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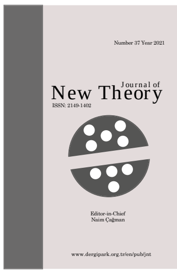

New Exact Solutions of Conformable Time-Fractional Bad and Good Modified Boussinesq Equations

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Abstract − The new exact solutions of the conformable time-fractional Bad and Good modified Boussinesq equations are obtained using the Exp-function method, which is different from previous literature works. These equations play a significant role in mathematical physics, engineering sciences and applied mathematics. Plentiful exact solutions with arbitrary parameters are effectively obtained by the method. The obtained solutions are shown graphically. It is shown that the Exp-function method provides a simpler but more effective mathematical tool for constructing exact solutions of non-linear evolution equations.

Keywords − *Conformable time fractional Bad and Good modified Boussinesq equations, conformable fractional derivative, exact solution, Exp-function method*

Mathematics Subject Classification (2020) – 35R11, 34A08

1. Introduction

Many phenomena in the real world are governed by nonlinear evolution equations (NLEEs). Hence, it is essential to obtain exact solutions to these equations, and many methods have been proposed to get exact solutions. The most known methods are the Exp-function method [1], the tanh-method [2], the homogeneous balance method [3], the trial function method [4], the $(\frac{G'}{G})$ expansion methods [5], the Kudryashov method [6]. The Exp-function method was used by many researchers to solutions of various NLEEs [7-10]. Also, new exact solutions of the nonlinear evolution equations may be obtained using different methods.

Travelling waves arise naturally in many physical systems, usually described by partial differential equations. Solitary waves, also known as 'solitons', are a particular travelling wave class with some special properties. Solitons can usually propagate over large distances without dispersion due to certain nonlinear effects cancelling out dispersive effects. They also have the additional property that they can interact with other solitons such that they emerge following a collision without changing shape, apart from a small phase change.

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Traditional real problems are defined with integer-order nonlinear evolution equations, characterized by theirs. However, the nonlinear evolution equations with integer-order derivatives are ideal classic events, which are not suitable for describing irregular phenomena. On the other hand, Fractional differential equations have become preferable, despite the difficulty in calculations, since they give more real results than standard integer order nonlinear evolution equations. Difficulties in fractional calculus have begun to be overcome thanks to new fractional derivative definitions and theorems made in recent years. The most popular definitions of the fractional derivative can be listed as the conformable derivative [11], the Caputo derivative [12], Riemann-Liouville derivative [12], Atangana-Baleanu derivative [13]. The new trends in exact solution research are to find new exact solutions and to develop new solution mechanisms.

Lately, Khalil et al. defined a limit-based fractional derivative in 2014 [11], named the conformable fractional derivative. The structure of this new definition of fractional derivative is simpler than that of other popular fractional derivatives.

The definition of conformable fractional derivative is given as follow.

Let f: $[0,\infty]\to\mathbb{R}$. Then, the conformable fractional derivative of order α is defined by

$$T_\alpha\big(f(t)\big) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1}$$

for all $t > 0$ and $\alpha \in (0,1]$. If f is α-differentiable in some $(0,\alpha)$, $\alpha > 0$ and $\lim_{t\to 0^+} T_\alpha\big(f(t)\big)$ exists, then

$$(T_\alpha f)(0) = \lim_{t\to 0^+} T_\alpha\big(f(t)\big) \tag{2}$$

In addition to this definition, it is known that $T_\alpha\big(f(t)\big) = \lim_{\varepsilon\to 0}\frac{f^{\lceil\alpha-1\rceil}(t+\varepsilon t^{\lceil\alpha\rceil-\alpha}) - f^{\lceil\alpha-1\rceil}(t)}{\varepsilon}$ such that $\alpha \in (n, n+1]$ and function f is n-th order differentiable at a point for $t > 0$, where $\lceil . \rceil$ ceil function, $\lceil\alpha\rceil$ is the smallest integer no larger than α [11].

This paper applies the Exp-function method to new exact solutions of fractional Bad and Good modified Boussinesq equations with conformable derivative. The rest of this paper is organized as follows: Some useful properties of the conformable fractional derivative and mechanism of the Exp-function method are given in Section 2. Exact solutions of the fractional Bad and Good modified Boussinesq equations are obtained in Section 3. Finally, the conclusions of this paper are given in Section 4.

2. Preliminary and The Exp-Function Method

It is well known that most of the events that develop in mathematical, physics and engineering fields can be described by partial differential equations (PDEs). So, partial differential equations are useful tools for mathematical modelling. First, the Exp-function method is defined by He and Wu (2006) and applied to various applications by many scientists [14-20]. The exact solution of non-linear partial differential equations is obtained by the Exp-function method. First, the partial differential equation is reduced to the ordinary differential equation and referred to the exact solution using the exponential function method. The Exp-function method is an effective method for solutions of the non-linear evolution equations that emerges in mathematical physics, applied mathematics and engineering applications. The Exp-function method also gives generalized single and periodic solutions of the nonlinear evolution equation. Some solutions of fractional Bad and Good modified Boussinesq equations with the aid of auxiliary equation method are obtained by Durur et al. [21].

We consider a general nonlinear PDE in the form

$$P = (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0 \tag{3}$$

Let us introduce a complex variable $\xi = kx + wt$ where k and w are constants. We rewrite Equation (3) in the subsequent nonlinear ODE:

$$Q(u,u',u'',u''',\dots)=0 \tag{4}$$

where the prime denotes the derivation concerning ξ [4]. According to the Exp-function method, we assume that the solution can be expressed in the form

$$u\,(\xi)=\frac{\sum_{n=-c}^{d}a_n e^{(n\xi)}}{\sum_{m=-p}^{q}b_m\,e^{(m\xi)}} \tag{5}$$

where c,d,p and q are positive integers, a_n and b_m are unknown constants to be observed. Without loss of generality, if we take $d=q=c=p=1$ then Equation (5) can be written as follow,

$$u\,(\xi)=\frac{a_{-1}\xi+\ a_0+\ a_1e^{\xi}}{b_{-1}e^{-\xi}+\ b_0+\ e^{\xi}} \tag{6}$$

Here, the constant b_1 is taken as 1 for simplicity. It will arrive us to a set of algebraic equations for the unknowns a_0, a_1 , a_{-1}; b_0, b_{-1}; k,w. Some useful theorems are given by Ebaid [22] for this subject.

Theorem 2.1. Suppose that $u^{(r)}$ and $u^{(\gamma)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $u\,(\xi)=\frac{\sum_{n=-c}^{d}a_n e^{(n\xi)}}{\sum_{m=-p}^{q}b_m e^{(m\xi)}}$ leads to $c=d$ and $p=q$, $\forall\, r\geq 1$, $\forall\gamma\geq 1$ [14].

i) Suppose that $u^{(r)}$ and $u^{(s)}u^{(k)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s and k are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q,\forall r,s,k\geq 1$

ii) Let $u^{(r)}$ and $(u^{(s)})^{\gamma}$ be respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s and γ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall r,s\geq 1$, $\forall\gamma\geq 2$

iii) Suppose that $u^{(r)}$ and $(u^{(s)})^{\gamma}u^{(\lambda)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s,γ and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall r,s,\gamma,\lambda\geq 1$

Theorem 2.2. [11] If f and g functions are α -differentiable at a point $t>0$ for $\alpha\in(0,1]$, then

a) $T_\alpha\big(\lambda f(t)+\delta g(t)\big)=\lambda T_\alpha\big(f(t)\big)+\delta T_\alpha\big(g(t)\big)$, for all $\delta,\lambda\in\mathbb{R}$

b) $T_\alpha(t^p)=pt^{p-\alpha}$, for all $p\in\mathbb{R}$

c) $T_\alpha(c)=0$ for all constant c.

d) $T_\alpha\big(f(t)g(t)\big)=f(t)T_\alpha\big(g(t)\big)+g(t)T_\alpha\big(f(t)\big)$

e) $T_\alpha\left(\frac{f(t)}{g(t)}\right)=\frac{g(t)T_\alpha(f(t))-f(t)T_\alpha(g(t))}{(g(t))^2}$

f) If the function f is differentiable, $T_\alpha\big(f(t)\big)=t^{1-\alpha}\frac{df}{dt}$.

