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Research Article

Orlicz algebras associated to a Banach function space

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Abstract

In this paper, we study the spaces $\mathfrak{X}^\Phi$ as Banach algebras, where $\mathfrak{X}$ is a quasi-Banach function space and Φ is a Young function, and extend some well-known facts regarding Lebesgue and Orlicz spaces on this new structure. Also, for each $p \geq 1$, we give some necessary condition for the space $\mathfrak{X}^p$ to be a Banach algebra under the pointwise product.

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1. Introduction and preliminaries

This is well-known that $L^1(G,m)$ is a Banach algebra with the convolution product, where G is a locally compact group and m is a left Haar measure on G. Also, for each $1 < p < \infty$, $L^p(G,m)$ is a convolution Banach algebra if and only if G is compact; see [20, 21, 23]. Moreover, for an abelian group G and $1 < p, q < \infty$, $L^p(G,m) * L^q(G,m) \subseteq L^p(G,m)$ if and only if G is compact [19, Corollary 1.4]. In [24] a version of this result for weighted Orlicz spaces on locally compact hypergroups is given. Orlicz spaces are a significant extension of Lebesgue spaces. The weighted version of this structure in the context of locally compact groups and hypergroups was studied in [15, 16]. In [1] we give some necessary and sufficient conditions for a weighted Orlicz space to be a convolution Banach algebra. In particular, for a class of locally compact groups, we prove that if the weighted Orlicz space $L^\Phi_w(G)$ is a convolution Banach algebra, then $\ell^\Phi_w(H)$ is a convolution Banach algebra too, where H is a countable discrete subgroup of G, w is a weight on G and the Young function $\Phi \in \Delta_2$ i.e. there are some constants $c > 0$ and $x_0 \geq 0$ such that for each $x \geq x_0$, $\Phi(2x) \leq c\,\Phi(x)$. In [25], for a compactly generated abelian group G, it is proved that if Φ is a Young function with Δ_2-condition and satisfies some sequence condition, then the Orlicz space $L^\Phi(G)$ is a convolution Banach algebra if and only if $f * g$ exists a.e. for all $f, g \in L^\Phi(G)$. See also [3, 11, 12] for more researches on this topic

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in the context of quotient spaces and hypergroups. Motivated from Lebesgue spaces on locally compact groups, H. Hudzik, A. Kamiska and J. Musielak in [9, Theorem 2] gave the following interesting equivalent conditions for an Orlicz space $L^\Phi(G)$ to be a convolution Banach algebra:

Theorem 1.1. *If G is a locally compact abelian group and Φ is a Young function satisfying Δ_2-condition, then the followings are equivalent:*

(1) *$L^\Phi(G)$ is a Banach algebra under convolution product;*
(2) *$L^\Phi(G) \subseteq L^1(G)$;*
(3) *$\lim_{x\to 0^+} \frac{\Phi(x)}{x} > 0$ or G is compact.*

H. Hudzik also presented some conditions in [8] for an Orlicz space to be a Banach algebra with the pointwise product. Recently in [5] the authors introduced Orlicz spaces $\mathcal{X}^\Phi$ associated to a Banach function space $\mathcal{X}$, where Φ is a Young function; see also [26]. This structure is a huge generalization of the classical Orlicz spaces. Previously, other versions of this structure were given and studied in [10,13,17,18]. In fact, setting $\mathcal{X} := L^1$, we have $(L^1)^\Phi = L^\Phi$, where Φ is a Young function. Motivated by the above background, in this paper we intend to give some conditions under which $\mathcal{X}^\Phi$ is a Banach algebra. In Section 2, we assume that $\mathcal{X}$ is a Banach algebra with a product $\bullet$ and for each $h, k \in \mathcal{X}$ and $v \in \mathcal{S}_\Psi := \{v \in \mathcal{M}_0(\Omega) : \Psi(|v|) \in \mathcal{X}, \|\Psi(|v|)\|_\mathcal{X} \le 1\}$, $(h \bullet k)\, v = h \bullet (kv)$, and show that if $\mathcal{X}^\Phi \subseteq \mathcal{X}$, then $\mathcal{X}^\Phi$ is a Banach algebra with $\bullet$. In particular, if the underlying measure space is finite and $\mathcal{X}$ is a Banach algebra with pointwise product, then $\mathcal{X}^\Phi$ is a Banach algebra with pointwise product. Also, we show that if $\mathcal{X}$ is a solid Banach function space and $\mathcal{X}^\Phi$ is an algebra with a positive product $\diamond$ satisfying the following conditions:

(1) $|f \diamond g| \le |f| \diamond |g|$ for all $f, g \in \mathcal{X}^\Phi$,
(2) for each $f_i, g_i \in \mathcal{X}^\Phi$ $(i = 1, 2)$, if $0 \le f_i \le g_i$, then $f_1 \diamond f_2 \le g_1 \diamond g_2$,

then $\mathcal{X}^\Phi$ is a Banach algebra. In section 3, we focus on pointwise product and among other results we show that for each $p \ge 1$, if $\mathcal{X}^p$ is closed under the pointwise product, then $\inf\{\|\chi_A\|_\mathcal{X} : \mu(A) > 0\} > 0$.

