Bingöl Üniversitesi Teknik Bilimler Dergisi
Bingol University Journal of Technical Science
Cilt:2, Sayı: 1, Sayfa: 16-24, 2021 *Volume:2, Number: 1, Page: 16-24, 2021*
Araştırma Makalesi *Research Article*

Non-invariant Hypersurfaces of Hyperbolic Sasakian Manifolds

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(Arrival: 01.06.2021, Acceptance: 01.07.2021, Published: 09.07.2021)

Abstract

The object of this paper is to study non-invariant hypersurfaces of hyperbolic Sasakian manifolds equipped with $(f, g, u, v, \lambda) -$ structure. Some properties obeyed by this structure are obtained. The necessary and sufficient conditions also have been obtained for totally umbilical non -invariant hypersurfaces with $(f, g, u, v, \lambda) -$ structure of hyperbolic Sasakian manifolds to be totally geodesic. The second fundamental form of a non-invariant hypersurface of hyperbolic Sasakian manifolds with (f, g, u, v, λ) - structure has been traced under the condition when f is parallel.

Keywords and Phrases: Hyperbolic Sasakian manifold, totally geodesic, totally umbilical.
2000 Mathematics Subject Classification*: 53D05, 53D25, 53D12.*

1. INTRODUCTION

Blair and Ludden [4] studied the hypersurfaces in an almost contact manifolds in 1969. They also proved that there does not exist invariant hypersurface of a contact manifold. In 1970, *S. I. Goldberg et. al* [2] introduced the notion of a non-invariant hypersurfaces of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the (1, 1) structure tensor field *f* defining the almost contact structure is never tangent to the hypersurface and also proved that there always exists a (f, g, u, v, λ) - structure on a non-invariant hypersurface of an almost contact metric manifold. The notion of (f, g, u, v, λ) - structure was given by *Yano and Okumura* [3]. Sinha and Sharma [8] studied the hypersurfaces of an almost paracontact metric manifold with para (f, g, u, v, λ) – structure.

The notion of geodesic plays an important role in the theory of relativity [5]. *Upadhyay and Dubey* [14] studied an almost hypersurfaces contact (f, ξ, η, g) –structure. *R. Prasad* [12] studied the non-invariant hypersurfaces of trans-Sasakian manifolds. *T. Khan* [11] studied the non-invariant hypersurfaces of Nearly Kenmotsu manifold. *Ahmed at. el.* [13] studied the non-invariant hypersurfaces of nearly hyperbolic Sasakian manifold. In the present paper, we study the non-invariant hypersurfaces of hyperbolic Sasakian manifolds.

This paper is organized as follows. In section 2, we give a brief description of hyperbolic Sasakian manifolds. In section 3, introduce the non-invariant hypersurfaces and induced (f, g, u, v, λ) - structure on non-invariant hypersurface *M* getting some equation. Some results of non-invariant hypersurfaces with (f, g, u, v, λ) - structure of hyperbolic Sasakian manifolds. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurfaces with (f, g, u, v, λ) - structure of hyperbolic Sasakian manifolds to be totally geodesic.

2. PRELIMINARIES

Let $\widehat{M}$be a complete real differentiable manifold of dimension $(2n+1)$. Let there exist a tensor field ϕ of type $(1,1)$, a vector field ξ and a $1-$ form η satisfying

$$\phi^2 X = X + \eta(X)\xi \tag{2.1}$$

$$\eta(\phi X) = 0 \tag{2.2}$$

for arbitrary vector fields $X, Y \in TM$. Then $\widehat{M}$ is called a hyperbolic contact manifold ([8], [14]). From the above equation we can easily prove that

$$\phi\xi = 0 \tag{2.3}$$

$$\eta(\xi) = -1 \tag{2.4}$$

Let the hyperbolic contact manifold $\widehat{M}$ be an endowed with a Riemannian metric g such that

$$\Phi(X,Y) = g(\phi X, Y) \tag{2.5}$$

$$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.6}$$

$$g(X,\xi) = \eta(X) \tag{2.7}$$

A hypersurfaces contact structure satisfying the equations (2.1) to (2.6) is said to be a hyperbolic contact metric manifold [14].

