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# SPECTRAL SINGULARITIES OF AN IMPULSIVE STURM–LIOUVILLE OPERATORS

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Abstract. In this paper, we handle an impulsive Sturm–Liouville equation with complex potential on the semi axis. The objective of this work is to examine some spectral properties of this impulsive Sturm–Liouville equation. By the help of a transfer matrix $B$, we obtain Jost solution of this problem. Furthermore, using Jost solution, we find Green function and resolvent operator of this equation. Finally, we consider two unperturbated impulsive Sturm–Liouville operators. We examine the eigenvalues and spectral singularities of these problems.

## 1. Introduction

The modeling of most of the problems encountered in the fields of mathematics, physics, mechanics and engineering in daily life is done with boundary value or initial value problems in applied mathematics and spectral analysis. Operator theory is used to solve these problems in spectral theory. First, many physicists and mathematicians studied the spectral theory of differential operators. The Sturm–Liouville operator, which is the equivalent of the one dimensional Schrödinger operator, has gained a wide place in the literature. Let us shortly give information about the literature of spectral theory of Sturm–Liouville operator. Spectral analysis of the nonself-adjoint Schrödinger operator was first investigated by Naimark in 1960 [20]. He proved that the spectrum of this operator consists of eigenvalues, continuous spectrum and spectral singularities. Furthermore, he discovered that the spectral singularities are poles of the resolvent operator's kernel on the continuous spectrum but not the eigenvalues of the operator. Kemp extended the results obtained by

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Naimark to differential operators defined on the whole real axis [16]. Gasymov also extended these results to three-dimensional Schrödinger operators [12]. Then, Schwartz investigated the spectral singularities of a certain class of abstract linear operators in Hilbert space and proved that self-adjoint operators have no spectral singularity [23]. Furthermore, these equations were studied under different initial and boundary conditions by Pavlov, Guseinov and Bairamov et al. [7, 9, 10, 14, 22].

On the other hand, in some processes, instant changes are encountered due to external factors. These are short term sudden changes and can be neglected compared to the whole process. Ordinary differential equations are not sufficient to model these processes. For this reason, impulsive differential equations are used to explain these processes mathematically. Unlike the Schrödinger equation, differential equations with impulsive conditions do not have a long history in the literature. Impulsive differential systems were first studied by Myshkis and Mil'man [18]. After, these equations were investigated by Bainov, Simenov and Lakshmikantham [3, 4]. Recently, many authors have examined impulsive differential equations in detail, because impulsive differential equations have been used in many scientific phenomena such as heart beat, population dynamics, atomic physics, mathematical economics, ecology, engineering, medicine and so forth [13, 15, 19]. Bairamov et al, Yardimci and Erdal investigated scattering analysis and spectral theory of different kinds of impulsive Sturm–Liouville equations [2, 5, 6, 8, 11, 24]. Different from these studies, in this paper, we consider the Sturm–Liouville equation with complex valued potential and impulsive condition in matrix form. Therefore, it creates different perspective.

Let us introduce the Sturm–Liouville operator $T$ in $L_2(0,\infty)$, generated by the equation

$$-\upsilon'' + q(z)\upsilon = \lambda^2\upsilon, \quad z \in [0, z_0) \cup (z_0, \infty) \tag{1}$$

with the boundary condition

$$(\eta_0 + \eta_1\lambda)\,\upsilon'(0) + (\zeta_0 + \zeta_1\lambda)\,\upsilon(0) = 0 \tag{2}$$

and the impulsive condition

$$\begin{bmatrix} \upsilon\left(z_0^+\right) \\ \upsilon'\left(z_0^+\right) \end{bmatrix} = B \begin{bmatrix} \upsilon\left(z_0^-\right) \\ \upsilon'\left(z_0^-\right) \end{bmatrix}, \qquad B = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}, \tag{3}$$

where $\beta_i$, $\eta_j$, $\zeta_j$, $i = 1,2,3,4$, $j = 0,1$ are complex numbers such that $\det B \neq 0$ and $\eta_0\zeta_1 - \eta_1\zeta_0 \neq 0$, $z_0$ is a positive real constant and $q$ is a complex valued function satisfying the following condition

$$\int_0^\infty (1+z)|q(z)|dz < \infty. \tag{4}$$

Throughout the paper, we will show impulsive boundary value problem (1)-(3) by ISBVP, shortly.

This paper is organized as follows: This study consists of five chapters including the introduction. In the next Section, we give basic solutions and definitions. Unlike other studies in the literature, we examine the effect of the impulsive condition on the Sturm–Liouville equation with complex potential in Section 3. We find the Jost solution of ISBVP (1)-(3). In Section 4, we obtain the set of eigenvalues and spectral singularities of (1)-(3). Also, we present an asymptotic equation to obtain the properties of eigenvalues. Then, we get the resolvent operator of the Sturm–Liouville operator $T$. Finally, we handle two different problems to apply our main results in Section 5.