First, we recall some basic notions regarding $\mathcal{X}^\Phi$ spaces.

Throughout, $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space, and the set of all $\mathcal{A}$-measurable complex-valued functions on Ω is denoted by $\mathcal{M}_0(\Omega)$.

Definition 1.2. Let $\mathcal{X}$ be a linear subspace of $\mathcal{M}_0(\Omega)$. If $\mathcal{X}$ equipped with a given norm $\|\cdot\|_\mathcal{X}$ is a complete space, we say that $\mathcal{X}$ is a *Banach function space* or simply *BFS* on Ω. In this case, $\mathcal{X}$ is called *solid* if for each $f \in \mathcal{X}$ and $g \in \mathcal{M}_0(\Omega)$ satisfying $|g| \le |f|$ we have $g \in \mathcal{X}$ and $\|g\|_\mathcal{X} \le \|f\|_\mathcal{X}$. We say that a Banach function space $\mathcal{X}$ satisfies property $(*)$ if for each $A \in \mathcal{A}$ with $\mu(A) < \infty$, we have $\chi_A \in \mathcal{X}$.

In this paper, we assume that Φ is a Young function with a corresponding complementary function Ψ.

Definition 1.3. Let $\mathcal{X}$ be a Banach function space on Ω. The set of all functions $f \in \mathcal{M}_0(\Omega)$ such that for some $\lambda > 0$, $\Phi(\frac{|f|}{\lambda}) \in \mathcal{X}$, is denoted by $\mathcal{X}^\Phi$. For each $f \in \mathcal{X}^\Phi$ we put

$$\|f\|_\Phi := \sup\{\| \, |fv| \, \|_\mathcal{X} : v \in \mathcal{S}_\Psi\}, \tag{1.1}$$

where

$$\mathcal{S}_\Psi := \{v \in \mathcal{M}_0(\Omega) : \Psi(|v|) \in \mathcal{X}, \ \|\Psi(|v|)\|_\mathcal{X} \le 1\}$$

and also,

$$\|f\|_\Phi^\circ := \inf\{\lambda > 0 : \Phi(\frac{|f|}{\lambda}) \in \mathcal{X}, \ \Big\|\Phi(\frac{|f|}{\lambda})\Big\|_\mathcal{X} \le 1\}. \tag{1.2}$$

2. Main results

We recall the following concept from [4].

Definition 2.1. A Banach function space $\mathcal{X}$ on Ω is called a *PCS-space* if for each sequence (f_n) in $\mathcal{X}$, if $f_n \to f$ in $\mathcal{X}$, then there is a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ a.e.

Remark 2.2. Note that any solid BFS on a σ-finite measure space is PCS-space. Since in this paper we have assumed that the measure space is always σ-finite, $\mathcal{X}^\Phi$ would be a PCS-space, where $\mathcal{X}$ is a solid BFS.

Lemma 2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be solid BFS's on Ω. Then, if $\mathcal{X} \subseteq \mathcal{Y}$, then there exists a constant $C > 0$ such that for each $f \in \mathcal{X}$, $\|f\|_{\mathcal{Y}} \leq C\,\|f\|_{\mathcal{X}}$.*

Proof. Let $\mathcal{X} \subseteq \mathcal{Y}$. Consider the inclusion mapping

$$T : \mathcal{X} \to \mathcal{Y}, \qquad T(f) := f, \qquad (f \in \mathcal{X}),$$

and put $\mathrm{Gr} := \{(f, f) : f \in \mathcal{X}\}$. Let $\{(f_n, f_n)\}_n$ be a sequence in Gr and let $(f_n, f_n) \to (f, g)$ in $\mathcal{X} \times \mathcal{Y}$. So, since $\mathcal{X}$ and $\mathcal{Y}$ are PCS spaces, there exists a subsequence $\{f_{n_j}\}_j$ such that $f_{n_j} \to f$ a.e. and $f_{n_j} \to g$ a.e. This implies that $(f, g) \in \mathrm{Gr}$, and thanks to The Closed Graph Theorem, the proof is complete. □