In addition, α-th order conformable fractional derivatives of some functions are given as,

i) $T_\alpha(e^{at})=at^{1-\alpha}e^{at}$, $\forall a\in\mathbb{R}$

ii) $T_\alpha(\sin\,(bt))=bt^{1-\alpha}\cos\,(bt)$, $\forall b\in\mathbb{R}$

iii) $T_\alpha(\cos\,(ct))=-ct^{1-\alpha}\sin(ct)$, $\forall c\in\mathbb{R}$

iv) $T_\alpha\left(\frac{t^\alpha}{\alpha}\right)=1$

v) $T_\alpha\left(\sin\left(\frac{t^\alpha}{\alpha}\right)\right) = \cos\left(\frac{t^\alpha}{\alpha}\right)$

vi) $T_\alpha\left(\cos\left(\frac{t^\alpha}{\alpha}\right)\right) = -\sin\left(\frac{t^\alpha}{\alpha}\right)$

vii) $T_\alpha\left(e^{\frac{t^\alpha}{\alpha}}\right) = e^{\frac{t^\alpha}{\alpha}}$

3. Fractional Bad and Good Modified Boussinesq Equations

The Boussinesq equation is first discovered by Joseph Boussinesq in 1870 [23]. This equation is one of the non-linear partial differential equations in plasma that have applications in many areas, such as ion sound waves, shallow water waves modelling, longitudinal propagation waves in elastic bars, suppressed waves in liquid-gas foam mixtures and the propagation model of these waves. Then, the Boussinesq equation is modified to adapt to deeper water problems and hence there are many new forms of this equation in the literature [21,23].

Time-Fractional Bad Modified Boussinesq Equation is denoted by

$$D_t^{(2\alpha)}u - D_x^2 u - D_x^4 u - 3D_x^2(u^2) + 3D_x(u^2 D_x u) = 0 \tag{7}$$

Time-Fractional Good Modified Boussinesq Equation is denoted by

$$D_t^{(2\alpha)}u - D_x^2 u + D_x^4 u - 3D_x^2(u^2) + 3D_x(u^2 D_x u) = 0 \tag{8}$$

We are now ready for solutions of Time-Fractional Bad Modified Boussinesq Equation using the Exp-function method to produce a solution set. In view of the Exp-function method, we assume that the solutions of Equation (7) can be expressed in the form,

$$u\,(\xi) = \frac{\sum_{n=-c}^{d} a_n e^{(n\xi)}}{\sum_{m=-p}^{q} b_m\, e^{(m\xi)}}$$

By using Theorem 2. 2. (f) and the travelling wave transformation as follow,

$$u(x,t) = u(\xi)\,, \xi = x - w\frac{t^\alpha}{\alpha} \tag{9}$$

and therefore, the Equation (7) convert to an ordinary differential equation. Here a_n and b_m are unknown constants, w denotes the velocity of the wave, and prime denotes the derivative of the functions concerning ξ.

By applying the wave transform in (9), the equation in (7) is obtained to the following ordinary differential equation.

$$(-\mathrm{w}^2 - 1)u'' - u^{iv} - 6u'^2 - 6uu'' + 6uu'^2 + 3u^2u'' = 0 \tag{10}$$

By applying the wave transform in (9), the equation in (8) is obtained to the following ordinary differential equation.

$$(-\mathrm{w}^2 - 1)u'' + u^{iv} - 6u'^2 - 6uu'' + 6uu'^2 + 3u^2u'' = 0 \tag{11}$$

Firstly, substitute Equation (6) into Equation (10) and Equation (11), then the unknown parameters are obtained by using Maple Software.

Hence, we obtain all the solutions related to Bad and Good Modified Boussinesq Equations cases. We balance the linear term of the highest order of Equation (10) u^{iv} with the highest order nonlinear term u^2u'',we set $p = c = 1$ and $q = d = 1$; then the trial solution of Equation (10), reduces to

$$\frac{-1}{A}\left[C_4e^{4\xi} + C_3e^{3\xi} + C_2e^{2\xi} + C_1e^{\xi} + C_0 + C_{-1}e^{-\xi} + C_{-2}e^{-2\xi} + C_{-3}e^{-3\xi} + C_{-4}e^{-4\xi}\right] = 0 \tag{12}$$

and all coefficients must be zero for the solutions of Equation (12). Hence, we get

$$A = b_{-1}(e^{-\xi} + b_0 + e^{\xi})^5 ;$$

$$C_4 = -2a_0 + 3a_1^2a_0 - 6a_0a_1 + 6a_1^2b_0 - 3a_1^3b_0 + 2a_1b_0 - w^2a_0 + w^2a_1b_0;$$

$$C_3 = -20a_{-1} + 24a_1^2b_{-1} + 20a_1b_{-1} + 12a_1^2a_{-1} + 9a_1^3b_0^2 - 6a_1^2b_0^2 - 12a_1^3b_{-1} + 12a_1a_0^2$$
$$-24a_1a_{-1} - 4w^2a_{-1} - 10a_1b_0^2 + 10a_0b_0 + 4w^2a_1b_{-1} + 18a_1b_0a_0 - w^2a_0b_0 + w^2a_1b_0^2$$
$$-21a_1^2a_0b_0 - 12a_0^2;$$

$$C_2 = 9a_0^3 + 4w^2a_0b_{-1} - 70a_1b_{-1}b_0 - 12a_1^2b_0b_{-1} - w^2a_1b_0^3 - 12a_1a_{-1}b_0 - 21a_1^2a_{-1}b_0$$
$$-21a_1b_0a_0^2 + 12a_0a_1^2b_0^2 + +18a_1a_0b_0^2 + 78a_1a_0b_{-1} - 11w^2a_{-1}b_0 - 66a_0a_1^2b_{-1} - 10a_0b_0^2$$
$$+80a_0b_{-1} - 54a_{-1}a_0 - 10a_{-1}b_0 - 12a_1^2b_0^3 + +33a_1^3b_0b_{-1} + w^2a_0b_0^2 + 7w^2a_1b_0b_{-1}$$
$$+10a_1b_0^3 - 6a_0^2b_0 + 54a_1a_0a_{-1};$$

$$C_1 = -w^2a_1b_0^4 - 84a_1a_0^2b_{-1} + 4w^2a_1b_{-1}^2 - 24a_1^2b_{-1}^2 + 56a_1b_0^2b_{-1} - 6a_0a_1b_0^3 - 78a_{-1}a_0b_0$$
$$+3a_0^2a_1b_0^2 + 3a_{-1}a_1^2b_0^2 - 4w^2a_{-1}b_{-1} + 12a_{-1}a_1b_0^2 + 72a_{-1}a_1b_{-1} - 54a_1^2b_{-1}b_0^2 - 11w^2a_{-1}b_0^2$$
$$+w^2a_0b_0^3 - 84a_1^2a_{-1}b_{-1} \pm 34a_0b_0b_{-1} + 36a_0^2b_{-1} - 48a_{-1}^2 + 2a_0b_0^3 + 48a_{-1}a_0^2 - 2a_1b_0^4$$
$$-172a_1b_{-1}^2 - 3a_0^3b_0 + 48a_1a_{-1}^2 + 36a_1^3b_{-1}^2 + +6a_0^2b_0^2 + 172a_{-1}b_{-1} - 22a_{-1}b_0^2$$
$$+51a_0a_1^2b_0b_{-1} + 84a_0a_1b_0\,b_{-1} + 13w^2a_0b_0b_{-1} - 2w^2a_1b_{-1}b_0^2 - 18a_{-1}a_0a_1b_0;$$

