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Asymptotics of the solution of the hyperbolic system with a small parameter

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ABSTRACT

Asymptotic study of singularly perturbed differential equations of hyperbolic type has received relatively little attention from researchers. In this paper, the asymptotic solution of the singularly perturbed Cauchy problem for a hyperbolic system is constructed. In addition, the regularization method for singularly perturbed problems of S. A. Lomov is used for the first time for the asymptotic solution of a hyperbolic system. It is shown that this approach greatly simplifies the construction of the asymptotics of the solution for singularly perturbed differential equations of hyperbolic type.

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1. Introduction

As mentioned above, the asymptotic study of singularly perturbed problems of hyperbolic type has been studied not so much. Almost half of the researchers paid attention to the following equation:

$$\varepsilon^2(\partial_t^2 u - \partial_x^2 u) + \varepsilon^k a(x,t)\frac{\partial u}{\partial t} + b(x,t)u = f(x,t) \qquad \text{(A)}$$

The case when k = 0 was studied in [1], where the asymptotics of the solution of the Cauchy problem and the mixed problem was constructed, which contains exponential and parabolic boundary layer functions.

The work [2] is devoted to the construction of the asymptotics of the solution to equation (A) in the case when the small parameter is absent at the time derivative and a(x, t) = 0. It is shown that the solution contains only a hyperbolic boundary layer.

The work [4] is devoted to the study of the boundary value problem for the equation:

$$\varepsilon^2(a(x)\partial_t^2 u - b(x)\partial_x^2 u) + u_x + u_t = 0,$$

here the second derivative of the solution to the degenerate equation has a discontinuity in the characteristic of the boundary emerging from the corner point (0,0). This leads to the appearance of an inner transition layer. The smoothing procedure is used to construct the asymptotics of the solution.

The Darboux problem for equation (A) in the case k = 0 was studied in [6], where the asymptotics of the solution of the boundary layer type was constructed. The works [7] - [12] are devoted to systems of hyperbolic equations.

In [7], a system of two equations with constant coefficients is studied, one of the eigenvalues is equal to zero and has an inner transition layer in the vicinity of the discontinuity of the solution of the degenerate equation.

A feature of the system studied in [8], [9] is the fact that the small parameter $\varepsilon > 0$ in the first equation is contained in the derivative with respect to t, and in the second equation - in the derivative with respect to x. This leads to interesting features of the solution and its asymptotics as $\varepsilon \to 0$.In [9], the same problem with a nonlinear term is studied and an existence theorem for a classical solution is proved. As a result the asymptotic solution of arbitrary order with respect to a small parameter is constructed.

Papers [10] - [14] are devoted to the study of scalar and multidimensional systems of hyperbolic equations in the critical case. In [10] a formal asymptotic representation of the solution of an initial-boundary value problem for a singularly perturbed system of first order partial differential equations with small nonlinearity is constructed. The aim of the work is to construct a formal asymptotic representation of the solution to the initial-boundary value problem.

The work [11] is devoted to the construction of an asymptotic expansion of the solution of the Cauchy problem for a singularly perturbed hyperbolic system of linear equations in the critical case. A feature of the problem is the presence of asymptotic terms described by parabolic equations. The degenerate solution has a discontinuity along the line due to the corner points of the boundaries. In the vicinity of such discontinuity lines, the solution has the character of a transition layer, the form of which can change depending on the initial setting.

In [12] a complete asymptotic expansion of the solution to the initial problem for a singularly perturbed hyperbolic system of equations was constructed and justified.
In the solution of this problem, splash zones arise, in the vicinity of which the asymptotics is described by a parabolic equation.

In [13] the boundary layer method is generalized to systems with an arbitrary number of spatial variables and the behavior of problem solving in the far field is described for large values of x, t.

The work [14] is devoted to the construction of an asymptotic expansion of the solution of the Cauchy problem for one class of hyperbolic weakly nonlinear systems with many spatial variables. A parabolic quasilinear equation is obtained, which describes the behavior of the solution for asymptotically large values of independent variables.

2. Asymptotıc constructıon

2.1. Statement of the problem

In this work, we study the Cauchy problem for a hyperbolic system:

$$\varepsilon(\partial_t u + A(x)\partial_x u) + B(t)u = f(x,t), \qquad x \in \Omega, \\ u|_{t=0} = u^0(x), \tag{1}$$

where $\varepsilon > 0$ – is a small parameter, $\Omega = (0 \le x \le 1) \times (0 < t \le T)$, $A(x)$, $B(t)$, $f(x,t)$- are known.