## 2. Preliminaries

Let $S\left(z,\lambda^2\right)$ and $C\left(z,\lambda^2\right)$ be the fundamental solutions of (1) in the interval $[0,z_0)$ satisfying the initial conditions

$$S(0,\lambda^2)=0, \quad S'(0,\lambda^2)=1,$$
$$C(0,\lambda^2)=1, \quad C'(0,\lambda^2)=0.$$

It is evident that the solutions $S\left(z,\lambda^2\right)$ and $C\left(z,\lambda^2\right)$ are entire functions of $\lambda$ and

$$W[S(z,\lambda^2),C(z,\lambda^2)]=-1, \qquad \lambda\in\mathbb{C},$$

where $W\left[\upsilon_1,\upsilon_2\right]$ denotes the Wronskian of the solutions $\upsilon_1$ and $\upsilon_2$ of the equation (1). The integral representations of $S\left(z,\lambda^2\right)$ and $C\left(z,\lambda^2\right)$ are well known in the literature as

$$S(z,\lambda^2)=\frac{\sin\lambda z}{\lambda}+\int_0^z Q(z,t)\frac{\sin\lambda t}{\lambda}dt \tag{5}$$

$$C(z,\lambda^2)=\cos\lambda z+\int_0^z R(z,t)\cos\lambda t dt, \tag{6}$$

where $Q(z,t)$ and $R(z,t)$ are expressed in terms of the potential function $q$ [17].

On the other hand, $e(z,\lambda)$ is bounded solution of the equation (1) in the interval $(z_0,\infty)$ fulfilling the following condition

$$\lim_{z\to\infty} e(z,\lambda)e^{-i\lambda z}=1, \quad \lambda\in\overline{\mathbb{C}}_+ := \{\lambda\in\mathbb{C} : \mathrm{Im}\lambda\geq 0\}$$

and it has an integral representation

$$e(z,\lambda)=e^{i\lambda z}+\int_z^\infty K(z,t)e^{i\lambda t}dt, \quad \lambda\in\overline{\mathbb{C}}_+, \tag{7}$$

where $K(z,t)$ is defined by the potential function $q$ [1]. The bounded solution $e(z,\lambda)$ is analytic with respect to $\lambda$ in $\mathbb{C}_+ := \{\lambda\in\mathbb{C} : \mathrm{Im}\lambda > 0\}$ and continuous up to the real axis. Similarly, $e\left(z,-\lambda\right)$ is bounded solution of (1) in $(z_0,\infty)$ satisfying the condition

$$\lim_{z\to\infty} e(z,-\lambda)e^{i\lambda z}=1, \quad \lambda\in\overline{\mathbb{C}}_- := \{\lambda\in\mathbb{C} : \mathrm{Im}\lambda\leq 0\}.$$

It is well known that

$$W\left[e\left(z,\lambda\right),e\left(z,-\lambda\right)\right]=-2i\lambda,\qquad \lambda\in\mathbb{R}\backslash\{0\}.$$

Furthermore, $\breve{e}(z,\lambda)$ is unbounded solution of (1) in $(z_0,\infty)$ subjecting the conditions [21]

$$\lim_{z\to\infty}\breve{e}(z,\lambda)e^{i\lambda z}=1,\qquad \lim_{z\to\infty}\breve{e}'(z,\lambda)e^{i\lambda z}=-i\lambda,\qquad \lambda\in\overline{\mathbb{C}}_+.$$

It is clear that

$$W\left[e\left(z,\lambda\right),\breve{e}\left(z,\lambda\right)\right]=-2i\lambda,\qquad z\in\left(z_0,\infty\right),\qquad \lambda\in\overline{\mathbb{C}}_+.$$

## 3. Solutions of Impulsive Sturm–Liouville Equation

By the help of linearly independent solutions (1), we will define the general solutions of (1) for $\lambda\in\mathbb{R}\backslash\{0\}$,

$$\Psi_1\left(z,\lambda\right)=\begin{cases}\upsilon_1^-\left(z,\lambda\right)=a^-(\lambda)S\left(z,\lambda^2\right)+b^-(\lambda)C\left(z,\lambda^2\right); & 0\le z<z_0\\ \upsilon_1^+\left(z,\lambda\right)=a^+(\lambda)e\left(z,\lambda\right)+b^+(\lambda)e\left(z,-\lambda\right); & z_0<z<\infty,\end{cases}\tag{8}$$

$$\Psi_2\left(z,\lambda\right)=\begin{cases}\upsilon_2^-\left(z,\lambda\right)=c^-(\lambda)S\left(z,\lambda^2\right)+d^-(\lambda)C\left(z,\lambda^2\right); & 0\le z<z_0\\ \upsilon_2^+\left(z,\lambda\right)=c^+(\lambda)e\left(z,\lambda\right)+d^+(\lambda)e\left(z,-\lambda\right); & z_0<z<\infty\end{cases}\tag{9}$$

and for $\lambda\in\overline{\mathbb{C}}_+\backslash\{0\}$,

$$\Psi_3\left(z,\lambda\right)=\begin{cases}\upsilon_3^-\left(z,\lambda\right)=f^-(\lambda)S\left(z,\lambda^2\right)+h^-(\lambda)C\left(z,\lambda^2\right); & 0\le z<z_0\\ \upsilon_3^+\left(z,\lambda\right)=f^+(\lambda)e\left(z,\lambda\right)+h^+(\lambda)\breve{e}\left(z,\lambda\right); & z_0<z<\infty,\end{cases}\tag{10}$$

respectively.
Using (3) and (8), we obtain

$$\begin{bmatrix}a^+(\lambda)\\ b^+(\lambda)\end{bmatrix}=N\begin{bmatrix}a^-(\lambda)\\ b^-(\lambda)\end{bmatrix},\tag{11}$$