$$C_0 = 110a_{-1}b_{-1}b_0 - 90a_1^2b_{-1}^2b_0 + 15a_{-1}b_0a_0^2 + 15a_{-1}a_1^2b_{-1}b_0 + 75a_0a_1^2b_{-1}^2 + 30a_0a_1b_{-1}^2$$
$$-5w^2a_{-1}b_{-1}b_0 + 10w^2a_0b_{-1}b_0^2 + 15a_1b_0a_0^2b_{-1} + 10w^2a_0b_{-1}^2 - 10a_{-1}b_0^3 - 220a_0b_{-1}^2$$
$$+30a_{-1}a_0b_{-1} - 5w^2a_{-1}b_0^3 + 15a_{-1}^2a_1b_0 + 110a_1b_{-1}^2b_0 - 10a_1b_{-1}b_0^3 + 20a_0b_{-1}b_0^2 - 30a_0^3b_{-1}$$
$$-30a_1a_0b_0^2b_{-1} - 30a_{-1}b_0^2a_0 - 5w^2a_1b_{-1}^2b_0 + 60a_0^2b_{-1}b_0 - 90a_{-1}^2b_0 + 120a_{-1}a_1b_{-1}b_0$$
$$-5w^2a_1b_{-1}b_0^3 - 180a_{-1}a_0a_1b_{-1} + 75a_0a_{-1}^2;$$

$$C_{-1} = -172a_{-1}b_{-1}^2 + 172a_1b_{-1}^3 - 24a_{-1}^2b_{-1} + 36a_0^2b_{-1}^2 - 2a_{-1}b_0^4 - 48a_1^2b_{-1}^3 - 54a_{-1}^2b_0^2$$
$$+48a_1a_0^2b_{-1}^2 - 22a_1b_{-1}^2b_0^2 + 6a_0^2b_{-1}b_0^2 - 84a_{-1}a_0^2b_{-1} - 6a_0a_{-1}b_0^3 - 34a_0b_{-1}^2b_0$$
$$+48a_{-1}a_1^2b_{-1}^2 + 3a_{-1}^2b_0^2a_1 + 3a_0^2a_{-1}b_0^2 + 4w^2a_{-1}b_{-1}^2 - 3a_0^3b_{-1}b_0 + 72a_1a_{-1}b_{-1}^2 - w^2a_{-1}b_0^4$$
$$+51a_0a_{-1}^2b_0 - 4w^2a_1b_{-1}^3 + w^2a_0b_{-1}b_0^3 + 12a_{-1}b_0^2a_1b_{-1} + 13w^2a_0b_{-1}^2b_0 - 2w^2a_{-1}b_0^2b_{-1}$$
$$-11w^2a_1b_0^2b_{-1}^2 + 2a_0b_{-1}b_0^3 - 84a_{-1}^2a_1b_{-1} + 84a_{-1}a_0b_{-1}b_0 - 78a_1b_{-1}^2a_0b_0$$
$$-18a_{-1}a_0a_1b_{-1}b_0 + 36a_{-1}^3 + 56a_{-1}b_{-1}b_0^2;$$

$$C_{-2} = 9a_0^3b_{-1}^2 - 12a_{-1}^3b_0^3 + 80a_0b_{-1}^3 + 33a_{-1}^3b_0 + 12a_0a_{-1}^2b_0^2 + 4w^2a_0b_{-1}^3 + 10a_{-1}b_{-1}b_0^3$$
$$-54a_0a_1b_{-1}^3 - 70a_{-1}b_{-1}^2b_0 - 12a_{-1}^2b_{-1}b_0 - 66a_{-1}^2a_0b_{-1} + 78a_{-1}a_0b_{-1}^2 - 10a_1b_{-1}^3b_0$$

$$-6a_0^2b_{-1}^2b_0 - 10a_0b_{-1}^2b_0^2 + 18a_{-1}b_0^2a_0b_{-1} + w^2a_0b_{-1}^2b_0^2 - 12a_{-1}b_0a_1b_{-1}^2 - 21a_{-1}b_0a_0^2b_{-1}$$

$$-11w^2a_1b_{-1}^3b_0 + 54a_{-1}a_1a_0b_{-1}^2 - w^2a_{-1}b_0^3b_{-1} - 21a_{-1}^2a_1b_0b_{-1} + 7w^2a_{-1}b_{-1}^2b_0;$$

$$C_{-3} = 18a_{-1}b_0a_0b_{-1}^2 - 21a_{-1}^2a_0b_0b_{-1} - 12a_{-1}^3b_{-1} + 24a_{-1}^2b_{-1}^2 - 20a_1b_{-1}^4 + 10a_0b_{-1}^3b_0$$

$$-w^2a_0b_{-1}^3b_0 + w^2a_{-1}b_{-1}^2b_0^2 - 12a_0^2b_{-1}^3 + 9a_{-1}^3b_0^2 + 20a_{-1}b_{-1}^3 - 10a_{-1}b_{-1}^2b_0^2 + 4w^2a_{-1}b_{-1}^3$$

$$+12a_{-1}^2a_1b_{-1}^2 - 4w^2a_1b_{-1}^4 - 6a_{-1}^2b_0^2b_{-1} + 12a_{-1}a_0^2b_{-1}^2 - 24a_{-1}a_1b_{-1}^3;$$

$$C_{-4} = w^2a_{-1}b_{-1}^3b_0 - 2a_0b_{-1}^4 + 6a_{-1}^2b_{-1}^2b_0 + 2a_{-1}b_{-1}^3b_0 - 3a_{-1}^3b_0b_{-1} + 3a_{-1}^2a_0b_{-1}^2 - 6a_{-1}a_0b_{-1}^3$$

$$-w^2a_0b_{-1}^4\,;$$

All the coefficients of $e^{n\xi}$ must be zero. Hence, we produce a system of algebraic equations which the Maple can tackle to produce the subsequent cases of solutions:

Case 1:

$$a_0 = a_1b_0\,, a_{-1} = a_1b_{-1} \tag{13}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = a_1 \tag{14}$$

where a_1 is a free parameter.

