|\beta'(t)\|}, \\ N(t) &= B(t) \times T(t), \\ B(t) &= \frac{\beta'(t) \times \beta''(t)}{\|\beta'(t) \times \beta''(t)\|}, \\ \kappa(t) &= \frac{\|\beta'(t) \times \beta''(t)\|}{\|\beta'(t)\|^3}, \\ \tau(t) &= \frac{det(\beta'(t), \beta''(t), \beta'''(t))}{\|\beta'(t) \times \beta''(t)\|^2}. \end{aligned} \tag{2.3}$$

**Theorem 2.4** ( [5]). *The cubic T-Bézier curves (2.1) have the following properties at $r(0) = P_0$:*

$$\begin{aligned} r'(0) &= \frac{\pi}{2}(2+\lambda)(P_1 - P_0), \\ r''(0) &= \frac{\pi^2}{2}[(2\lambda+1)(P_1 - P_0) + (P_2 - P_1)], \\ r'''(0) &= \frac{\pi^3}{8}(5\lambda - 2)(P_1 - P_0). \end{aligned} \tag{2.4}$$

**Theorem 2.5** ( [5]). *The cubic T-Bézier curves (2.1) have the following properties at $r(1) = P_3$:*

$$\begin{aligned} r'(1) &= \frac{\pi}{2}(2+\mu)(P_3 - P_2), \\ r''(1) &= \frac{\pi^2}{2}[(2\mu+1)(P_3 - P_2) + (P_2 - P_1)], \\ r'''(1) &= \frac{\pi^3}{8}(5\mu - 2)(P_3 - P_2). \end{aligned} \tag{2.5}$$

## 3. The Serret-Frenet Frame of a T-Bézier Curve in Euclidean 2-Space

In this section, the Serret-Frenet frame and curvature of a cubic planar T-Bézier curve are given at starting and ending points.

**Theorem 3.1.** *A cubic planar T-Bézier curve with control points $P_0, P_1, P_2, P_3$ has the following Serret-Frenet frame $\{T(t), N(t)\}$ and curvature $\kappa(t)$ defined by (2.1) for $t = 0$ are*

$$\begin{aligned} T(0) &= \frac{\triangle P_0}{\|\triangle P_0\|}, \\ N(0) &= \frac{(P_0^y - P_1^y, P_1^x - P_0^x)}{\sqrt{(P_1^x - P_0^x)^2 + (P_1^y - P_0^y)^2}}, \\ \kappa(0) &= \frac{2[(P_2^x - P_1^x)(P_0^y - P_1^y) + (P_2^y - P_1^y)(P_1^x - P_0^x)]}{\pi(2+\lambda)^2(\sqrt{(P_1^x - P_0^x)^2 + (P_1^y - P_0^y)^2})^3}, \end{aligned}$$

*where $\triangle P_0 = P_1 - P_0 \neq 0$, $\sqrt{(P_1^x - P_0^x)^2 + (P_1^y - P_0^y)^2} \neq 0$ and $\lambda \neq -2$.*

*Proof.* From the equations (2.2) and terminal properties (2.4), the proof of above theorem is clear. □

In the following theorem, we get the invariants of the Serret-Frenet frame of a cubic planar T-Bézier curve at ending point.

**Theorem 3.2.** *A cubic planar T-Bézier curve with control points $P_0, P_1, P_2, P_3$ has the following Serret-Frenet frame $\{T(t), N(t)\}$ and curvature $\kappa(t)$ defined by (2.1) for $t = 1$ are*

$$\begin{aligned} T(1) &= \frac{\triangle P_2}{\|P_2\|}, \\ N(1) &= \frac{(P_2^y - P_3^y, P_3^x - P_2^x)}{\sqrt{(P_3^x - P_2^x)^2 + (P_3^y - P_2^y)^2}}, \\ \kappa(1) &= \frac{2[(P_2^x - P_1^x)(P_2^y - P_3^y) + (P_2^y - P_1^y)(P_3^x - P_2^x)]}{\pi(2+\mu)^2(\sqrt{(P_3^x - P_2^x)^2 + (P_3^y - P_2^y)^2})^3}, \end{aligned}$$

*where* $\triangle P_2 = P_3 - P_2 \neq 0$, $\sqrt{(P_3^x - P_2^x)^2 + (P_3^y - P_2^y)^2} \neq 0$ *and* $\mu \neq -2$.