The problem is solved under the following assumptions: $A(x) \in C^\infty([0,1], \mathbb{C}^{n\times n}), B(t) \in C^\infty([0,T], \mathbb{C}^{n\times n}), f(x,t) \in C^\infty(\bar{\Omega}, C^n)$,

$B(t)\psi_i(t) = \lambda_i(t)\psi_i(t), Re\,\lambda_i(t) \ge 0, \lambda_i(t) \ne \lambda_j(t)$, $\forall\, i \ne j, \quad t \in [0,T], i,j = \overline{1,n}$.

2.2. Regularization of the problem

Under such assumptions for the regularization of the problem (1) we introduce regularizing independent variables using methods described in [15]:

$$\xi_i = \frac{1}{\varepsilon}\varphi_i(t), i = \overline{1,n}, \varphi_i(0) = 0.$$

Here, for now, $\varphi_i(t)$-is an arbitrary function that satisfies $\varphi_i(0) = 0$. Then for the extended function $\tilde{u}(x,t,\xi,\varepsilon)|_{\xi=\varepsilon^{-1}\varphi(t)} \equiv u(x,t,\varepsilon)$ we set the problem as:

$$T_0\tilde{u} \equiv \sum_{i=1}^{n} \varphi_i'(t)\partial_{\xi_i}\tilde{u} + B(t)\tilde{u} \\ = f(x,t) - \varepsilon\partial_t\tilde{u} - \varepsilon A(x)\,\partial_x\tilde{u}, \\ \tilde{u}|_{\xi=0} = c(x,t), c(x,t)|_{t=0} = u^0(x). \tag{2}$$

2.3. Solution of iterative problems

The solution of this extended problem is determined in the form of a series:

$$\tilde{u}(x,t,\xi,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(x,t,\xi). \tag{3}$$

On the basis of (2) for the coefficients of series (3) we obtain the following iterative problems:

$$T_0 u_0 = f(x,t), \\ T_0 u_k = -\partial_t u_{k-1} - A(x)\partial_x u_{k-1}, \\ u|_{\xi=0} = d_0(x,t), d_0(x,0) = u^0(x), \\ u_k|_{\xi=0} = d_k(x,t), d_k(x,0) \\ = 0, \tag{4}$$

here $d_k(x,t) -$ is an arbitrary function.

Where $T_0 \equiv \sum_{i=1}^{n} \varphi_i'(t)\partial_{\xi_i} + B(t),\ k \geq 0.$

We introduce a class of functions in which the iterative problems (4) are solved:

$$U = \left\{ u(x,t,\varepsilon): u = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} c_{ij}(x,t) exp\left(-\xi_j\right) + v_{ii}(x,t)\right]\psi_i(t)\right\},$$

$c_{ij}(x,t), v_i(x,t) \epsilon C^{\infty}(\bar{\Omega}).$

The vector function $\psi_i(t)$ included in U is an eigenvector of the matrix $B(t)$ which corresponds to the eigenvalue $\lambda_i(t) \quad i = \overline{1,n}$.

2.4. Solvability of intermediate tasks

Let the eigenvectors $\psi_i^*(t)$ of the matrix $B^*(t) -$ matrix is conjugate to $B(t)$ which corresponds to the eigenvalue $\bar{\lambda}_i(t)$:

$B^*(t)\psi_i^*(t) = \bar{\lambda}_i(t)\psi_i^*(t), i = \overline{1,n},$

moreover:

$$\left(\psi_i(t), \psi_j^*(t)\right) = \delta_{ij}, \quad i,j = \overline{1,n}.$$

The scalar product of elements $u \epsilon U$ and elements $v \epsilon U^*$ is conjugate space which is defined as:

$\langle u, v\rangle = \sum_{i,j=1}^{n} u_{ij}\overline{v_{ij}} + \sum_{i=1}^{n} u_i \bar{v}_i.$

Theorem 1. Suppose the above assumption and $H(x,t,\xi) \in U$ are satisfied. Then equation (5) is solvable

$$T_0 u(M) = H(M),\ M = (x,t,\xi) \tag{4}$$

in U if and only if $H(x,t,\xi) \perp Ker\, T_0^*$.