A hyperbolic contact metric manifold is said to be a hyperbolic cosymplectic metric manifold if the structure tensor ϕ and the $1-$form η are parallel with respect to a symmetric affine connection $\hat{\nabla}$ on $\widehat{M}$. Since $\phi^2 = I + \eta \otimes \xi$, the vector field ξ is also parallel with respect to ξ, i.e.

$$(\hat{\nabla}_X \phi)Y = 0 \tag{2.5}$$

$$(\hat{\nabla}_X \eta)Y = 0 \tag{2.6}$$

$$\hat{\nabla}_X \xi = 0 \tag{2.7}$$

A hyperbolic contact metric manifold $\widehat{M}$in which

$$-2\Phi = d\eta \tag{2.11}$$

is satisfied, called an almost hyperbolic Sasakian manifold.

An almost hyperbolic Sasakian manifold $\widehat{M}$, for which ξ is Killing vector, i.e.

$$(\hat{\nabla}_X \eta)Y + (\hat{\nabla}_Y \eta)X = 0 \tag{2.12}$$

where $\hat{\nabla}$ is the Riemannian connection, is called a hyperbolic K-contact Riemannian manifold.

In a hyperbolic K-contact Riemannian manifold, the following relation hold

$$\Phi(X,Y) = -(\hat{\nabla}_X\eta)Y = (\hat{\nabla}_Y\eta)X \tag{2.13}$$

A hyperbolic K-contact Riemannian manifold $\widehat{M}$ is called a hyperbolic Sasakian manifold [7], if

$$(\hat{\nabla}_X\phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.14}$$

$$\hat{\nabla}_X\xi = -\phi X \tag{2.15}$$

A hypersurfaces of an almost contact metric manifold $\widehat{M}$is called a non-invariant hypersurfaces, if the transform of a tangent vector of the hypersurfaces under the action of $(1,1)$ tensor field ϕ defining the contact structure is never tangent to the hypersurfaces. Let X be tangent vector on non-invariant hypersurfaces of an almost contact metric manifold $\widehat{M}$, then ϕX is never to tangent of the hypersurfaces.

Let $\widehat{M}$ be a non-invariant hypersurface of an almost contact metric manifold, Now, we define the following:

$$\phi X = fX + u(X)\widehat{N} \tag{2.16}$$

$$\phi\widehat{N} = -U \tag{2.17}$$

$$\xi = V + \lambda\widehat{N}, \quad \lambda = \eta(\widehat{N}) \tag{2.18}$$

$$\eta(X) = v(X) \tag{2.19}$$

where f is $(1,1)$ tensor field, u and v are $1-$form, $\widehat{N}$ is a unit normal to the hypersurface, $X \in TM$ and $u(X) \neq 0$, then we get an induced $(f,g,u,v,\lambda)-$ structure on $\widehat{M}$ satisfying the conditions

$$\begin{cases} f^2 = I + u\otimes U + v\otimes V, \\ u\circ f = \lambda v, v\circ f = -\lambda u, \\ v(V) = -1-\lambda^2, u(V) = 0 = v(U), u(U) = -1-\lambda^2, \\ fV = \lambda U, fU = \lambda V, \\ u(X) = g(X,U), v(X) = g(X,V), \\ g(fX,fY) = -g(X,Y) - u(X)u(Y) - v(X)v(Y) \end{cases} \tag{2.20}$$

Using equation (2.5) and (2.6), we have

$$\Phi(X, fY) - \Phi(fX, Y) = 2g(fX, fY) + v(X)v(Y) \tag{2.21}$$

for all $X, Y \in TM$ & $\lambda = \eta(\widehat{N})$.

The Gauss and Weingarten formulae are given by

$$\widehat{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)\widehat{N} \tag{2.22}$$

$$\widehat{\nabla}_X \widehat{N} = -A_{\widehat{N}} X \tag{2.23}$$

for all $X, Y \in TM$, where $\widehat{\nabla}$ and ∇ are the Riemannian and induced connection on $\widehat{M}$ and M respectively and $\widehat{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$. In this formula σ is the second fundamental form on M related to $A_{\widehat{N}}$ by

$$\sigma(X, Y) = g(A_{\widehat{N}} X, Y) \tag{2.24}$$

for all $X, Y \in TM$.