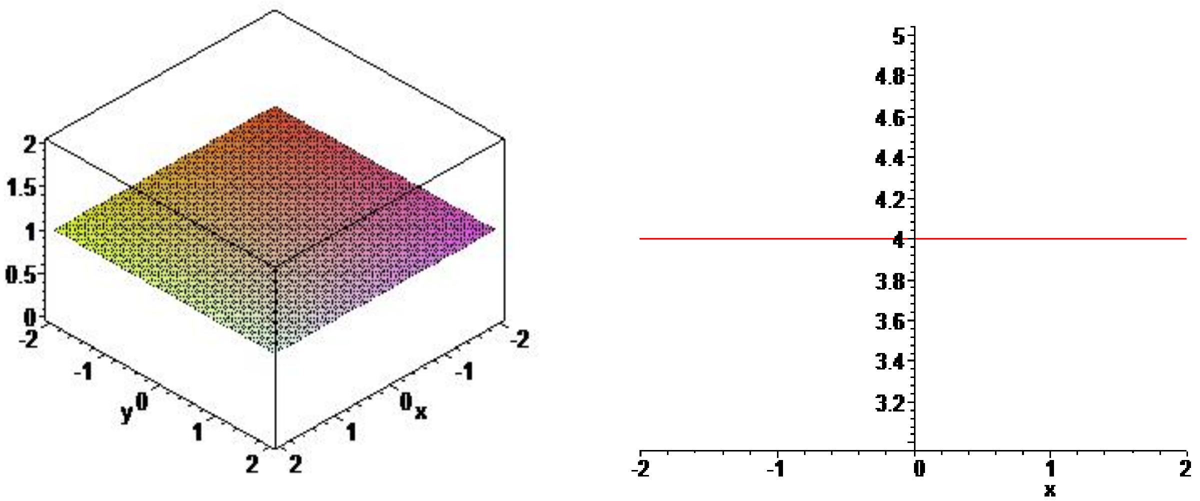


Fig. 1. 3D and 2D plots of travelling wave solutions (Case 1)

The plots indicate the wave solutions for $a_1 = 4$ in Equation (14).

Case 2:

$$a_0 = 0, \qquad b_0 = 0, a_1 = 1, w = 2\sqrt{2}I, w = -2\sqrt{2}I,\ b_{-1} = 0 \tag{15}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{a_{-1}e^{-\xi} + e^{\xi}}{e^{\xi}} \tag{16}$$

where a_{-1} is a free parameter.

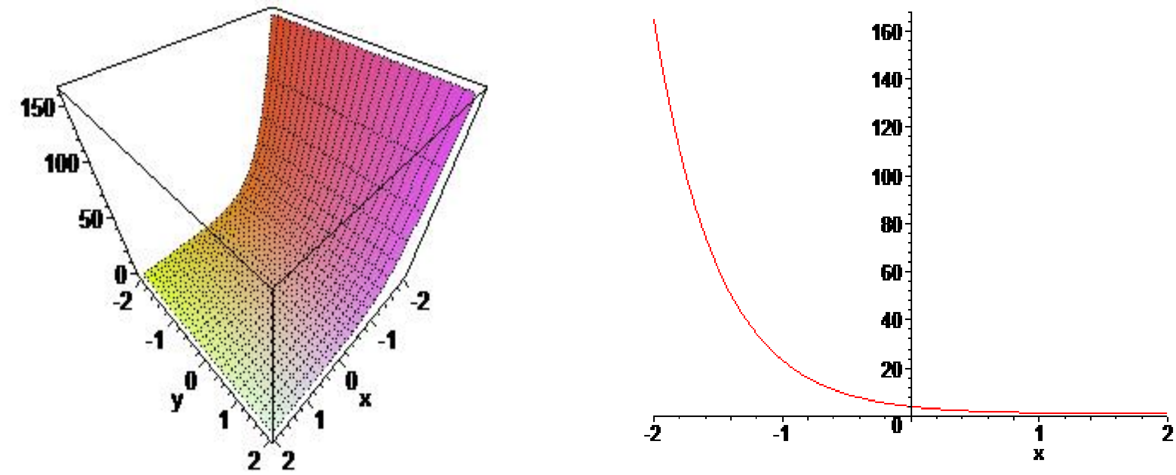


Fig. 2. 3D and 2D plots of travelling wave solutions (Case 2)

The plots indicate the wave solutions for $a_{-1} = 3$ in Equation (16).

<u>Case 3:</u>

$$a_0 = 0, b_0 = 0, a_1 = 1 + \sqrt{2},\ a_1 = 1 - \sqrt{2}\,, w = I\sqrt{2}\,, w = -I\sqrt{2} \tag{17}$$

$$a_{-1} = -\left(1 + \sqrt{2}\right) b_{-1} + 2b_{-1},\ a_{-1} = -\left(1 - \sqrt{2}\right) b_{-1} + 2b_{-1}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{\left(-\left(1 + \sqrt{2}\right) b_{-1} + 2b_{-1}\right)e^{-\xi} + (1 + \sqrt{2})e^{\xi}}{b_{-1}e^{-\xi} + e^{\xi}} \tag{18}$$

where b_{-1} is a free parameter.

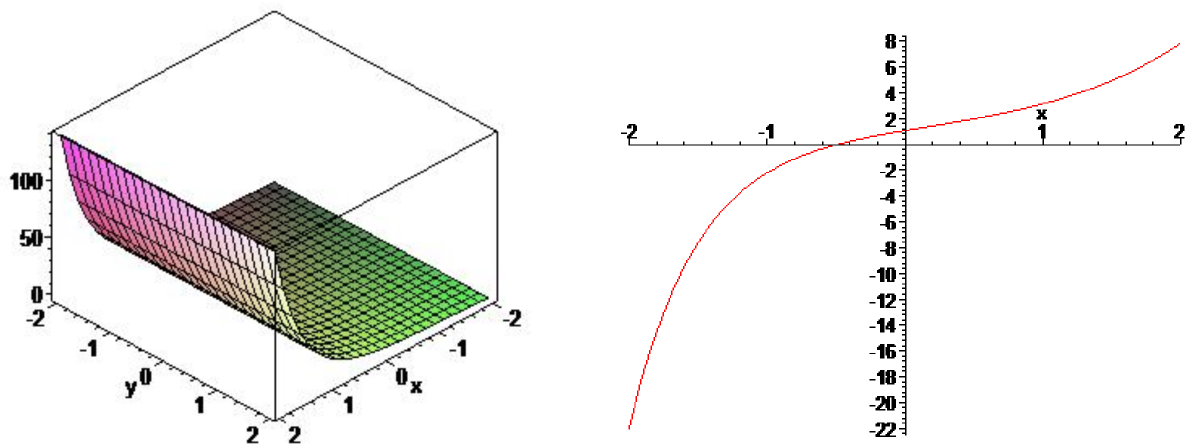


Fig. 3. 3D and 2D plots of travelling wave solutions (Case 3)

The plots indicate the wave solutions for $b_{-1} = 5$ in Equation (18).

<u>Case 4:</u>

$$a_{-1} = 0, b_{-1} = 0, a_1 = 1 + \frac{1}{2}\sqrt{2},\ a_1 = 1 - \frac{1}{2}\sqrt{2}\,, w = \frac{1}{2} I\sqrt{14} \tag{19}$$

$$w = -\frac{1}{2}I\sqrt{14},\ a_0 = -\left(1-\frac{1}{2}\sqrt{2}\right)b_0 + 2b_0, a_0 = -\left(1+\frac{1}{2}\sqrt{2}\right)b_0 + 2b_0$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{-\left(1-\frac{1}{2}\sqrt{2}\right)b_0 + 2b_0 + (1-\frac{1}{2}\sqrt{2})e^{\xi}}{b_0 + e^{\xi}} \tag{20}$$

where b_0 is a free parameter.

