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Semi-Invariant Submanifolds A Lorentzian Kenmotsu Manifold With Semi-Symmetric Metric Connection

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Abstract

In this paper, semi-invariant submanifolds of a lorentzian Kenmotsu manifold endowed with a semi-symmetric metric connection are studied. Necessary and sufficient conditions are given on a submanifold of a lorentzian Kenmotsu manifold to be semi-invarinat submanifold with the semi-symmetric metric connection. Moreover, the parallel conditions of the distribution on semi-invariant submanifolds of a lorentzian Kenmotsu manifold with the semi-symmetric metric.

Key words: Lorentzian Kenmotsu manifold, semi invariant submanifold, semi symmetric metric connection.

1. INTRODUCTION

To study manifolds with negative curvature, Bishop and O'Neill introduced the notion of warped product as a generalization of Riemannian product [1]. In 1960's and 1970's, when almost contact manifolds were studied as an odd dimensional counterpart of almost complex manifolds, the warped product was used to make examples of almost contact manifolds.

Kenmotsu studied a class of almost contact Riemannian manifold. He showed normal an almost contact Riemannian manifold with [4] but not quasi Sasakian hence not Sasakian.

At the same time, in the year 1969, Takahashi [11] has introduced the Sasakian manifolds with Pseudo-Riemannian metric and prove that one can study the Lorentzian Sasakian structure with an indefinite metric. Furthermore, in 1990, K. L. Duggal [2] has initiated the space time manifolds with contact structure and analyzed the paper of Takahashi. In 1991, Roşça introduced Lorentzian Kenmotsu manifold [7]. Many authors studied on Lorentzian Kenmotsu manifold [8,9].

Semi-invariant submanifolds are studied by some authours (for examples, M. Kobayashi [5], B.B. Sinha, A.K. Srivastava [10] and A.Turgut Vanlı, R. Sarı [12]). In [6] S. A. Nirmala and R.C. Mangala have introduced a semi-symmetric non-metric connection, they studied some properties of the curvature tensor with respect to the semi-symmetric non-metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\overline{M}$ $\cdot$ The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$ otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. In [3], A. Friedmann and J. A.Schouten introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form

$T(X,Y) = \eta(Y)X - \eta(X)Y$

where η is a 1-form. In [13], K. Yano studied some properties of semi-symmetric metric connections.

The paper is organized as follows : In section 2, we give a brief introduction of lorentzian Kenmotsu manifold. We defined a lorentzian Kenmotsu manifold with a semi-symmetric metric connection. In section 3, we give some basic results for semi-invariant submanifolds of lorentzian Kenmotsu manifold with a semi-symmetric metric connection. In last section, we obtained some necessary and sufficient conditions for parallel of certain distributions on semi-invariant submanifolds of lorentzian Kenmotsu manifold with a semi-symmetric metric connection.

2. PRELIMINARIES

Let $\bar{M}$ be a $(2n+1)$-dimensional differentiable manifold, φ is (1,1)-tensor field, η is a 1-form, ξ is a vector field and g is a semi-Riemannian metric on $\bar{M}$. If for all $X, Y \in \Gamma(T\bar{M})$ following conditions are satifeied then $\bar{M}$ called Lorentzian almost para contact metric manifold

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1 \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2}$$

In addition, we have

$$\varphi(\xi) = 0, \ \eta o \varphi = 0, \ \eta(X) = -g(X,\xi). \tag{3}$$

Moreover, a Lorentzian almost contact metric manifold is normal if $[\varphi,\varphi] - 2d\eta \otimes \xi = 0$ where $[\varphi,\varphi]$ is denoting the Nijenhuis tensor field associated to φ. A normal Lorentzian almost contact metric manifold is called Lorentzian contact metric manifold.

Definition 2.1 Let $(\bar{M}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Lorentzian almost contact metric manifold. $\bar{M}$ is said to be a Lorentzian almost Kenmotsu manifold if 1-form η are closed and $d\Phi = -2\eta \wedge \Phi$.

If $\bar{M}$ is also normal then we call $\bar{M}$ is called a Lorentzian Kenmotsu manifold. The following theorem gives us the neccesary and sufficient condition for $\bar{M}$ to be Loretnzian Kenmotsu manifold.