3. NON-INVARIANT HYPERSURFACES

Lemma 3.1. If M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$ structure of hyperbolic Sasakian manifold $\widehat{M}$, then

$$(\widehat{\nabla}_X \eta)Y + (\widehat{\nabla}_Y \eta)X = (\nabla_X v)Y + (\nabla_Y v)X - 2\lambda\sigma(X, Y) \tag{3.1}$$

$$\widehat{\nabla}_X \xi = \nabla_X V - \lambda A_{\widehat{N}} X + (\sigma(X, V) + X\lambda)\widehat{N}. \tag{3.2}$$

for all $X, Y \in TM$.

Proof. After computations similar to Lemma 3.1[4], lemma follows.

Theorem 3.2. If M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$ structure of hyperbolic Sasakian manifold $\widehat{M}$, then

$$(\nabla_X f)Y = g(X, Y)V + u(Y)A_{\widehat{N}} X - v(Y)X - \sigma(X, Y)U \tag{3.3}$$

$$(\nabla_X u)Y = \lambda g(X, Y) - \sigma(X, fY) \tag{3.4}$$

for all $X, Y \in TM$.

Proof: By covariant differentiation, we know that

$$(\widehat{\nabla}_X \phi)Y = \widehat{\nabla}_X \phi Y - \phi(\widehat{\nabla}_X Y)$$

Using equation (2.16) in (2.22), we have

$$(\widehat{\nabla}_X \phi)\, Y = \widehat{\nabla}_X (fY + u(Y)\widehat{N}) - \phi(\nabla_X Y + \sigma(X,Y)\widehat{N})$$

$$(\widehat{\nabla}_X \phi)\, Y = \widehat{\nabla}_X fY + \widehat{\nabla}_X \big(u(Y)\widehat{N}\big) - \phi \nabla_X Y - \sigma(X,Y)\phi\widehat{N}$$

Using (2.16) and (2.17), we have

$$(\widehat{\nabla}_X \phi)\, Y = \nabla_X fY + \sigma(X, fY)\widehat{N} + u(Y)\big(\widehat{\nabla}_X \widehat{N}\big) + \left(\widehat{\nabla}_X u(Y)\right)\widehat{N} - f\nabla_X Y - u(\nabla_X Y)\widehat{N} \\ + \sigma(X,Y)U$$

Using (2.23), we have

$$(\widehat{\nabla}_X \phi)\, Y = \nabla_X fY - f\nabla_X Y + \sigma(X, fY)\widehat{N} - u(Y)A_{\widehat{N}} X + \sigma(X,Y)U \\ + \big(\nabla_X u(Y) + \sigma(X, u(Y))\widehat{N}\big)\widehat{N} - u(\nabla_X Y)\widehat{N}$$

$$(\widehat{\nabla}_X \phi)\, Y = (\nabla_X f)Y - u(Y)A_{\widehat{N}} X + \sigma(X,Y)U + \sigma(X, fY)\widehat{N} + \big(\nabla_X u(Y) - u(\nabla_X Y)\big)\widehat{N}$$

$$(\widehat{\nabla}_X \phi)\, Y = (\nabla_X f)Y - u(Y)A_{\widehat{N}} X + \sigma(X,Y)U + \sigma(X, fY)\widehat{N} + (\nabla_X u)Y)\widehat{N}$$

$$(\nabla_X f)Y = g(X,Y)V + u(Y)A_{\widehat{N}}X - v(Y)X - \sigma(X,Y)U \tag{3.5}$$

Now, using (2.18) and (2.19) in (2.14), we have

$$\big(\widehat{\nabla}_X \phi\big)Y = g(X,Y)\big(V + \lambda\widehat{N}\big) - v(Y)X$$
$$\big(\widehat{\nabla}_X \phi\big)Y = g(X,Y)V + \lambda g(X,Y)\widehat{N} - v(Y)X \tag{3.6}$$

Comparing (3.5) and (3.6), we have

$$(\nabla_X f)Y - u(Y)A_{\widehat{N}} X + \sigma(X,Y)U + \big((\nabla_X u)Y + \sigma(X, fY)\big)\widehat{N} \\ = g(X,Y)V + \lambda g(X,Y)\widehat{N} - v(Y)X$$

Equating tangential and normal part, we have the required results.