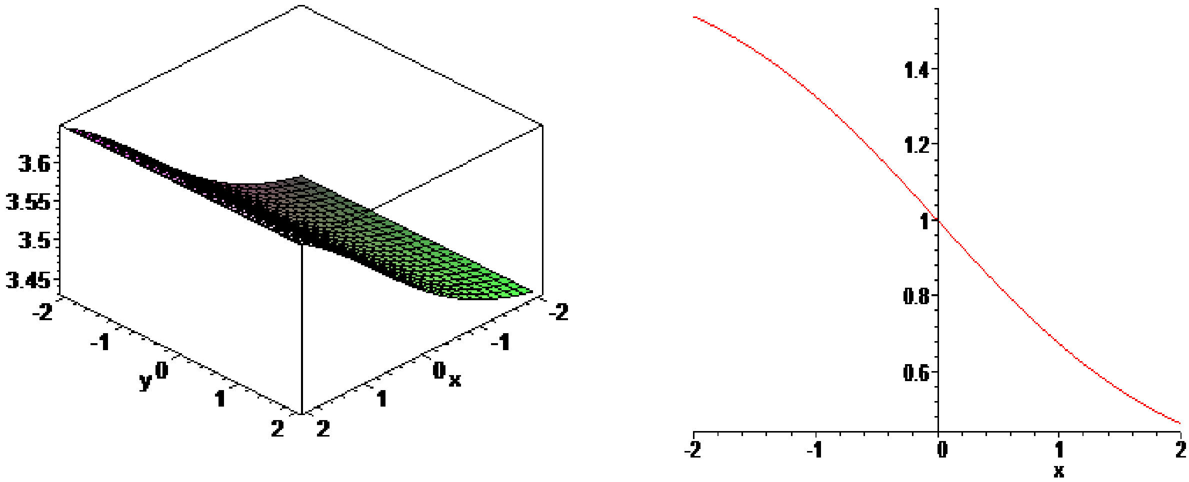


Fig. 4. 3D and 2D plots of travelling wave solutions (Case 4)

The plots indicate the wave solutions for $b_0 = 2$ in Equation (20).

<u>Case 5:</u>

$$a_{-1} = 0, b_0 = a_0, a_1 = 0, w = I\sqrt{2}\,, w = -I\sqrt{2},\ b_{-1} = \frac{1}{8}{a_0}^2 \tag{21}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{a_0}{\frac{1}{8}{a_0}^2 e^{-\xi} + a_0 + e^{\xi}} \tag{22}$$

where a_0 is a free parameter.

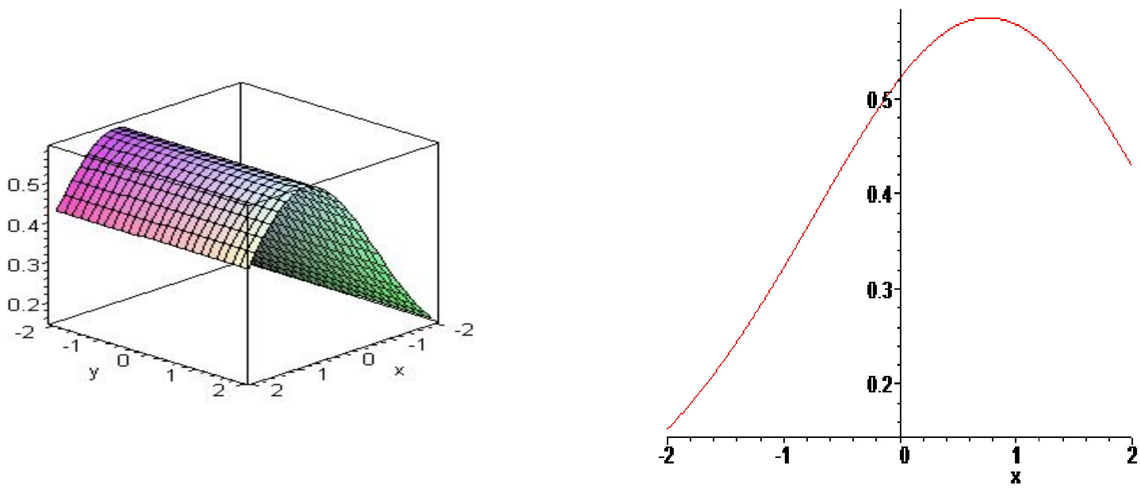


Fig. 5. 3D and 2D plots of travelling wave solutions (Case 5)

The plots indicate the wave solutions for $a_0 = 6$ in Equation (22).

<u>Case 6:</u>

$$a_{-1} = -a_0^2 + \frac{1}{2}\left(1 - \frac{1}{2}\sqrt{2}\right)a_0^2 - \left(1 - \frac{1}{2}\sqrt{2}\right)b_0 a_0 + \frac{1}{4}\left(1 - \frac{1}{2}\sqrt{2}\right)b_0^2 + 2b_0 a_0 - \frac{1}{2}b_0^2$$

$$a_{-1} = -a_0^2 + \frac{1}{2}\left(1 + \frac{1}{2}\sqrt{2}\right)a_0^2 - \left(1 + \frac{1}{2}\sqrt{2}\right)b_0 a_0 + \frac{1}{4}\left(1 + \frac{1}{2}\sqrt{2}\right)b_0^2 + 2b_0 a_0 - \frac{1}{2}b_0^2 \quad (23)$$

$$b_{-1} = b_0 a_0 - \frac{1}{2}a_0^2 - \frac{1}{4}b_0^2, w = \frac{1}{2}I\sqrt{14}, w = -\frac{1}{2}I\sqrt{14}, a_1 = 1 - \frac{1}{2}\sqrt{2}, a_1 = 1 + \frac{1}{2}\sqrt{2}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{\left(-a_0^2 + \frac{1}{2}\left(1 - \frac{1}{2}\sqrt{2}\right)a_0^2 - \left(1 - \frac{1}{2}\sqrt{2}\right)b_0 a_0 + \frac{1}{4}\left(1 - \frac{1}{2}\sqrt{2}\right)b_0^2 + 2b_0 a_0 - \frac{1}{2}b_0^2\right)e^{-\xi} + a_0 + \left(1 - \frac{1}{2}\sqrt{2}\right)e^{\xi}}{\left(b_0 a_0 - \frac{1}{2}a_0^2 - \frac{1}{4}b_0^2\right)e^{-\xi} + b_0 + e^{\xi}} \quad (24)$$

where b_0 and a_0 are free parameters.

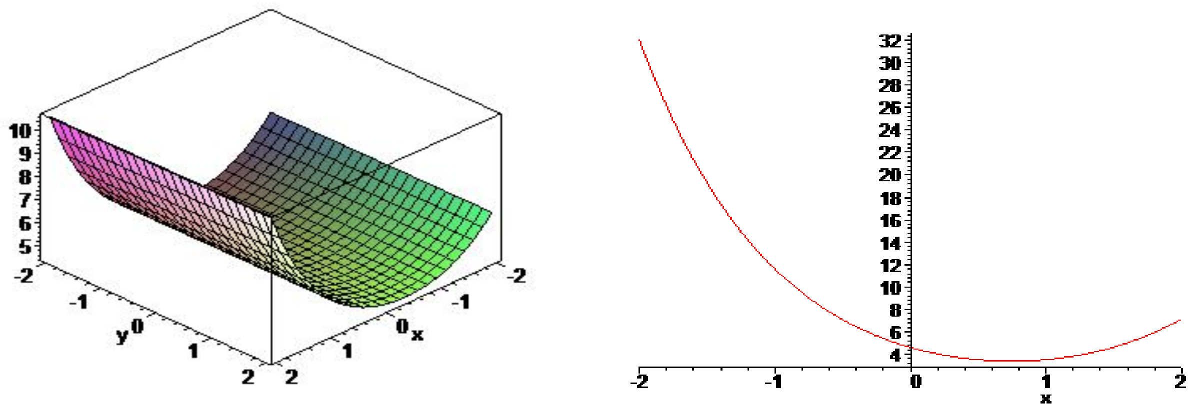


Fig. 6. 3D and 2D plots of travelling wave solutions (Case 6)

The plots indicate the wave solutions for $a_0 = 1, b_0 = 1$ in Equation (24).