Theorem 2.4. *Let $\mathcal{X}$ be a solid BFS. Suppose that $\mathcal{X}$ is a Banach algebra by a product $\bullet$ such that for each $h \in \mathcal{X}$ and $v \in \mathcal{S}_\Psi$, we have $hv \in \mathcal{X}$ and for each $k \in \mathcal{X}$,*

$$(h \bullet k)\, v = h \bullet (kv). \tag{2.1}$$

If $\mathcal{X}^\Phi \subseteq \mathcal{X}$, then $\mathcal{X}^\Phi$ is a Banach algebra.

Proof. Assume that $\mathcal{X}^\Phi \subseteq \mathcal{X}$. By Lemma 2.3 there is a constant $C > 0$ such that for each $f \in \mathcal{X}^\Phi$, $\|f\|_{\mathcal{X}} \leq C\,\|f\|_\Phi$. Therefore, for each $f, g \in \mathcal{X}^\Phi$ we have

$$\begin{aligned}
\|f \bullet g\|_\Phi &= \sup\{\|(f \bullet g)\, v\|_{\mathcal{X}} : v \in \mathcal{S}_\Psi\} \\
&= \sup\{\|f \bullet (gv)\|_{\mathcal{X}} : v \in \mathcal{S}_\Psi\} \\
&\leq \|f\|_{\mathcal{X}} \sup\{\|gv\|_{\mathcal{X}} : v \in \mathcal{S}_\Psi\} \\
&= \|f\|_{\mathcal{X}}\, \|g\|_\Phi \\
&\leq C\, \|f\|_\Phi\, \|g\|_\Phi.
\end{aligned}$$

□

Corollary 2.5. *Let $\mathcal{X}$ be a solid BFS on Ω, and $\mu(\Omega) < \infty$. If $\mathcal{X}$ is a Banach algebra with a product $\bullet$ satisfying the relation* (2.1)*, then $\mathcal{X}^\Phi$ is a Banach algebra with $\bullet$ too. In particular, if $\mathcal{X}$ is a Banach algebra with pointwise product, then $\mathcal{X}^\Phi$ is a Banach algebra with pointwise product.*

Proof. Just note that by [5, Proposition 4.4], the assumption $\mu(\Omega) < \infty$ implies that $\mathcal{X}^\Phi \subseteq \mathcal{X}$. Now, the conclusion follows from Theorem 2.4 and [14, Proposition 2.2 (i)]. □

In the following result we give some sufficient condition for $\mathcal{X}^\Phi \subseteq \mathcal{X}$. The main idea for the proof of this theorem comes from [9, Theorem 2].

Theorem 2.6. *Let $\mathcal{X}$ be a solid BFS on Ω. If $\lim_{x\to 0^+} \frac{\Phi(x)}{x} > 0$ or $\mathbf{1} \in \mathcal{X}$, then $\mathcal{X}^\Phi \subseteq \mathcal{X}$.*

Proof. Suppose that $f \in \mathcal{X}^\Phi$. Then, there exists some $\lambda > 0$ such that

$$\Phi(\frac{|f|}{\lambda}) \in \mathcal{X}. \tag{2.2}$$

Case 1. Assume that $\lim_{x\to 0^+} \frac{\Phi(x)}{x} > 0$. Then, since Φ is convex, there exists a constant $c > 0$ such that

$$cx \leq \Phi(x), \qquad (x \geq 0). \tag{2.3}$$

Since $\mathcal{X}$ is solid, by (2.3) and (2.2) we have $f \in \mathcal{X}$.
Case 2. Assume that $\mathbf{1} \in \mathcal{X}$. By convexity of Φ, there are $x_0 > 0$ and $k > 0$ such that

$$kx \leq \Phi(x), \qquad (x \geq x_0). \tag{2.4}$$

Put

$$A := \{x \in \Omega : \frac{1}{\lambda}|f(x)| \leq x_0\}.$$

Since $\mathcal{X}$ is solid, we have $f\chi_A \in \mathcal{X}$ because

$$\frac{1}{\lambda}|f|\,\chi_A \leq x_0\,\chi_A \leq x_0\mathbf{1} \in \mathcal{X}.$$

Also, by relations (2.4) and (2.2) we have

$$\frac{k\,|f|}{\lambda}\,\chi_{A^c} \leq \Phi(\frac{|f|}{\lambda}) \in \mathcal{X}.$$