Theorem 2.2 Let $(\bar{M}, \varphi, \xi, \eta, g)$ be a Lorentzian almost contact metric manifold. $\bar{M}$ is a Loretnzian Kenmotsu manifold if and only if

$$(\bar{\nabla}_X \varphi)Y = -g(\varphi X, Y)\xi + \eta(Y)\varphi X \tag{4}$$

for all $X, Y \in \Gamma(T\bar{M})$.

Corollary 2.3 Let $\bar{M}$ be (2n+1)-dimensional a Lorentzian Kenmotsu manifold with structure (φ, ξ, η, g). Then we have

$$\bar{\nabla}_X \xi = -\varphi^2 X \quad (5)$$

for all $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{\Gamma}(\boldsymbol{T\bar{M}})$.

Now, we define a connection $\tilde{\nabla}$ as

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi \quad (6)$$

Theorem 2.4. Let $\bar{\nabla}$ be the Riemannian connection on a Lorentzian Kenmotsu manifold $\bar{M}$. Then the linear connection which is defined as

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi \quad (7)$$

is a semi-symmetric non metric connection on $\bar{M}$.

Proof. Let $\tilde{T}$ be the torsion tensor of $\bar{\nabla}$. Then,

$$\tilde{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$
$$= \eta(Y)X - \eta(X)Y.$$

Moreover we get,

$$(\tilde{\nabla}_X g)(Y,Z) = X[g(Y,Z)] - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z).$$

Theorem 2.5. Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu manifold $\bar{M}$ with semi-symmetric metric connection. Then

$$(\tilde{\nabla}_X \varphi)Y = 0 \quad (8)$$

for all $X, Y \in \Gamma(TM)$.

Corollary 2.6. Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu manifold $\bar{M}$ with semi-symmetric metric connection, then

$$\tilde{\nabla}_X \xi = 0 \quad (9)$$

for all $X \in \Gamma(TM)$.

3. SEMI INVARIANT SUBMANIFOLDS OF LORENTZIAN KENMOTSU MANIFOLD

Definition 3.1 An *(2n+1)*-dimensional Riemannian submanifold *M* of a Lorentzian Kenmotsu manifold $\bar{M}$ is called a semi-invariant submanifold if ξ are tangent to $\bar{M}$ and there exists on *M* a pair of orthogonal distribution $\{D, D^{\perp}\}$ such that

i. $TM = D \oplus D^{\perp} \oplus Sp\{\xi\}$,

ii. The distribution D is invariant under φ, $\varphi D = D$,

iii. The distribution $D^{\perp}$ is anti invariant under φ, that is $\varphi D^{\perp} \subset TM^{\perp}$.

A semi-invariant submanifold M is said to be an invariant (resp. anti-invariant) submanifold if we have $D^{\perp} = \{0\}$ (resp. D={0}). We say that M is a proper semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ with respect to the induced metric g. Then Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla^*{}_X Y - h(X, Y) \tag{10}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{*\perp}_X Y \tag{11}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. $\nabla^{*\perp}$ is the connection in the normal bundle, h is the second fundamental from of $\overline{M}$ and A_V is the Weingarten endomorphism associated with V. The second fundamental form h and the shape operator A related by

$$g(h(X, Y), V) = g(A_V X, Y) \tag{12}$$

We denote by same symbol g both metrices on $\overline{M}$ and M. Let $\widetilde{\nabla}$ be the semi-symmetric metric connection on $\overline{M}$ and ∇ be the induced connection on M with respect to unit normal N. Then,

$$\widetilde{\nabla}_X Y = \nabla_X Y + m(X, Y) \tag{13}$$

where m is a tensor field of type (0,2) on semi-invariant submanifold M. Using (6) and (10) we have,

$$\nabla_X Y + m(X, Y) = \nabla^*{}_X Y + h(X, Y) + \eta(Y)X$$

So equation tangential and normal components from both the sides, we get

$m(X, Y) = h(X, Y)$ and

$$\widetilde{\nabla}_X Y = \nabla^*{}_X Y + \eta(Y)X \tag{14}$$

From (14) and (11)

$$\begin{aligned} \widetilde{\nabla}_X N &= \nabla^*{}_X N + \eta(N)X \\ &= -A_N X + \eta(N)X \\ &= (-A_N + a)X \end{aligned}$$

where is $a = \eta(N)X$ function on M.