Theorem 3.3. If M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$ structure of hyperbolic Sasakian manifold $\widehat{M}$, then

$$\sigma(X,\xi)U = f^2X + f(\nabla_X \xi) - u(X)U \tag{3.7}$$

$$u(\nabla_X \xi) = -u(fX) \tag{3.8}$$

for all $X, Y \in TM$.

Proof: By covariant differentiation, we know that

$$(\hat{\nabla}_X \phi)\xi = \hat{\nabla}_X \phi\xi - \phi(\hat{\nabla}_X \xi)$$

Using (2.3), we have

$$(\hat{\nabla}_X\phi)\,\xi = -\phi(\hat{\nabla}_X\,\xi) \tag{3.9}$$

Using equation (2.15) in above, we have

$$(\hat{\nabla}_X\phi)\,\xi = -\phi(-\phi X)$$

$$(\hat{\nabla}_X\phi)\xi = \phi(\phi X)$$

Using (2.16) in above, we have

$$(\hat{\nabla}_X\phi)\xi = \phi(fX + u(X)\hat{N})$$

$$(\hat{\nabla}_X\phi)\xi = \phi(fX) + u(X)\phi\hat{N}$$

Using (2.16) and (2.17), we get

$$\begin{aligned}(\hat{\nabla}_X\phi)\xi &= f(fX) + u(fX)\hat{N} - u(X)U \\ (\hat{\nabla}_X\phi)\xi &= f^2X + u(fX)\hat{N} - u(X)\end{aligned} \tag{3.10}$$

Using (2.22) in (3.9), we have

$$(\hat{\nabla}_X\phi)\xi = -\phi(\nabla_X\,\xi) + \sigma(X,\xi)\hat{N}$$

$$(\hat{\nabla}_X\phi)\,\xi = -\phi(\nabla_X\,\xi) - \sigma(X,\xi)\phi\hat{N}$$

Using (2.16) and (2.17), we get

$$(\hat{\nabla}_X\phi)\xi = -f(\nabla_X\,\xi) - u(\nabla_X\,\xi)\hat{N} + \sigma(X,\xi)U \tag{3.11}$$

Comparing (3.10) and (3.11), we have

$$f^2X + u(fX)\hat{N} - u(X)U = -f(\nabla_X\,\xi) - u(\nabla_X\,\xi)\hat{N} + \sigma(X,\xi)U$$

Equating tangential and normal part, we have required results.

Theorem 3.4. If M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of hyperbolic Sasakian manifold $\hat{M}$, then

$$\nabla_X V = \lambda A_{\hat{N}}X - fX, \tag{3.12}$$

$$\sigma(X,V) = -u(X) - X\lambda. \tag{3.13}$$

If M be a totally umbilical non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of hyperbolic Sasakian manifold $\hat{M}$, then it is totally geodesic if and only if,

$$u(X) + X\lambda = 0 \tag{3.14}$$

for all $X, Y \in TM$.

Proof: Using (2.16) in (2.15), we get

$$\hat{\nabla}_X \xi = -fX - u(X)\hat{N} \tag{3.15}$$

Comparing (3.2) and (3.15), we get

$$\nabla_X V - \lambda A_{\hat{N}} X + (\sigma(X, V) + X\lambda)\hat{N} = -fX - u(X)\hat{N}$$

Equating tangential and normal part, we get desired results.