where

$$N:=\begin{bmatrix}N_{11}(\lambda) & N_{12}(\lambda)\\ N_{21}(\lambda) & N_{22}(\lambda)\end{bmatrix}=L^-BM\tag{12}$$

such that

$$L=\begin{bmatrix}e\left(z_0,\lambda\right) & e\left(z_0,-\lambda\right)\\ e'\left(z_0,\lambda\right) & e'\left(z_0,-\lambda\right)\end{bmatrix}$$

and

$$M=\begin{bmatrix}S\left(z_0,\lambda^2\right) & C\left(z_0,\lambda^2\right)\\ S'\left(z_0,\lambda^2\right) & C'\left(z_0,\lambda^2\right)\end{bmatrix}.$$

Since $\det L=-2i\lambda$, in accordance with (12), we find that

$$\begin{aligned}N_{21}(\lambda)=\frac{i}{2\lambda}&[-e'(z_0,\lambda)\left(\beta_1 S\left(z_0,\lambda^2\right)+\beta_2 S'\left(z_0,\lambda^2\right)\right)\\ &+e\left(z_0,\lambda\right)\left(\beta_3 S\left(z_0,\lambda^2\right)+\beta_4 S'\left(z_0,\lambda^2\right)\right)]\end{aligned}\tag{13}$$

$$N_{22}(\lambda) = \frac{i}{2\lambda}[-e'(z_0,\lambda)\left(\beta_1 C\left(z_0,\lambda^2\right) + \beta_2 C'\left(z_0,\lambda^2\right)\right)$$
$$+e\left(z_0,\lambda\right)\left(\beta_3 C\left(z_0,\lambda^2\right) + \beta_4 C'\left(z_0,\lambda^2\right)\right)]. \tag{14}$$

Now, we shall consider the Jost solution of ISBVP (1)-(3) and denote by $E$. Thus, by using (8), the coefficients $a^+(\lambda)$ and $b^+(\lambda)$ turn into 1 and 0, respectively. For $\lambda \in \overline{\mathbb{C}}_+$, we write the following solution of (1)-(3)

$$E(z,\lambda) = \begin{cases} a^-(\lambda)S(z,\lambda^2) + b^-(\lambda)C(z,\lambda^2); & z \in [0,z_0) \\ e(z,\lambda); & z \in (z_0,\infty). \end{cases}$$

By the help of (11) and (12), we easily obtain the coefficients $a^-(\lambda)$ and $b^-(\lambda)$ as follows

$$a^-(\lambda) = \frac{N_{22}(\lambda)}{\det N}, \qquad b^-(\lambda) = -\frac{N_{21}(\lambda)}{\det N}. \tag{15}$$

Let us consider the solution of (1)-(3) satisfying the boundary condition (2) and denote by $F$. By (2) and (9), the following can be easily seen

$$c^-(\lambda) = \left(\zeta_0 + \zeta_1\lambda\right), \qquad d^-(\lambda) = \left(\eta_0 + \eta_1\lambda\right).$$

For $\lambda \in \mathbb{R}\backslash\{0\}$, we will consider the following solution of ISBVP (1)-(3)

$$F(z,\lambda) = \begin{cases} -\left(\zeta_0 + \zeta_1\lambda\right)S(z,\lambda^2) + \left(\eta_0 + \eta_1\lambda\right)C(z,\lambda^2); & z \in [0,z_0) \\ c^+(\lambda)e(z,\lambda) + d^+(\lambda)e(z,-\lambda); & z \in (z_0,\infty). \end{cases}$$

From (3) and (12), we get

$$c^+(\lambda) = -\left(\zeta_0 + \zeta_1\lambda\right)N_{11}(\lambda) + \left(\eta_0 + \eta_1\lambda\right)N_{12}(\lambda) \tag{16}$$

$$d^+(\lambda) = -\left(\zeta_0 + \zeta_1\lambda\right)N_{21}(\lambda) + \left(\eta_0 + \eta_1\lambda\right)N_{22}(\lambda), \tag{17}$$

respectively.

**Lemma 1.** *For $\lambda \in \mathbb{R}\backslash\{0\}$, the Wronskian of the solutions $E\left(z,\lambda\right)$ and $F\left(z,\lambda\right)$ is given by*

$$W[E(z,\lambda),F(z,\lambda)] = \begin{cases} H\left(\lambda\right); & z \in [0,z_0) \\ 2i\lambda H\left(\lambda\right)\det N; & z \in (z_0,\infty), \end{cases}$$

*where*

$$H\left(\lambda\right) := \frac{\left(\zeta_0 + \zeta_1\lambda\right)N_{21}(\lambda) - \left(\eta_0 + \eta_1\lambda\right)N_{22}(\lambda)}{\det N}. \tag{18}$$

*Proof.* Using the definition of Wronskian for $z \in [0,z_0)$, we find

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right] = -\left(\zeta_0 + \zeta_1\lambda\right)b^-(\lambda) - \left(\eta_0 + \eta_1\lambda\right)a^-(\lambda).$$

By using (15), the following can be easily seen

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right] = H(\lambda)$$

for $z \in [0, z_0)$.
Similarly, we write

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right] = -2i\lambda d^{+}(\lambda), \quad z \in (z_0, \infty).$$

By the help of (17), it is clear that

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right] = 2i\lambda H\left(\lambda\right)\det N$$

for $z \in (z_0, \infty)$.
This completes the proof. □

Since $H$ is composed of $e\left(z,\lambda\right), C\left(z,\lambda^2\right)$ and $S\left(z,\lambda^2\right)$, it is analytic in $\mathbb{C}_+$ and continuous up to the real axis.