<u>Case 7:</u>

$$a_0 = \frac{b_0(a_1^2 - a_1 - 1)}{-1 + a_1}, w = \sqrt{-6\,a_1 + 3a_1^2 - 2}, w = -\sqrt{-6\,a_1 + 3a_1^2 - 2} \quad (25)$$

$$b_{-1} = \frac{1}{8}\frac{b_0^2(2a_1^2 - 4a_1 + 1)}{(-1 + a_1)^2}, a_{-1} = \frac{1}{8}\frac{b_0^2(2a_1^2 - 4a_1 + 1)a_1}{(-1 + a_1)^2}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{\left(\frac{1}{8}\frac{b_0^2(2a_1^2 - 4a_1 + 1)a_1}{(-1 + a_1)^2}\right)e^{-\xi} + \frac{b_0(a_1^2 - a_1 - 1)}{-1 + a_1} + a_1 e^{\xi}}{\left(\frac{1}{8}\frac{b_0^2(2a_1^2 - 4a_1 + 1)}{(-1 + a_1)^2}\right)e^{-\xi} + b_0 + e^{\xi}} \quad (26)$$

where b_0 and a_1 are free parameters.

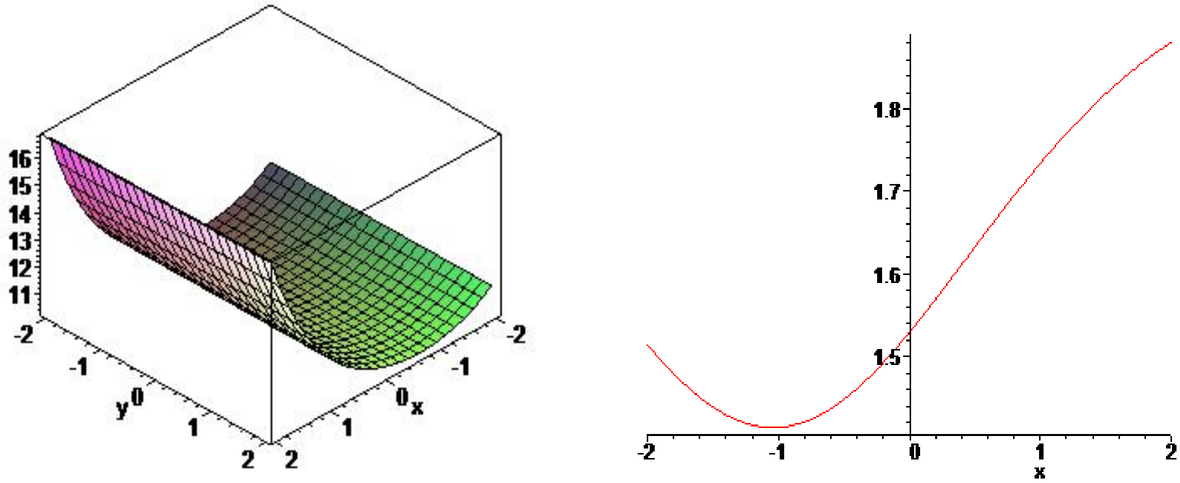


Fig. 7. 3D and 2D plots of travelling wave solutions (Case 7)

The plots indicate the wave solutions for $a_1 = 2, b_0 = 1$ in Equation (26).

<u>Case 8:</u>

$$a_{-1} = -\frac{1}{8}{a_0}^2, b_0 = 0, a_1 = 1, w = I\sqrt{5}, w = -I\sqrt{5}, b_{-1} = \frac{1}{8}{a_0}^2 \tag{27}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{-{a_0}^2 e^{-\xi} + 8a_0 + 8e^{\xi}}{{a_0}^2 e^{-\xi} + 8e^{\xi}} \tag{28}$$

where a_0 is a free parameter.

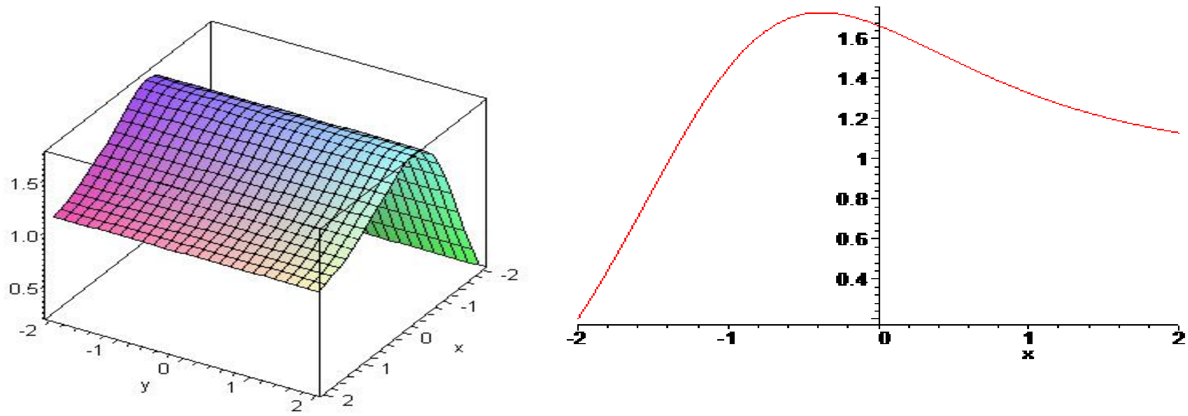


Fig. 8. 3D and 2D plots of travelling wave solutions (Case 8)

The plots indicate the wave solutions for $a_0 = 1$ in Equation (28).

Therefore, the full solutions of the Time-Fractional Bad Modified Boussinesq equation for the above conditions have been obtained. Now let's solve the Time-Fractional Good Modified Boussinesq equation using the Exp-function method.

We balance the linear term of the highest order of Equation (11) u^{iv} with the highest order nonlinear term u^2u'', we set $p = c = 1$ and $q = d = 1$; then the trial solution, Equation (11), reduces to

$$\frac{-1}{B}\left[D_4e^{4\xi}+D_3e^{3\xi}+D_2e^{2\xi}+D_1e^{\xi}+D_0+D_{-1}e^{-\xi}+D_{-2}e^{-2\xi}+D_{-3}e^{-3\xi}+D_{-4}e^{-4\xi}\right]=0 \qquad (29)$$

and for the solutions of Equation (29), all coefficients must be zero.