This implies that $f\,\chi_{A^c} \in \mathcal{X}$, so

$$f = f\,\chi_A + f\,\chi_{A^c} \in \mathcal{X},$$

and the proof is complete. □

The following result covers the convolution Banach algebras $L^\Phi(G)$, where G is a locally compact group; see [9, Theorem 2]. Let $\alpha : \Omega \times \Omega \to \Omega$ be a measurable function. Then, for each $g \in \mathcal{X}^\Phi$ and $v \in \mathcal{S}_\Psi$, we define

$$T_\alpha(g, v)(x) := g(x)\,(v \circ \alpha)(x, \cdot), \qquad (x \in \Omega).$$

Theorem 2.7. *Let $\mathcal{X}$ be a solid BFS on Ω, $\mathcal{X}^\Phi \subseteq \mathcal{X}$, and $\alpha : \Omega \times \Omega \to \Omega$ be a measurable function such that for each $g \in \mathcal{X}^\Phi$ and $v \in \mathcal{S}_\Psi$, $(v \circ \alpha)(x, \cdot) \in \mathcal{S}_\Psi$ and for all $x \in \Omega$, $T_\alpha(g, v)(x) \in \mathcal{X}$. If*

$$\|(f \diamond g)\,v\|_{\mathcal{X}} \leq \Big\|f(\cdot)\,\|T_\alpha(g, v)(\cdot)\|_{\mathcal{X}}\Big\|_{\mathcal{X}} \tag{2.5}$$

for all $f, g \in \mathcal{X}^\Phi$ and $v \in \mathcal{S}_\Psi$, then $(\mathcal{X}^\Phi, \diamond)$ is a Banach algebra with the product $\diamond$.

Proof. Let $f, g \in \mathcal{X}^\Phi$. Then,

$$\begin{aligned}
\|f \diamond g\|_\Phi &= \sup\{\|(f \diamond g)\,v\|_{\mathcal{X}} : v \in \mathcal{S}_\Psi\} \\
&\leq \sup\{\Big\|f(\cdot)\,\|T(g, v)(\cdot)\|_{\mathcal{X}}\Big\|_{\mathcal{X}} : v \in \mathcal{S}_\Psi\} \\
&\leq \sup\{\|f\|_{\mathcal{X}}\,\|g\|_\Phi : v \in \mathcal{S}_\Psi\} \\
&= \|f\|_{\mathcal{X}}\,\|g\|_\Phi \\
&\leq C\,\|f\|_\Phi\,\|g\|_\Phi,
\end{aligned}$$

for some constant $C > 0$, thanks to Lemma 2.3. □

The next result would be an extension of the implication (1) ⇒ (3) of [9, Theorem 1].

Theorem 2.8. *Assume that $\mathcal{X}$ is a solid BFS and $\mathcal{X}^\Phi$ is an algebra with a positive product $\diamond$ such that $|f \diamond g| \leq |f| \diamond |g|$ for all $f, g \in \mathcal{X}^\Phi$. Also, assume that for each $f_i, g_i \in \mathcal{X}^\Phi$ $(i = 1, 2)$, $f_1 \diamond f_2 \leq g_1 \diamond g_2$ whenever $0 \leq f_i \leq g_i$. Then, there is a constant $C > 0$ such that for each $f, g \in \mathcal{X}^\Phi$,*

$$\|f \diamond g\|_\Phi \leq C\,\|f\|_\Phi\,\|g\|_\Phi.$$

Proof. In contrast, let $(\mathcal{X}^\Phi, \diamond)$ be an algebra and for each $n \in \mathbb{N}$, there are $f_n, g_n \in X^\Phi$ with $\|f_n\| = \|g_n\| = 1$ such that $\|f_n \diamond g_n\|_\Phi \geq n4^n$. So, for each $n \in \mathbb{N}$, there is a number $0 < \lambda_n < \frac{3}{2}$ such that

$$\Phi\left(\frac{|f_n|}{\lambda_n}\right), \Phi\left(\frac{|g_n|}{\lambda_n}\right) \in \mathcal{X}.$$

For each $m, k \in \mathbb{N}$ with $m > k$ we have

$$\sum_{n=k}^{m} \frac{\|f_n\|_\Phi}{2^n \lambda_n} \leq \sum_{n=k}^{m} \frac{1}{2^n}.$$