## 4. Eigenvalues, Spectral Singularities And Resolvent Operator of $T$

From Lemma 1, a necessary and sufficient condition to investigate the eigenvalues and spectral singularities of the Sturm–Liouville operator $T$ with impulsive condition (3) is to investigate the zeros of the function $H$.
The set of eigenvalues and spectral singularities of the operator $T$ are defined as

$$\sigma_d\left(T\right) = \{\mu = \lambda^2 : \operatorname{Im}\lambda > 0 \text{ and } H(\lambda) = 0\},$$

$$\sigma_{ss}\left(T\right) = \{\mu = \lambda^2, \operatorname{Im}\lambda = 0, \lambda \neq 0 \text{ and } H(\lambda) = 0\},$$

respectively.

**Theorem 1.** *Under the condition* (4)*, the function $H$ satisfies the following asymptotic equation*

$$H\left(\lambda\right) = \frac{\mu_1\beta_2\lambda^2}{\det N}\left(\frac{i}{4} + O\left(\frac{1}{\lambda}\right)\right), \quad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \to \infty,$$

*where* $\mu_1\beta_2 \neq 0$.

*Proof.* By means of (5)-(7), we easily find for $\lambda \in \mathbb{C}$

$$S'\left(z_0,\lambda^2\right) = \cos\lambda z_0 + Q\left(z_0,z_0\right)\frac{\sin\lambda z_0}{\lambda} + \int\limits_0^{z_0} Q(z_0,t)\frac{\sin\lambda t}{\lambda}dt \tag{19}$$

$$C'\left(z_0,\lambda^2\right) = -\lambda\sin\lambda z_0 + R\left(z_0,z_0\right)\cos\lambda z_0 + \int\limits_0^{z_0} R(z_0,t)\cos\lambda t dt \tag{20}$$

and for $\lambda \in \overline{\mathbb{C}}_+$

$$e'(z_0,\lambda) = i\lambda e^{i\lambda z_0} - K\left(z_0,z_0\right)e^{i\lambda z_0} + \int\limits_{z_0}^{\infty} K_z(z_0,t)e^{i\lambda t}dt. \tag{21}$$

From (5)-(7), we get

$$\begin{aligned} S\left(z_0,\lambda^2\right) &= \frac{e^{-i\lambda z_0}}{\lambda}\left(\frac{i}{2}+o\left(1\right)\right) \\ C\left(z_0,\lambda^2\right) &= e^{-i\lambda z_0}\left(\frac{1}{2}+o\left(1\right)\right) \\ e\left(z_0,\lambda\right) &= e^{i\lambda z_0}\left(1+o\left(1\right)\right), \end{aligned} \tag{22}$$

where $\lambda \in \overline{\mathbb{C}}_+$ and $|\lambda| \to \infty$.
In a similar way, by using (19)-(21), we obtain for $\lambda \in \overline{\mathbb{C}}_+$ and $|\lambda| \to \infty$

$$\begin{aligned} S'\left(z_0,\lambda^2\right) &= e^{-i\lambda z_0}\left(\frac{1}{2}+O\left(\frac{1}{\lambda}\right)\right) \\ C'\left(z_0,\lambda^2\right) &= \lambda e^{-i\lambda z_0}\left(-\frac{i}{2}+O\left(\frac{1}{\lambda}\right)\right) \\ e'\left(z_0,\lambda\right) &= \lambda e^{i\lambda z_0}\left(i+O\left(\frac{1}{\lambda}\right)\right). \end{aligned} \tag{23}$$

By means of (22) and (23), it is obvious that $H\left(\lambda\right)$ satisfies the asymptotic equation given in Theorem 1. This completes the proof. □

Now, let us define another solution of (1)-(3)

$$G\left(z,\lambda\right) = \begin{cases} -\left(\zeta_0+\zeta_1\lambda\right)S(z,\lambda^2)+\left(\eta_0+\eta_1\lambda\right)C(z,\lambda^2); & z\in[0,z_0) \\ f^+(\lambda)e(z,\lambda)+h^+(\lambda)\breve{e}(z,\lambda); & z\in(z_0,\infty) \end{cases}$$

for all $\lambda \in \overline{\mathbb{C}}_+\backslash\{0\}$. By the help of (3), we obtain that

$$\begin{bmatrix} f^+(\lambda) \\ h^+(\lambda) \end{bmatrix} = V\begin{bmatrix} -\left(\zeta_0+\zeta_1\lambda\right) \\ \left(\eta_0+\eta_1\lambda\right) \end{bmatrix}, \tag{24}$$

where

$$V := \begin{bmatrix} V_{11}(\lambda) & V_{12}(\lambda) \\ V_{21}(\lambda) & V_{22}(\lambda) \end{bmatrix} = U^{-}BM \tag{25}$$

with

$$U = \begin{bmatrix} e\left(z_0,\lambda\right) & \breve{e}\left(z_0,\lambda\right) \\ e'\left(z_0,\lambda\right) & \breve{e}'\left(z_0,\lambda\right) \end{bmatrix}. \tag{26}$$