Now, if M is totally umbilical, then $A_{\bar{N}} = \zeta I$, where ζ is Kahlerian metric, then (2.24) reduces as

$$\sigma(X, Y) = g(A_{\hat{N}} X, Y) = g(\zeta X, Y) = \zeta g(X, Y)$$

therefore,

$$\sigma(X, V) = \zeta g(X, V) = \zeta v(X)$$

Using (2.18) and above equation in (3.14), we have

$$\zeta v(X) = -X\lambda - u(X)$$

If M is totally geodesic, i.e. $\zeta = 0$, then from above equation, we have

$$u(X) + X\lambda = 0.$$

Theorem 3.5. If M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$ structure of hyperbolic Sasakian manifold $\hat{M}$ and $U$is parallel, then

$$\lambda X + f(A_{\hat{N}} X) = 0 \tag{3.16}$$

for all $X, Y \in TM$.

Proof: Consider covariant differentiation, we have

$$(\hat{\nabla}_X \phi)\, \hat{N} = \hat{\nabla}_X \phi\, \hat{N} - \phi(\hat{\nabla}_X \hat{N}) \tag{3.17}$$

Using equation (2.16), (2.17), (2.22) and (2.23) in above, we have

$$(\hat{\nabla}_X \phi)\, \hat{N} = \nabla_X \phi\, \hat{N} + h\big(X, \phi\, \hat{N}\big)\hat{N} - f(\hat{\nabla}_X \hat{N}) - u(\hat{\nabla}_X \hat{N})\hat{N}$$

$$\big(\hat{\nabla}_X \phi\big)\hat{N} = -\nabla_X U + f(A_{\hat{N}} X) \tag{3.18}$$

Putting $Y = \hat{N}$ in (2.14), we have

$$(\widehat{\nabla}_X \phi)\widehat{N} = g(X, \widehat{N})\xi - \eta(\widehat{N})X$$

Using (2.18), we have

$$(\widehat{\nabla}_X \phi)\widehat{N} = -\lambda X \tag{3.19}$$

From (3.18) and (3.19), we have

$$-\nabla_X U + f(A_{\widehat{N}} X) = -\lambda X$$

$$\nabla_X U = \lambda X + f(A_{\widehat{N}} X)$$

If U is parallel then, $\nabla_X U = 0$, so from above equation, we have

$$\lambda X + f(A_{\widehat{N}} X) = 0.$$

Theorem 3.6. If M be a non-invariant hypersurface with $(f, g, u, v, \lambda) -$ structure of hyperbolic Sasakian manifold $\widehat{M}$ and f is parallel, then

$$\sigma(X, U) = u(A_{\widehat{N}}\, X) \tag{3.19}$$

$$X\lambda = 0 \tag{3.20}$$

for all $X, Y \in TM$.

Proof. As f is parallel, then from (3.3), we have

$$\sigma(X, Y)U = g(X, Y)V + u(Y)A_{\widehat{N}}\, X - v(Y)X$$

Applying u both sides, we get

$$\sigma(X, Y)u(U) = g(X, Y)u(V) + u(Y)u(A_{\widehat{N}}\, X) - v(Y)u(X)$$

Using (2.20), we have

$$\begin{aligned} (-1 - \lambda^2)\sigma(X, Y) &= 0 + u(Y)u(A_{\widehat{N}}\, X) - v(Y)u(X) \\ -(1 + \lambda^2)\sigma(X, Y) &= u(Y)u(A_{\widehat{N}}\, X) - v(Y)u(X) \end{aligned} \tag{3.21}$$

Replacing $Y = U$, we have

$$\begin{aligned} (1 + \lambda^2)\sigma(X, U) &= u(U)u(A_{\widehat{N}}\, X) - v(U)u(X) \\ -(1 + \lambda^2)\sigma(X, U) &= -(1 + \lambda^2)u(A_{\widehat{N}}\, X) \\ \sigma(X, U) &= u(A_{\widehat{N}}\, X) \end{aligned} \tag{3.22}$$

Now, putting $Y = V$ in (3.21), we get

$$-(1+\lambda^2)\sigma(X,V) = u(V)u(A_{\widetilde{N}}\, X) - v(V)u(X)$$

Using (2.20), we have

$$-(1+\lambda^2)\sigma(X,V) = (1+\lambda^2)u(X)$$
$$\sigma(X,V) = -u(X) \tag{3.23}$$

Comparing (3.13) and (3.23), we get desired result.

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