This implies that the function $F := \sum_{n=1}^{\infty} \frac{|f_n|}{2^n \lambda_n}$ belongs to $\mathfrak{X}^\Phi$. Similarly, $G := \sum_{n=1}^{\infty} \frac{|g_n|}{2^n \lambda_n} \in \mathfrak{X}^\Phi$. But, for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \|F \diamond G\|_\Phi &\geq \left\| \frac{|f_n| \diamond |g_n|}{4^n \lambda_n^2} \right\|_\Phi \\ &\geq \frac{1}{4^n \lambda_n^2} \, \| \, |f_n \diamond g_n| \, \|_\Phi \\ &= \frac{1}{4^n \lambda_n^2} \, \|f_n \diamond g_n\|_\Phi \\ &\geq \frac{n \, 4^n}{4^n \lambda_n^2} \geq \frac{4n}{9}, \end{aligned}$$

and so, $F \diamond G \notin \mathfrak{X}^\Phi$, a contradiction. □

Sequence conditions. We say that two Young functions Φ_1 and Φ_2 satisfy the *sequence conditions* if there are sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of nonnegative real numbers such that

$$\sum_{n=1}^{\infty} \alpha_n \, \beta_n = \infty, \quad \sum_{n=1}^{\infty} \Phi_1(\alpha_n) < \infty, \quad \sum_{n=1}^{\infty} \Phi_2(\beta_n) < \infty.$$

Theorem 2.9. *Let $\mathfrak{X}$ be a solid BFS on Ω, and two Young functions Φ_1 and Φ_2 satisfy the above sequence conditions. Assume that (V_n) and (W_n) are sequences of elements of $\mathcal{A}$ satisfying the following properties:*

(1) $\sup_n \|\chi_{V_n}\|_{\mathfrak{X}} < \infty$ *and* $\sup_n \|\chi_{W_n}\|_{\mathfrak{X}} < \infty$;
(2) *for each distinct $m, n \in \mathbb{N}$, $V_n \cap V_m = W_n \cap W_m = \varnothing$;*
(3) *there is a set $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for some $M > 0$ and for each $x \in A$ and $n \in \mathbb{N}$, we have $M \leq (\chi_{V_n} \diamond \chi_{W_n})(x)$, where $\diamond$ is a product on characteristic functions;*
(4) *for each distinct $m, n \in \mathbb{N}$, we have $\chi_{V_n} \diamond \chi_{W_m} = 0$ on A.*

Then, $\diamond$ does not have a natural extension on $\mathfrak{X}^{\Phi_1} \times \mathfrak{X}^{\Phi_2}$.

Proof. Define

$$f := \sum_{n=1}^{\infty} \alpha_n \, \chi_{V_n} \quad \text{and} \quad g := \sum_{n=1}^{\infty} \beta_n \, \chi_{W_n}. \tag{2.6}$$

Then, since $\Phi_1(0) = 0$ we have

$$\Phi_1(f) = \sum_{n=1}^{\infty} \Phi_1(\alpha_n \chi_{V_n}) = \sum_{n=1}^{\infty} \Phi_1(\alpha_n) \, \chi_{V_n}.$$

For each $m, k \in \mathbb{N}$ with $k > m$ we have

$$\left\| \sum_{n=m}^{k} \Phi_1(\alpha_n) \, \chi_{V_n} \right\|_X \leq \sum_{n=m}^{k} \Phi_1(\alpha_n) \, \|\chi_{V_n}\|.$$

This implies that $f \in \mathfrak{X}^{\Phi_1}$ since $\mathfrak{X}$ is complete and $\sup_n \|\chi_{V_n}\|_{\mathfrak{X}} < \infty$. Similarly, we have $g \in \mathfrak{X}^{\Phi_2}$. On the other hand, for each $x \in A$,

$$\begin{aligned}(f \diamond g)(x) &= \sum_{n=1}^{\infty} \alpha_n \, (\chi_{V_n} \diamond g)(x) \\ &= \sum_{n=1}^{\infty} \alpha_n \sum_{m=1}^{\infty} \beta_m \, (\chi_{V_n} \diamond \chi_{W_m})(x) \\ &= \sum_{n=1}^{\infty} \alpha_n \, \beta_n (\chi_{V_n} \diamond \chi_{W_n})(x) \\ &\geq M \sum_{n=1}^{\infty} \alpha_n \, \beta_n = \infty.\end{aligned}$$

□

Corollary 2.10. *Let $\mathfrak{X}$ be a solid BFS and $(\mathfrak{X}, \diamond)$ be an algebra such that the hypothesis $(1) - (4)$ in Theorem 2.9 hold. Then, Φ does not satisfy the sequence conditions.*

This statement is a development of the following result which has been proved in [25].