From (25) and (26), the following equations can be found as

$$\begin{aligned} V_{21}(\lambda) = \frac{i}{2\lambda}[&-e'(z_0,\lambda)\left(\beta_1 S\left(z_0,\lambda^2\right)+\beta_2 S'\left(z_0,\lambda^2\right)\right) \\ &+e\left(z_0,\lambda\right)\left(\beta_3 S\left(z_0,\lambda^2\right)+\beta_4 S'\left(z_0,\lambda^2\right)\right)] \end{aligned} \tag{27}$$

$$\begin{aligned} V_{22}(\lambda) = \frac{i}{2\lambda}[&-e'(z_0,\lambda)\left(\beta_1 C\left(z_0,\lambda^2\right)+\beta_2 C'\left(z_0,\lambda^2\right)\right) \\ &+e\left(z_0,\lambda\right)\left(\beta_3 C\left(z_0,\lambda^2\right)+\beta_4 C'\left(z_0,\lambda^2\right)\right)]. \end{aligned} \tag{28}$$

By using (24), the coefficients $f^{+}(\lambda)$ and $h^{+}(\lambda)$ must be as follows

$$f^{+}(\lambda) = -(\zeta_0 + \zeta_1\lambda) V_{11}(\lambda) + (\eta_0 + \eta_1\lambda) V_{12}(\lambda)$$

$$h^{+}(\lambda) = -(\zeta_0 + \zeta_1\lambda) V_{21}(\lambda) + (\eta_0 + \eta_1\lambda) V_{22}(\lambda).$$

By using (13), (14), (27) and (28), it is clear that

$$N_{21}(\lambda) = V_{21}(\lambda), \qquad N_{22}(\lambda) = V_{22}(\lambda).$$

Therefore, using (18), we rewrite $h^{+}(\lambda)$ as

$$h^{+}(\lambda) = -H(\lambda) \det N. \tag{29}$$

In view of (29), we obtain that

$$W[E(z,\lambda), G(z,\lambda)] = \begin{cases} H(\lambda); & z \in [0, z_0) \\ 2i\lambda H(\lambda) \det N; & z \in (z_0, \infty) \end{cases}$$

for $\lambda \in \overline{\mathbb{C}}_{+} \backslash \{0\}$.

**Theorem 2.** *Assume* (4). *Then the resolvent operator of $T$ is defined by*

$$\mathbb{R}_\lambda \phi = \int_0^\infty R(z,t;\lambda)\phi(t)dt,$$

*where*

$$R(z,t;\lambda) = \begin{cases} \dfrac{E(z,\lambda)G(t,\lambda)}{W[E(z,\lambda), G(z,\lambda)]}; & 0 \le t < z \\ \dfrac{G(z,\lambda)E(t,\lambda)}{W[E(z,\lambda), G(z,\lambda)]}; & z \le t < \infty \end{cases}$$

*is the Green function of* (1)-(3) *for $z \neq z_0$, $t \neq z_0$.*

*Proof.* Let us consider the following equation

$$-\upsilon'' + q(z)\upsilon - \lambda^2\upsilon = \phi(z), \quad z \in [0, z_0) \cup (z_0, \infty). \tag{30}$$

By using the solutions $E(z,\lambda)$ and $G(z,\lambda)$, we write the solution of (30)

$$\phi(z,\lambda) = \theta_1(z)E(z,\lambda) + \theta_2(z)G(z,\lambda).$$

Using the method of variation of parameters, we get the coefficients $\theta_1(z)$ and $\theta_2(z)$ as follows

$$\theta_1(z) = k + \int_0^z \frac{\phi(t)G(t,\lambda)}{W[E(z,\lambda), G(z,\lambda)]} dt$$

$$\theta_2(z) = m + \int_z^\infty \frac{\phi(t)E(t,\lambda)}{W[E(z,\lambda), G(z,\lambda)]} dt,$$

where $k$ and $m$ are real numbers. Let us write the coefficients $\theta_1(z)$ and $\theta_2(z)$ in solution $\phi(z,\lambda)$

$$\phi(z,\lambda) = kE(z,\lambda) + \int_0^z \frac{\phi(t)G(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}dtE(z,\lambda)$$

$$+mG(z,\lambda) + \int_z^\infty \frac{\phi(t)E(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}dtG(z,\lambda).$$

Since the solution $\phi(z,\lambda)$ is in $L_2(0,\infty)$, $m$ becomes zero. In accordance with the boundary condition (2), we also find that $k$ is equal to zero. The proof is completed. □

## 5. Unperturbated Impulsive Operators

In this section, we will investigate two unperturbated impulsive Sturm–Liouville operators.