$$B=(b_{-1}e^{-\xi}+\ b_0+\ e^{\xi})^5;$$

$$D_4\ =w^2a_1b_0-3a_1^3b_0-6a_1a_0+6a_1^2b_0-w^2a_0+3a_1^2a_0;$$

$$\begin{aligned}D_3\ =\ &24a_1^2b_{-1}-12a_1b_{-1}-24a_1a_{-1}+12a_1^2a_{-1}+12a_1b_0^2+12a_1a_0^2-12a_1^3b_{-1}-4w^2a_1\\&+9a_1^3b_0^2-21a_0a_1^2b_0-6a_1^2b_0^2-12a_0b_0+4w^2a_1b_{-1}-w^2a_0b_0+w^2a_1b_0^2-12a_0^2+12a_{-1}\\&+18a_1b_0a_0;\end{aligned}$$

$$\begin{aligned}D_2\ =\ &-12a_1b_0^3-72a_0b_{-1}-12a_1^2b_0^3-12a_{-1}b_0-66a_1^2b_{-1}a_0+12a_0b_0^2-54a_{-1}a_0+9a_0^3\\&+18a_1b_0^2a_0-11w^2a_{-1}b_0+33a_1^3b_{-1}b_0+12a_0a_1^2b_0^2-21a_1^2b_{-1}b_0+54a_{-1}a_0a_1-12a_{-1}a_1b_0\\&+84a_1b_{-1}b_0+4w^2a_0b_{-1}+w^2a_0b_0^2-21a_1b_0a_0^2-w^2a_1b_0^3+78a_1b_{-1}a_0-12a_1^2b_{-1}b_0\\&-6a_0^2b_0+7w^2a_1b_{-1}b_0;\end{aligned}$$

$$\begin{aligned}D_1\ =\ &-3a_0^3b_0+180a_1b_{-1}^2+36a_0^2b_{-1}+48a_{-1}a_0^2-24a_1^2b_{-1}^2+6a_0^2b_0^2+48a_1a_{-1}^2+36a_1^3b_{-1}^2\\&-180a_{-1}b_{-1}+3a_{-1}a_1^2b_0^2-18a_{-1}a_0a_1b_0-2w^2a_1b_0^2b_{-1}-11w^2a_{-1}b_0^2-78a_{-1}a_0b_0\\&+13w^2a_0b_0b_{-1}+84a_1b_{-1}a_0b_0+51a_0a_1^2b_{-1}b_0+72a_{-1}a_1b_{-1}-84a_1b_{-1}a_0^2-54a_1^2b_{-1}b_0^2\\&+60a_0b_{-1}b_0+4w^2a_1b_{-1}^2-6a_0a_1b_0^3-48a_{-1}^2+w^2a_0b_0^3+12a_{-1}a_1b_0^2-84a_1^2a_{-1}b_{-1}\\&+3a_0^2a_1b_0^2-w^2a_1b_0^4-60a_1b_{-1}b_0^2-4w^2a_{-1}b_{-1};\end{aligned}$$

$$\begin{aligned}D_0\ =\ &-30a_0a_1b_{-1}^2+75a_0a_1^2b_{-1}^2+10w^2a_0b_{-1}^2+240a_0b_{-1}^2+120a_{-1}a_1b_{-1}b_0+15a_1^2a_{-1}b_{-1}b_0\\&-90a_1^2b_{-1}^2b_0-90a_{-1}^2b_0-30a_{-1}^2a_0b_0^2+15a_{-1}b_0a_0^2-120a_1b_{-1}^2b_0-120a_{-1}b_{-1}b_0\\&-30a_1b_0^2a_0b_{-1}+15a_1a_{-1}^2b_0-180a_0a_{-1}a_1b_{-1}+10w^2a_0b_0^2b_{-1}+15a_1b_0a_0^2b_{-1}+75a_0a_{-1}^2\\&-5w^2a_{-1}b_{-1}b_0-5w^2a_1b_0^3b_{-1}-5w^2a_1b_0b_{-1}^2-5w^2a_{-1}b_0^3-30a_0^3b_{-1}+60a_0^2b_{-1}b_0\\&+30a_0a_{-1}b_{-1};\end{aligned}$$

$$\begin{aligned}D_{-1}=\ &36a_{-1}^3-6a_0a_{-1}b_0^3-4w^2a_1b_{-1}^3-3a_0^3b_{-1}b_0+3a_0^2a_{-1}b_0^2+6a_0^2b_{-1}b_0^2-w^2a_{-1}b_0^4\\&-84a_{-1}^2a_1b_{-1}-84a_0^2a_{-1}b_{-1}+48a_1a_0^2b_{-1}^2+60a_0b_{-1}^2b_0-60a_{-1}b_{-1}b_0^2+3a_{-1}^2a_1b_0^2\\&+4w^2a_{-1}b_{-1}^2+51a_0a_{-1}^2b_0+48a_{-1}a_1^2b_{-1}^2+72a_{-1}a_1b_{-1}^2+180a_{-1}b_{-1}^2-180a_1b_{-1}^3-48a_1^2b_{-1}^3\\&-54a_{-1}^2b_0^2-24a_{-1}^2b_{-1}+36a_0^2b_{-1}^2+w^2a_0b_{-1}b_0^3+13w^2a_0b_{-1}^2b_0-78a_1b_{-1}^2a_0b_0\\&-11w^2a_1b_0^2b_{-1}^2-18a_0a_{-1}a_1b_0b_{-1}+12a_{-1}b_0^2a_1b_{-1}-2w^2a_{-1}b_0^2b_{-1}+84a_{-1}a_0b_0b_{-1};\end{aligned}$$

$$D_{-2}=w^2a_0b_{-1}^2b_0^2+54a_{-1}a_0a_1b_{-1}^2-21a_{-1}a_0^2b_0b_{-1}-11w^2a_1b_{-1}^3b_0+18a_{-1}b_0^2a_0b_{-1}$$

$$+7w^2a_{-1}b_{-1}^2b_0-12a_{-1}b_0a_1b_{-1}^2-21a_{-1}^2a_1b_0b_{-1}-w^2a_{-1}b_0^3b_{-1}+9a_0^3b_{-1}^2-72a_0b_{-1}^3$$

$$+33a_{-1}^3b_0-12a_{-1}^2b_0^3-66a_{-1}^2a_0b_{-1}+84a_{-1}b_{-1}^2b_0-12a_1b_{-1}^3b_0+78a_{-1}a_0b_{-1}^2-6a_0^2b_{-1}^2b_0$$

$$+4w^2a_0b_{-1}^3-12a_{-1}b_{-1}b_0^3+12a_0b_{-1}^2b_0^2+12a_0a_{-1}^2b_0^2-54a_1b_{-1}^3a_0-12a_{-1}^2b_0b_{-1};$$

$$D_{-3}=24a_{-1}^2b_{-1}^2+12a_1b_{-1}^4-12a_{-1}b_{-1}^3+9a_{-1}^3b_0^2-12a_0^2b_{-1}^3-12a_{-1}^3b_{-1}+12a_{-1}a_0^2b_{-1}^2$$

$$+12a_{-1}b_{-1}^2b_0^2-12a_0b_{-1}^3b_0+w^2a_{-1}b_{-1}^2b_0^2-21a_{-1}^2a_0b_{-1}b_0+18a_{-1}a_0b_{-1}^2b_0-w^2a_0b_{-1}^3b_0$$

$$+12a_{-1}^2a_1b_{-1}^2-4w^2a_1b_{-1}^4-6a_{-1}^2b_0^2b_{-1}+4w^2a_{-1}b_{-1}^3-24a_{-1}a_1b_{-1}^3;$$

$$D_{-4}=w^2a_{-1}b_{-1}^3b_0-3a_{-1}^3b_{-1}b_0-w^2a_0b_{-1}^4-6a_{-1}a_0b_{-1}^3+6a_{-1}^2b_{-1}^2b_0+3a_{-1}^2a_0b_{-1}^2;$$

All the coefficients of $e^{n\xi}$ must be zero. Hence, we produce a system of algebraic equations which the Maple can tackle to produce the subsequent cases of solutions:

Case 1:

$$a_0 = a_1b_0\,, a_{-1} = a_1b_{-1} \tag{30}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = a_1 \tag{31}$$

where a_1 is a free parameter.