Corollary 2.11. *Let G be a compactly generated locally compact abelian group and Φ be a Young function with $\Phi \in \Delta_2$ satisfying the sequence condition and $\lim_{x \to 0^+} \frac{\Phi(x)}{x} = 0$. Then, $L^\Phi(G)$ is a Banach algebra under the convolution $*$ if and only if for each $f, g \in L^\Phi(G)$, $(f * g)(x)$ exists for almost every $x \in G$.*

3. On $\mathfrak{X}^p$ spaces

For each $p \geq 1$, the function Φ_p defined by $\Phi_p(x) := x^p$ for all $x \geq 0$, is a Young function. We denote $\mathfrak{X}^p := \mathfrak{X}^{\Phi_p}$ and $\| \cdot \|_{\mathfrak{X}^p} := \| \cdot \|^\circ_{\Phi_p}$. In fact, for each $f \in \mathfrak{X}^p$ we have $\|f\|_{\mathfrak{X}^p} = \| \, |f|^p \, \|_{\mathfrak{X}}^{\frac{1}{p}}$. Note that if $\mathfrak{X} := L^1(\mu)$, then $\mathfrak{X}^p = L^p(\mu)$, the classical Lebesgue spaces. In this section specially we study $\mathfrak{X}^p$ as an algebras with the poitwise product. First we give a general result regarding $\mathfrak{X}^\Phi$ spaces which is an extension of [22, Theorem 7 page 64].

Theorem 3.1. *Let $\mathfrak{X}$ be a solid BFS on Ω, and Φ_i be a strictly increasing Young function for $i = 1, 2, 3$ such that*

$$\Phi_1^{-1}(x) \, \Phi_2^{-1}(x) \leq \Phi_3^{-1}(x) \tag{3.1}$$

for all x. Then, $\mathfrak{X}^{\Phi_1} \cdot \mathfrak{X}^{\Phi_2} \subseteq \mathfrak{X}^{\Phi_3}$ and $\|fg\|^\circ_{\Phi_3} \leq \|f\|^\circ_{\Phi_1} \|g\|^\circ_{\Phi_2}$ for all $f \in \mathfrak{X}^{\Phi_1}$ and $g \in \mathfrak{X}^{\Phi_2}$.

Proof. Pick non-zero functions $f \in \mathfrak{X}^{\Phi_1}$ and $g \in \mathfrak{X}^{\Phi_2}$. Then, there are $\lambda_1, \lambda_2 > 0$ such that $\Phi_1(\frac{f}{\lambda_1}) \in \mathfrak{X}$ with $\left\|\Phi_1(\frac{f}{\lambda_1})\right\|_{\mathfrak{X}} \leq 1$ and $\Phi_2(\frac{g}{\lambda_2}) \in \mathfrak{X}$ with $\left\|\Phi_2(\frac{g}{\lambda_2})\right\|_{\mathfrak{X}} \leq 1$. Then, by [22, Lemma 6 page 63], the inequality (3.1) implies that

$$\Phi_3(\frac{|fg|}{2\lambda_1\lambda_2}) \leq \frac{1}{2} \left(\Phi_1(\frac{f}{\lambda_1}) + \Phi_2(\frac{g}{\lambda_2}) \right).$$

Consequently, by solidity of $\mathfrak{X}$ we have $fg \in \mathfrak{X}^{\Phi_3}$ and the requested inequality holds. □

Corollary 3.2. *Let $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r}$. Then,*

(1) $\mathfrak{X}^p \cdot \mathfrak{X}^q \subseteq \mathfrak{X}^r$.
(2) *The equality $\mathfrak{X}^p \cdot \mathfrak{X}^q = \mathfrak{X}^r$ holds if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

Proof. Part (1) directly follows from Theorem 3.1. For part (2), just note that if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then for each $f \in \mathfrak{X}^r$ we have $f^{\frac{p}{p+q}} \in \mathfrak{X}^q$, $f^{\frac{q}{p+q}} \in \mathfrak{X}^p$ and $f = f^{\frac{p}{p+q}} \cdot f^{\frac{q}{p+q}}$. □

A subset $\mathcal{E}$ of a topological vector space $\mathcal{X}$ is called spaceable if there is an infinitely dimensional closed subspace $\mathcal{Y}$ of $\mathcal{X}$ with $\mathcal{Y} \subseteq \mathcal{E} \cup \{0\}$.

For each function f on Ω, $E_f := \{x \in \Omega : f(x) \neq 0\}$.