**Example 1.** *Now, we consider the Sturm–Liouville operator $T_0$ in $L^2[0,\infty)$ corresponding to the following impulsive problem*

$$\begin{gathered} -\upsilon'' = \lambda^2\upsilon, \quad z\in[0,1)\cup(1,\infty) \\ (\eta_0+\eta_1\lambda)\,\upsilon'(0) + (\zeta_0+\zeta_1\lambda)\,\upsilon(0) = 0 \\ \begin{bmatrix}\upsilon(1^+)\\ \upsilon'(1^+)\end{bmatrix} = B\begin{bmatrix}\upsilon(1^-)\\ \upsilon'(1^-)\end{bmatrix}, \qquad B=\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}, \end{gathered} \tag{31}$$

*where $\gamma_1,\gamma_2$, $\eta_j$, $\zeta_j$, $j=0,1$ are complex numbers such that $\eta_0\zeta_1-\eta_1\zeta_0\neq 0$ and $\gamma_1\gamma_2\neq 0$. Since $q=0$, it is evident that*

$$e(z,\lambda) = e^{i\lambda z}, \quad C\left(z,\lambda^2\right) = \cos\lambda z, \quad S\left(z,\lambda^2\right) = \frac{\sin\lambda z}{\lambda}.$$

*By using* (18)*, we write that*

$$\begin{aligned} H(\lambda) = \frac{ie^{i\lambda}}{2\lambda\det N}&[(\eta_0+\eta_1\lambda)(i\gamma_1\lambda\cos\lambda+\gamma_2\lambda\sin\lambda) \\ &+(\zeta_0+\zeta_1\lambda)(\gamma_2\cos\lambda - i\gamma_1\sin\lambda)]. \end{aligned} \tag{32}$$

*To investigate the eigenvalues and spectral singularities of* (31)*, we examine the zeros of $H$. Let us choose $\zeta_1=\eta_0=1$ and $\zeta_0=\eta_1=0$ in* (32) *for the simplicity. Therefore, we rewrite the equation* (32)

$$H(\lambda) = \frac{ie^{i\lambda}}{2\det N}[i\gamma_1\cos\lambda+\gamma_2\sin\lambda-i\gamma_1\sin\lambda+\gamma_2\cos\lambda].$$

*We obtain that*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + \frac{1}{2}\mathrm{Arg}\left(\frac{1+D}{1-D}\right)+k\pi, \quad k\in\mathbb{Z},$$

*where* $D = \dfrac{\gamma_1 - i\gamma_2}{\gamma_2 - i\gamma_1}$. *There appear three cases:*

*Case1: Let* $D = \dfrac{e^{i\theta} - 1}{e^{i\theta} + 1}$ *such that* $\theta \in \mathbb{R}$. *Since* $D = \dfrac{e^{i\theta} - 1}{e^{i\theta} + 1}$, *it is easily seen that* $\operatorname{Arg}\left(\dfrac{1+D}{1-D}\right) = \theta$ *and* $\left|\dfrac{1+D}{1-D}\right| = 1$. *Then, we find that*

$$\lambda_k = \frac{\theta}{2} + k\pi, \quad k \in \mathbb{Z}.$$

*In this case,* $\lambda_k \in \mathbb{R}\backslash\{0\}$, $k \in \mathbb{Z}$ *are the spectral singularities of* (31). *However, there is no eigenvalues.*

*Case2: Let* $\operatorname{Im} D \neq 0$.

*2a: Let* $D$ *be purely imaginary. We obtain that*

$$\lambda_k = \frac{1}{2}\operatorname{Arg}\left(\frac{1+D}{1-D}\right) + k\pi, \quad k \in \mathbb{Z}.$$

*In this case, similar with Case1, the ISBVP* (31) *has no eigenvalues. But it has spectral singularity.*

*2b: Assume* $\operatorname{Re} D < 0$. *We get*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + \frac{1}{2}\operatorname{Arg}\left(\frac{1+D}{1-D}\right) + k\pi, \quad k \in \mathbb{Z}.$$

*Since* $0 < \left|\dfrac{1+D}{1-D}\right| < 1$, $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ *are the eigenvalues of* (31). *However, the operator* $T_0$ *doesn't have any spectral singularity.*

*2c: For* $0 < \operatorname{Re} D$, *the impulsive Sturm–Liouville boundary value problem* (31) *has no eigenvalues and spectral singularity.*

*Case3: Let* $D$ *be a real number.*

*3a: If* $0 < D < 1$, *then* $1 < \left|\dfrac{1+D}{1-D}\right|$. *Similar to the Case2c, the eigenvalues and spectral singularity of* (31) *are not existing.*

*3b: For* $1 < D < \infty$, *we see that*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}.$$

*Since* $\lambda_k \in \mathbb{C}_-$, *there are no eigenvalues and spectral singularity.*

*3c: Assume* $-1 < D < 0$. *We obtain that*

$$\lambda_k = -\frac{i}{2}\ln\left(\frac{1+D}{1-D}\right) + k\pi, \quad k \in \mathbb{Z}.$$

*Since* $0 < \left|\dfrac{1+D}{1-D}\right| < 1$, *there exists eigenvalues but the problem* (31) *has no spectral singularty.*

*3d: For $-\infty < D < 1$, we find that*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z},$$

*where $0 < \left|\frac{1+D}{1-D}\right| < 1$. Hence, $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ are the eigenvalues of $T_0$. But this operator has no spectral singularity.*