Proof. With using the (6)

$$H(x,t,\xi) = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} h_{ij}(x,t)\ \exp\left(-\xi_j\right) + h_i(x,t)\right]\psi_i(t), \tag{5}$$

and substituting the function $u(x,t,\xi) \in U$ into equation (5), with respect to $c_{ij}(x,t)$, $v_i(x,t)$, we get:

$$\sum_{ij=1}^{n}\left\{-\varphi_j{}'(t)\, c_{ij}(x,t) + \lambda_i(t)\, c_{ij}(x,t)\right\}\exp\left(-\xi_j\right)\ \psi_i(t) + \sum_{i=1}^{n} v_i(x,t)\, \lambda_i(t)\, \psi_i(t) = \sum_{i,j=1}^{n} h_{ij}(x,t)\exp\left(-\xi_j\right)\ \psi_i(t) + \sum_{i=1}^{n} h_i(x,t)\ \psi_i(t).$$

We choose a regularizing function in the form:

$\varphi_j'(t) = \lambda_j(t),\ \varphi_j(t) = \int_0^t \lambda_j(s)ds\,,\ j = \overline{1,n},\ ,$

then the previous ratio will be rewritten as:

$$\sum_{i,j=1}^{n}\left(\lambda_i(t) - \lambda_j(t)\right) c_{ij}(x,t)\, \psi_i(t)\exp\left(-\xi_j\right) + \sum_{i=1}^{n} \lambda_i\, v_i(x,t)\, \psi_i(t) = \sum_{i,j=1}^{n} h_{ij}(x,t)\, \psi_i(t)\exp\left(-\xi_j\right) + \sum_{i=1}^{n} h_i(x,t)\, \psi_i(t).$$

Multiplying the obtained relation by $\psi_i^*(t)\exp(-\xi_i) \in Ker\,(T_0^*)$, T_0^* is operator adjoint to T_0. We will make sure that the function $u(x,t,\xi) \in U$ is a solution of the equation (5) for arbitrary $c_{ii}(x,t)$, if and only if

$< H(x,t,\xi),\ \psi^*(t)\exp(-\xi_i) >\equiv 0 \Rightarrow h_{ii}(x,t) \equiv 0.$

The theorem is proved.

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied. Then equation (5) is uniquely solvable in U, which satisfying the following conditions:

$$u|_{t=\xi=0} = u^0(x), \qquad < \partial_t u + A(x)\partial_x u, u_i^* >= 0, \quad i = \overline{1,n}. \tag{6}$$

Proof.

By Theorem 1 the function $u(x,t,\xi) \in U$ is the solution of equation (5) with an accuracy up to arbitrary functions $c_{ii}(x,t)$. Subjecting it to the initial conditions from (7), we define $c_{ii}(x,0)$:

$$c_{ii}(x,0) = (u^0(x), \psi_i^*(0)) - \sum_{j=1\, i\neq j}^{n} c_{ij}(x,0) - v_i(x,0), \quad i = \overline{1,n}. \tag{7}$$

The second condition in (7) is written as:

$$\partial_t c_{ii}(x,t) + \sum_{k=1}^{n} \alpha_{ki}(x,t)\partial_x c_{kk}(x,t) + \sum_{k=1}^{n} \beta_{ki}(t) c_{kk}(x,t) = \rho_i(x,t), \quad 8)$$

where $\alpha_{ki}(x,t) = \left(A(x)\psi_k(t),\ \psi_i^*(t)\right)$, $\beta_{ki}(t) = \left(\psi_k'(t), \psi_k^*(t)\right)$, $\rho_i(x,t)$ – is a known function.

Having solved (8), (9) we uniquely define the function $u(x,t,\xi) \in U$.

The theorem is proved.

Applying Theorems 1 and 2, we successively determine the solutions of iterative problems (4).

2.5 Assessment of the remainder term.

Theorem 3. Suppose that the conditions of Theorems 1 and 2 are satisfied. For the remainder term $R_{\varepsilon n}(x,t,\xi)|_{\xi=\frac{\varphi(x)}{\varepsilon}} = u(x,t,\varepsilon) - \sum_{k=1}^{n} \varepsilon^k u_k(x,t,\xi)|_{\xi=\frac{\varphi(x)}{\varepsilon}}$ for sufficiently small $\varepsilon > 0$ the following estimate is fair:

$$\| R_{\varepsilon n}\left(x,t,\frac{\varphi(x)}{\varepsilon}\right) \| < c\varepsilon^{n+1}.$$

The proof of this theorem is based on the using of the energy integral [16].

3. Conclusion

The regularization method for singularly perturbed problems of S. A. Lomov was first used for the asymptotic solution of a hyperbolic system. By choosing a regularizing function as a solution of the Cauchy problem for a first-order partial differential equation, the process of constructing the asymptotic of a solution of any order of the problem is significantly simplified. This approach makes it possible to easily solve the studied problems in works [7]-[14].

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