Now, Gauss and Weingarten formulas for a semi-invariant submanifolds of a Lorentzian Kenmotsu manifold with a semi-symmetric non-metric connection is

$$\widetilde{\nabla}_X Y = \nabla_X Y - h(X, Y) \tag{15}$$

and

$$\widetilde{\nabla}_X N = (-A_N + a)X + \nabla^{*\perp}_X Y \tag{16}$$

for all $X, Y \in \Gamma(TM)$ $N \in \Gamma(TM^{\perp})$ h second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{17}$$

The projection morphisms of TM to D and $D^{\perp}$ are denoted by P and Q respectively. For any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ we have

$X = PX + QX + \sum_{i=1}^{s} \eta^i(X)\xi_i$ (18)

and

$\varphi V = tV + nV$ (19)

where tV (resp. nV) denotes the tangential (resp. normal) component of φN.

Theorem 3.5. The connection induced on semi-invariant submanifolds of a Lorentzian Kenmotsu manifold with semi-symmetric metric connection is also a semi-symmetric metric connection.

4. GEOMETRY OF DISTRIBUTIONS

Definition 4.1 The invariant (resp.anti-inavariant) distribution on D (resp. $D^{\perp}$) is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ (resp. $\nabla_Z W \in D^{\perp}$) for all $X, Y \in D$ (resp. $Z, W \in \Gamma(D^{\perp})$).

Theorem 4.2 Let M be a semi invariant submanifold of a Lorentzian Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection. Then, the horizontal distribution D is parallel if and only if

$h(X, \varphi Y) = h(\varphi X, Y) = \varphi h(X, Y)$

for all $X, Y \in D$.

Proof. Since every parallel is involutive then the first equality follows immediately. Now since D is parallel, for all $X, Y \in D$,we have

$\nabla_X \varphi Y \in D.$

After some calculations, we get

$th(X, Y) = 0.$

Then D is parallel if and only if $h(X, \varphi Y) = nh(X, Y)$.

On the other hand

$\varphi h(X, Y) = th(X, Y) + nh(X, Y).$

Then we have $\varphi h(X, Y) = nh(X, Y)$, which completes the proof.

Theorem 4.3 Let M be a semi invariant submanifold of a Lorentzian Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection. Then, the distribution $D^{\perp}$ is parallel if and only if

$-A_{\varphi W} Z = g(Z, W)\xi + th(Z, W)$

for all $Z, W \in D^{\perp}$.

Proof. For all $Z, W \in D^{\perp}$ we get,

$-A_{\varphi W} Z - \varphi P \nabla_Z W = g(Z, W)\xi + th(Z, W)$

Then, we have,

$\nabla_Z W \in D^{\perp} \Leftrightarrow P \nabla_Z W = 0,$

which gives our assertion.

Definition 4.4 A semi invariant submanifold is said to be mixed totaly geodesic $h(X,Y)=0$, for any $X \in \Gamma(D)$ and $Z \in D^{\perp}$.

Lemma 5 Let M be a semi invariant submanifold of a Lorentzian Kenmotsu manifold $\overline{M}$ with semi-symmetric metric connection. Then M is mixed totaly geodesic if and only if

$A_V X \in D, \qquad \forall X \in D, V \in \Gamma(TM^{\perp})$

and

$A_V Y \in D^{\perp}, \quad \forall Y \in D^{\perp}, V \in \Gamma(TM^{\perp}).$

Proof. For all $X \in D$, $V \in \Gamma(TM^{\perp})$ and $Y \in D^{\perp}$ we have

$$g(A_V X, Y) = g(h(X,Y), V). \tag{20}$$

Let M be mixed totally geodesic. Then we get

$g(h(X,Y),V)$=0.

On the other hand, using (20) we arrive

$g(A_V X, Y) = 0 \Leftrightarrow A_V X \in D.$

Hence, for all $\forall X \in D, V \in \Gamma(TM^{\perp})$, we obtain

$g(h(X,Y),V) = 0 \Leftrightarrow h(X,Y) = 0 \Leftrightarrow A_V X \in D.$

In a similar way is deduced relation second equation.

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