**Example 2.** *We investigate the Sturm–Liouville operator $T_1$ in $L^2[0,\infty)$ created by the following ISBVP*

$$\begin{aligned} -\upsilon'' &= \lambda^2 \rho(z)\upsilon, \quad z \in [0,1)\cup(1,\infty) \\ (\eta_0+\eta_1\lambda)\,\upsilon'(0) &+ (\zeta_0+\zeta_1\lambda)\,\upsilon(0) = 0 \\ \begin{bmatrix} \upsilon(1^+) \\ \upsilon'(1^+) \end{bmatrix} &= B\begin{bmatrix} \upsilon(1^-) \\ \upsilon'(1^-) \end{bmatrix}, \qquad B = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}, \end{aligned} \tag{33}$$

*where $\tau_1, \tau_2$, $\eta_j$, $\zeta_j$, $j = 0,1$ are complex numbers, $\eta_0\zeta_1 - \eta_1\zeta_0 \neq 0$, $\tau_1\tau_2 \neq 0$ and $\rho$ is density function defined as*

$$\rho(z) = \begin{cases} \omega^2; & 0 \le z < 1 \\ 1; & 1 < z \end{cases}$$

*such that $\omega \in \mathbb{C}\backslash\{-1,0,1\}$. It is evident that for this example*

$$e(z,\lambda) = e^{i\lambda z}, \quad C\left(z,\lambda^2\right) = \cos(\lambda\omega z), \quad S\left(z,\lambda^2\right) = \frac{\sin(\lambda\omega z)}{\lambda\omega}.$$

*From (18), we obtain that*

$$\begin{aligned} H(\lambda) = \frac{ie^{i\lambda}}{2\lambda \det N}[&(\eta_0+\eta_1\lambda)(i\tau_1\lambda\cos(\lambda\omega)+\tau_2\lambda\omega\sin(\lambda\omega)) \\ &+(\zeta_0+\zeta_1\lambda)(\tau_2\cos(\lambda\omega) - i\tau_1\frac{\sin(\lambda\omega)}{\omega})]. \end{aligned} \tag{34}$$

*For the simplicity on calculations, if we choose $\zeta_1 = \eta_0 = 1$ and $\zeta_0 = \eta_1 = 0$ in (34), we get*

$$H(\lambda) = \frac{ie^{i\lambda}}{2\det N}[i\tau_1\cos(\lambda\omega) + \tau_2\omega\sin(\lambda\omega) - i\tau_1\frac{\sin(\lambda\omega)}{\omega} + \tau_2\cos(\lambda\omega)].$$

*We easily find that*

$$\lambda_k = -\frac{i}{2\omega}\ln\left|\frac{1+P}{1-P}\right| + \frac{1}{2\omega}\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right], \quad k \in \mathbb{Z},$$

*where $P = \frac{\tau_1\omega - i\tau_2\omega}{\tau_2\omega^2 - i\tau_1}$. Let $\omega = m+in$. We can write the real and imaginary parts of $\lambda_k$ as follows*

$$\operatorname{Re}\lambda_k = \frac{1}{2|\omega|^2}\left\{m\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right] - n\ln\left|\frac{1+P}{1-P}\right|\right\}$$

*and*

$$\mathrm{Im}\lambda_k = -\frac{1}{2\left|\omega\right|^2}\left\{ m\ln\left|\frac{1+P}{1-P}\right| + n\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]\right\},$$

*respectively.*

*It is evident that if*

$$\left[m\ln\left|\frac{1+P}{1-P}\right| + n\left(\operatorname{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right)\right]=0$$

*then the operator $T_1$ has spectral singularities, and if*

$$\left[m\ln\left|\frac{1+P}{1-P}\right| + n\left(\operatorname{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right)\right]<0$$

*then the operator $T_1$ has eigenvalues.*

*Case1: If $P=\dfrac{e^{i\theta}-1}{e^{i\theta}+1}$, $\theta\in\mathbb{R}$, then $\operatorname{Arg}\left(\dfrac{1+P}{1-P}\right)=\theta$ and $\left|\dfrac{1+P}{1-P}\right|=1$. We find that*

$$\lambda_k=\frac{\theta+2k\pi}{2\omega},\quad k\in\mathbb{Z}.$$

*1a: Assume $\omega\in\mathbb{R}$. $\lambda_k\in\mathbb{R}\backslash\{0\}$, $k\in\mathbb{Z}$ are spectral singularities of the operator $T_1$ but ISBVP (33) has no eigenvalues.*

*2a: Assume $\omega\in\mathbb{C}$. We write*

$$\mathrm{Im}\lambda_k=-\frac{1}{2\left|\omega\right|^2}\left[n\left(\theta+2k\pi\right)\right],\quad k\in\mathbb{Z}.$$

*If $n\left(\theta+2k\pi\right)<0$, then $\lambda_k\in\mathbb{C}_+$, $k\in\mathbb{Z}$ are eigenvalues of this problem (33). Otherwise, the eigenvalues and spectral singularities of (33) are not existing.*

*Case2: Let $\mathrm{Im}P\neq 0$.*

*2a: Let $P$ be purely imaginary. We write*

$$\lambda_k=\frac{1}{2\omega}\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right],\quad k\in\mathbb{Z}.$$

*For $\omega\in\mathbb{R}$, $\lambda_k\in\mathbb{R}\backslash\{0\}$, $k\in\mathbb{Z}$ are spectral singularities of the operator $T_1$. However, the problem (33) has no eigenvalues.*