Definition 3.3. [2, Definition 4.1] Let $\mathcal{B}$ be a topological vector space. We say that a relation $\sim$ on $\mathcal{B}$ has property (D) if the following conditions hold:

(1) If (x_n) is a sequence in $\mathcal{B}$ such that $x_n \sim x_m$ for all distinct index m, n, then for each disjoint finite subsets A, B of $\mathbb{N}$ we have

$$\sum_{n\in A} \alpha_n x_n \sim \sum_{m\in B} \beta_m x_m,$$

where α_n and β_m's are arbitrary scalars.

(2) If a sequence (x_n) converges to x in $\mathcal{B}$ and for some $y \in \mathcal{B}$, $x_n \sim y$ for all $n \in \mathbb{N}$, then $x \sim y$.

Here, we recall the following theorem that is an applicable version of the useful theorem 3.3 in [4].

Theorem 3.4. *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $\sim$ be a relation on $\mathcal{B}$ with property D, and $\mathcal{K}$ be a nonempty cone subset of $\mathcal{B}$. Assume that:*

(1) *there is a constant $c > 0$ such that $\|x + y\| \geq c\,\|x\|$ for all $x, y \in \mathcal{B}$ with $x \sim y$;*

(2) *if $x, y \in \mathcal{B}$ such that $x + y \in \mathcal{K}$ and $x \sim y$ then $x, y \in \mathcal{K}$;*

(3) *there is an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{B} - \mathcal{K}$ such that for each distinct $m, n \in \mathbb{N}$, $x_m \sim x_n$.*

Then, $\mathcal{B} - \mathcal{K}$ is spaceable in $\mathcal{B}$.

Proof. See Theorem 4 in [2]. □

In the following as an application of Theorem 3.4 we generalized [7, Theorem 2] to $\mathcal{X}^p$ spaces. First we recall the following lemma from [2, Lemma 1].

Lemma 3.5. *Let $\mathcal{X}$ be a solid Banach function. Then, the followings are equivalent:*
a) $\inf\{\|\chi_F\|_{\mathcal{X}} : F \in \mathcal{A}$ *and* $\mu(F) > 0\} = 0$.
b) there is a pairwise disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that

$$0 < \|\chi_{A_n}\|_{\mathcal{X}} \leq \frac{1}{2^n}, \quad (n \in \mathbb{N}).$$

In next result, the set $\mathcal{X}^p \times \mathcal{X}^q$ is equipped with the norm $\|\cdot\|$ defined by

$$\|(f, g)\| := \|f\|_{\mathcal{X}^p} + \|g\|_{\mathcal{X}^q}$$

for all $(f, g) \in \mathcal{X}^p \times \mathcal{X}^q$.

Let $\inf\{\|\chi_A\|_{\mathcal{X}} : \mu(A) > 0\} = 0$, and assume that $\frac{1}{p} + \frac{1}{q} > \frac{1}{r}$. Then, $\frac{pq}{p+q} < r$ and so by [26, Theorem 2.1], we have $\mathcal{X}^{\frac{pq}{p+q}} \nsubseteq \mathcal{X}^r$. Pick $h \in \mathcal{X}^{\frac{pq}{p+q}} - \mathcal{X}^r$. Then, $\mathcal{X}^p \cdot \mathcal{X}^q \nsubseteq \mathcal{X}^r$ because $f := h^{\frac{q}{p+q}} \in \mathcal{X}^p$ and $g := h^{\frac{p}{p+q}} \in \mathcal{X}^q$, but $fg = h \notin \mathcal{X}^r$; see [6, Theorem 9] for the case of Lebesgue spaces. In next result we give a generalization of [7, Theorem 2] and we show that the difference $\mathcal{X}^p \cdot \mathcal{X}^q - \mathcal{X}^r$ is enough large.

Theorem 3.6. *Let $\frac{1}{p} + \frac{1}{q} > \frac{1}{r}$ and $\inf\{\|\chi_A\|_{\mathcal{X}} : \mu(A) > 0\} = 0$. Then, the set $\{(f, g) \in \mathcal{X}^p \times \mathcal{X}^q : fg \notin \mathcal{X}^r\}$ is spaceable in $\mathcal{X}^p \times \mathcal{X}^q$.*

Proof. We show that the requirements of Theorem 3.4 hold taking $\mathcal{B} := \mathcal{X}^p \times \mathcal{X}^q$, $\mathcal{K} := \{(f, g) \in \mathcal{X}^p \times \mathcal{X}^q : fg \in \mathcal{X}^r\}$, and the relation $\sim$ defined by

$$(f_1, g_1) \sim (f_2, g_2) \quad \text{if and only if} \quad E_{f_1} \cap E_{f_2} = E_{g_1} \cap E_{g_2} = \varnothing$$

for all $(f_1, g_1), (f_2, g_2) \in \mathcal{B}$. By some calculations one can see that the relation $\sim$ satisfied the property (D) in Definition 3.3. Also, one can easily see that items (1) and (2) in