*Proof.* From the equations (2.2) and terminal properties (2.5), it can be proved. □

## 4. The Serret-Frenet Frame of a T-Bézier Curve in Euclidean 3-Space

In this section, the Serret-Frenet frame and curvature of a cubic spatial T-Bézier curve are given at end points.

**Theorem 4.1.** *A cubic spatial T-Bézier curve with control points $P_0, P_1, P_2, P_3$ has the following Serret-Frenet frame $\{T(t), N(t)\ B(t)\}$, curvature $\kappa(t)$ and torsion $\tau$ for $t = 0$ are*

$$\begin{aligned} T(0) &= \frac{\triangle P_0}{\|\triangle P_0\|}, \\ N(0) &= \frac{\triangle P_0 \times \triangle P_1}{\|\triangle P_0 \times \triangle P_1\|} \times \frac{\triangle P_0}{\|\triangle P_0\|}, \\ B(0) &= \frac{\triangle P_0 \times \triangle P_1}{\|\triangle P_0 \times \triangle P_1\|}, \\ \kappa(0) &= \frac{2\|(\triangle P_0 \times \triangle P_1)\|}{(2+\lambda)^2\|\triangle P_0\|^3}, \\ \tau(0) &= 0, \end{aligned}$$

*where* $\triangle P_1 = P_2 - P_1$, $\triangle P_0 \neq 0$, $\|\triangle P_0 \times \triangle P_1\| \neq 0$ *and* $\lambda \neq 2$ .

*Proof.* From the equations (2.2) and terminal properties (2.4), it can be proved. □

**Theorem 4.2.** *A cubic spatial T-Bézier curve with control points $P_0, P_1, P_2, P_3$ has the following Serret-Frenet frame $\{T(t), N(t)\ B(t)\}$, curvature $\kappa(t)$ and torsion $\tau$ for $t = 1$ are*

$$\begin{aligned} T(1) &= \frac{\triangle P_2}{\|P_2\|}, \\ N(1) &= \frac{\triangle P_2 \times \triangle p_1}{\|\triangle p_2 \times \triangle P_1\|} \times \frac{\triangle P_2}{\|P_2\|}, \\ B(1) &= \frac{\triangle P_2 \times \triangle P_1}{\|\triangle P_2 \times \triangle P_1\|}, \\ \kappa(1) &= \frac{2\|(\triangle P_2 \times \triangle P_1)\|}{(2+\mu)^2\|\triangle P_2\|^3}, \\ \tau(1) &= 0, \end{aligned}$$

*where* $\triangle P_2 = P_3 - P_2$, $\triangle P_2 \neq 0$, $\|\triangle P_2 \times \triangle P_1\| \neq 0$ *and* $\mu \neq 2$ .

*Proof.* From the equations (2.3) and terminal properties (2.5), it can be proved. □

## 5. Conclusion

In this paper, the Frenet frames of trigonometric Bézier curves in Euclidean 2 and $3-$space are investigated at end points. Especially, the curvatures are investigated at beginning and the ending points. We know that the function of torsion of a differentiable curve in Euclidean 3-space measures how far the curve deviates from the osculating plane stretched by the unit tangent and unit normal of the curve. The results are shown that in the 3-dimensional case, T-Bézier curves lies on the osculating plane at these points. In the light of this study, the Frenet frame and curvatures functions along a trigonometric Bezier curve for each t can be studied.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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