*If $\omega\in\mathbb{C}$, then we find that*

$$\mathrm{Im}\lambda_k=-\frac{n}{2\left|\omega\right|^2}\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right],\quad k\in\mathbb{Z}.$$

*It is easily seen that, for $n\left[\operatorname{Arg}\left(\dfrac{1+P}{1-P}\right)+2k\pi\right]<0$, the impulsive Sturm–Liouville boundary value problem (33) has eigenvalues. Otherwise, the problem (33) has no eigenvalues and spectral singularities.*

*2b: Assume $ReA<0$. For $\omega\in\mathbb{R}$, we get*

$$\mathrm{Im}\lambda_k=-\frac{m}{2\left|\omega\right|^2}\left(\ln\left|\frac{1+P}{1-P}\right|\right),\quad k\in\mathbb{Z}.$$

*If $m > 0$, then the operator $T_1$ has eigenvalues. Otherwise, there are no eigenvalues and spectral singularities of* (33)*.*
*For $\omega \in \mathbb{C}$, we obtain that*

$$\mathrm{Im}\lambda_k = -\frac{1}{2|\omega|^2}\left\{m\ln\left|\frac{1+P}{1-P}\right| + n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right]\right\}, \quad k \in \mathbb{Z}.$$

*If $m > 0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right] < 0$, then $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ are eigenvalues of* (33)*. However, if $m < 0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right] > 0$ then the operator $T_1$ has no eigenvalues and spectral singularities.*
*2c: Assume ReP $> 0$. Similar with case2b, if $\omega \in \mathbb{R}$ and $m < 0$, then there exist eigenvalues of* (33)*. However, for $\omega \in \mathbb{R}$ and $m > 0$, there are no eigenvalues and spectral singularities of ISBVP* (33)*.*
*Let $\omega \in \mathbb{C}$, it is clear that if $m < 0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right] < 0$, then the problem* (33) *has eigenvalues. If $m > 0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right] > 0$, then the eigenvalues and spectral singularities of* (33) *are not existing.*
*Case3: Let $P$ be a real number.*
*3a: For $0 < P < 1$, we find that*

$$\lambda_k = -\frac{i}{2\omega}\ln\left(\frac{1+P}{1-P}\right) + \frac{k\pi}{\omega}, \quad k \in \mathbb{Z}.$$

*Assume $\omega \in \mathbb{R}$. If $m < 0$, then the operator $T_1$ has eigenvalues. However, if $m > 0$, then the problem* (33) *does not have any spectral singularity and eigenvalues.*
*Assume $\omega \in \mathbb{C}$. If $m < 0$ and $n(2k\pi) < 0$, then $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ are eigenvalues of ISBVP* (33) *but if $m > 0$ and $n(2k\pi) > 0$, then the operator $T_1$ has no eigenvalues and spectral singularity.*
*3b: For $1 < P < \infty$, it is evident that*

$$\lambda_k = -\frac{i}{2\omega}\ln\left|\frac{1+P}{1-P}\right| + \frac{1}{2\omega}\left[(2k+1)\pi\right], \quad k \in \mathbb{Z}.$$

*Let $\omega \in \mathbb{R}$. Similar with Case3a, for $m < 0$, the problem* (33) *has eigenvalues. Otherwise, the operator $T_1$ has no eigenvalues and spectral singularities.*
*Let $\omega \in \mathbb{C}$. If $m < 0$ and $n(2k+1)\pi < 0$, then there exists eigenvalues of* (33) *but if $m > 0$ and $n(2k+1)\pi > 0$, then there are no eigenvalues and spectral singularities.*
*3c: For $-1 < P < 0$, we obtain*

$$\lambda_k = -\frac{i}{2\omega}\ln\left(\frac{1+P}{1-P}\right) + \frac{k\pi}{2\omega}, \quad k \in \mathbb{Z}.$$

*Assume $\omega \in \mathbb{R}$. The operator $T_1$ has eigenvalues if and only if $m > 0$.*
*Assume $\omega \in \mathbb{C}$. If $m > 0$ and $n(2k\pi) < 0$, then the problem* (33) *has eigenvalues.*

*But if* $m < 0$ *and* $n(2k\pi) > 0$, *then ISBVP* (33) *has no eigenvalues and spectral singularities.*
*3d: For* $-\infty < P < 1$, *we get*

$$\lambda_k = -\frac{i}{2\omega}\ln\left|\frac{1+P}{1-P}\right| + \frac{1}{2\omega}\left[(2k+1)\pi\right], \quad k \in \mathbb{Z}.$$

*Let* $\omega \in \mathbb{R}$. $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ *are eigenvalues of this example* (33) *if and only if* $m > 0$.
*Let* $\omega \in \mathbb{C}$. *If* $m > 0$ *and* $n(2k+1)\pi < 0$, *then there exists eigenvalues of* (33). *If* $m < 0$ *and* $n(2k+1)\pi > 0$, *then the eigenvalues and spectral singularities of* (33) *are not existing.*
*Case4: Let* $\omega$ *be purely imaginary. We easily find that*

$$\mathrm{Im}\lambda_k = -\frac{n}{2|\omega|^2}\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right], \quad k \in \mathbb{Z}.$$

*The operator* $T_1$ *has spectral singularities if and only is*

$$\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi = 0.$$

*The problem* (33) *has eigenvalues if and only if*

$$n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right] < 0.$$

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