Theorem 3.4 hold. So, it would be sufficient to prove the condition (3). Since $\inf\{\|\chi_A\|_{\mathcal{X}} : \mu(A) > 0\} = 0$, by Lemma 3.5 there exists a pairwise disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ such that $0 < \|\chi_{A_n}\|_{\mathcal{X}} \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Let $\{N_n\}_{n\in\mathbb{N}}$ is a family of pairwise disjoint infinite subsets of $\mathbb{N}$. For each $n \in \mathbb{N}$ set

$$f_n := \sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{p}}}, \quad g_n := \sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{q}}}.$$

Since

$$\sum_{k\in N_n} \left\| \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{p}}} \right\|_{\mathcal{X}^p} = \Big(\sum_{k\in N_n} \frac{1}{k^2}\Big)^{\frac{1}{p}} < \infty,$$

the series $\sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{p}}}$ is Cauchy in $\mathcal{X}^p$, and so by completeness of $\mathcal{X}^p$ there exists an element $\tilde{f}_n \in \mathcal{X}^p$ such that $\tilde{f}_n = \sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{p}}}$ in $\mathcal{X}^p$. But $\mathcal{X}^p$ is a PCS-space, and so for each $n \in \mathbb{N}$ we have $f_n = \sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{p}}} = \tilde{f}_n$ a.e. Therefore, $f_n \in \mathcal{X}^p$ $(n \in \mathbb{N})$. Similarly we have $g_n \in \mathcal{X}^q$ for all $n \in \mathbb{N}$. On the other hand since A_k's are disjoint,

$$f_n g_n = \sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\frac{1}{p}+\frac{1}{q}}} \notin \mathcal{X}^r, \quad (n \in \mathbb{N}).$$

Indeed, if $f_n g_n \in \mathcal{X}^r$, then $|f_n g_n|^r \in \mathcal{X}$, but for each $s \in N_n$ we have

$$\begin{aligned}\left(\||f_n g_n|^r\|_{\mathcal{X}}\right)^{\frac{1}{r}} &= \Big(\| \sum_{k\in N_n} \frac{\chi_{A_k}}{\left(k^2\|\chi_{A_k}\|_{\mathcal{X}}\right)^{\left(\frac{1}{p}+\frac{1}{q}\right)r}} \|_{\mathcal{X}}\Big)^{\frac{1}{r}} \\ &\geq \frac{\|\chi_{A_s}\|_{\mathcal{X}}^{1-\left(\frac{1}{p}+\frac{1}{q}\right)r}}{s^{2r\left(\frac{1}{p}+\frac{1}{q}\right)}} \\ &\geq \frac{2^{\left[\left(\frac{1}{p}+\frac{1}{q}\right)r-1\right]\cdot s}}{s^{2r\left(\frac{1}{p}+\frac{1}{q}\right)}},\end{aligned}$$

and so $\||f_n g_n|^r\|_{\mathcal{X}} = \infty$, a contradiction. This implies that $\{(f_n, g_n)\}_{n=1}^{\infty}$ is an infinite sequence in $\mathcal{B} - \mathcal{K}$ with $(f_n, g_n) \sim (f_m, g_m)$ for all distinct $m, n \in \mathbb{N}$ because $N_n \cap N_m = \varnothing$. □

Setting $p = q = r$ in the above theorem we conclude the next fact.

Corollary 3.7. *For each $p \geq 1$, if $\mathcal{X}^p$ is closed under the pointwise product, then* $\inf\{\|\chi_A\|_{\mathcal{X}} : \mu(A) > 0\} > 0$.

Remark 3.8. In Corollary 3.7 replacing $\mathcal{X}$ by $\mathcal{X}^{\Phi}$ and setting $p := 1$ we conclude that if $\mathcal{X}^{\Phi}$ is closed under pointwise product, then

$$\inf \left\{ \frac{1}{\Phi^{-1}\left(\frac{1}{\|\chi_A\|_{\mathcal{X}}}\right)} : \mu(A) > 0 \right\} > 0,$$

and then equivalently we have $\inf\{\|\chi_A\|_{\mathcal{X}} : \mu(A) > 0\} > 0$. In particular, if $\inf\{\mu(A) : \mu(A) > 0\} > 0$, then the the Orlicz space L^{Φ} is not a Banach algebra with the pointwise product.

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