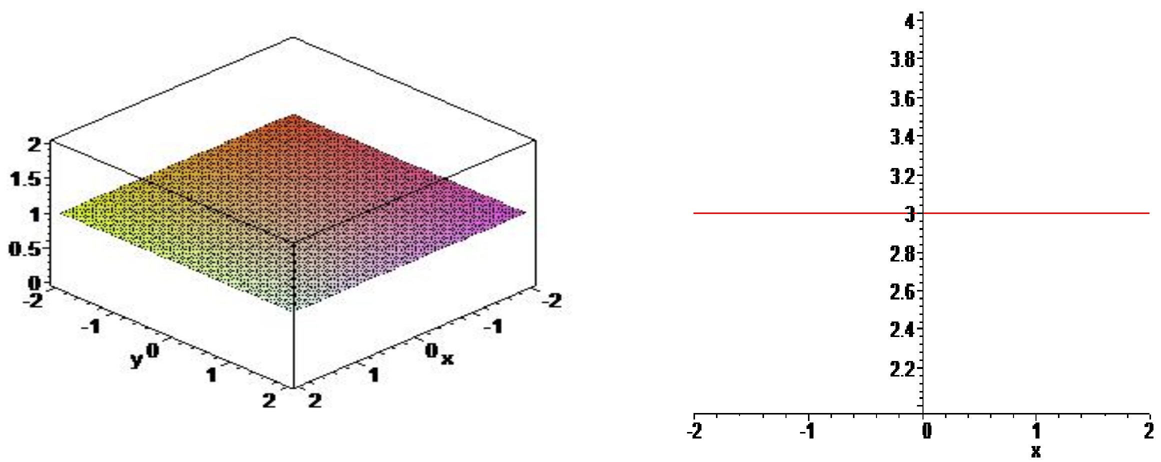


Fig. 9. 3D and 2D plots of travelling wave solutions (Case 1)

The plots indicate the wave solutions for $a_1 = 3$ in Equation (31).

Case 2:

$$a_0 = 0, b_0 = 0,\ a_1 = 1, w = 0,\ b_{-1} = 0 \tag{32}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{a_{-1}e^{-\xi} + e^{\xi}}{e^{\xi}} \tag{33}$$

where a_{-1} is a free parameter.

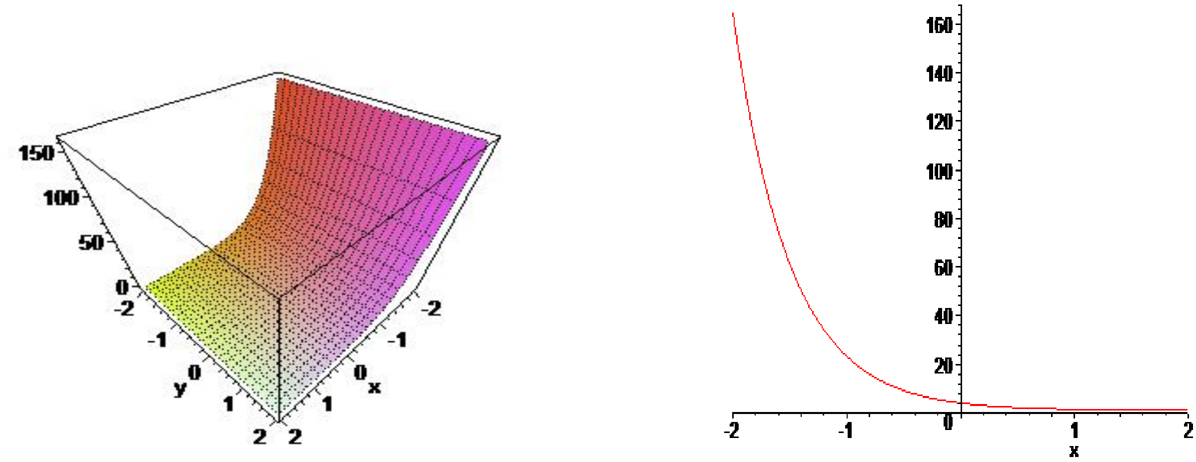


Fig. 10. 3D and 2D plots of travelling wave solutions (Case 2)

The plots indicate the wave solutions for $a_{-1} = 3$ in Equation (33).

Case 3:

$$a_0 = 0,\ b_0 = 0,\ a_1 = 1 + I\sqrt{2},\ a_1 = 1 - I\sqrt{2}\,, w = I\sqrt{6}\,, w = -I\sqrt{6} \tag{34}$$
$$a_{-1} = -\left(1 + I\sqrt{2}\right) b_{-1} + 2b_{-1},\ a_{-1} = -\left(1 - I\sqrt{2}\right) b_{-1} + 2b_{-1}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{\left(-\left(1 + I\sqrt{2}\right) b_{-1} + 2b_{-1}\right)e^{-\xi} + (1 + I\sqrt{2})e^{\xi}}{b_{-1}e^{-\xi} + e^{\xi}} \tag{35}$$

where b_{-1} is a free parameter.

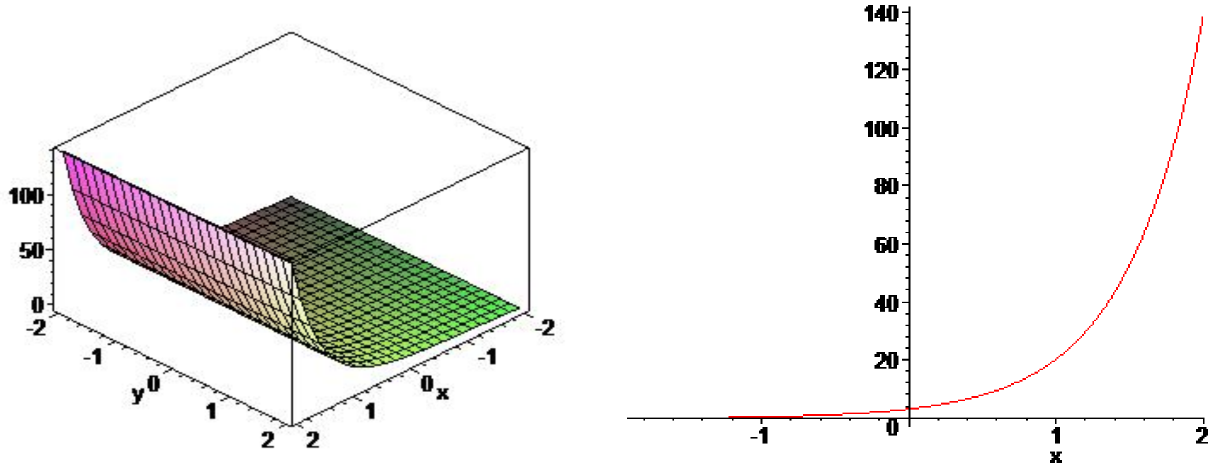


Fig. 11. 3D and 2D plots of travelling wave solutions (Case 3)

The plots indicate the wave solutions for $b_{-1} = 1$ in Equation (35).

Case 4:

$$a_{-1} = 0, b_{-1} = 0, \qquad a_1 = 1 + \frac{1}{2}I\sqrt{2},\ a_1 = 1 - \frac{1}{2}I\sqrt{2}\,, w = \frac{3}{2}I\sqrt{2} \tag{36}$$
$$w = -\frac{3}{2}I\sqrt{2},\ a_0 = -\left(1 - \frac{1}{2}I\sqrt{2}\right) b_0 + 2b_0,\ a_0 = -\left(1 + \frac{1}{2}I\sqrt{2}\right) b_0 + 2b_0$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{-\left(1-\frac{1}{2}\sqrt{2}\right)b_0 + 2b_0 + \left(1-\frac{1}{2}\sqrt{2}\right)e^{\xi}}{b_0 + e^{\xi}} \tag{37}$$

where b_0 is a free parameter.

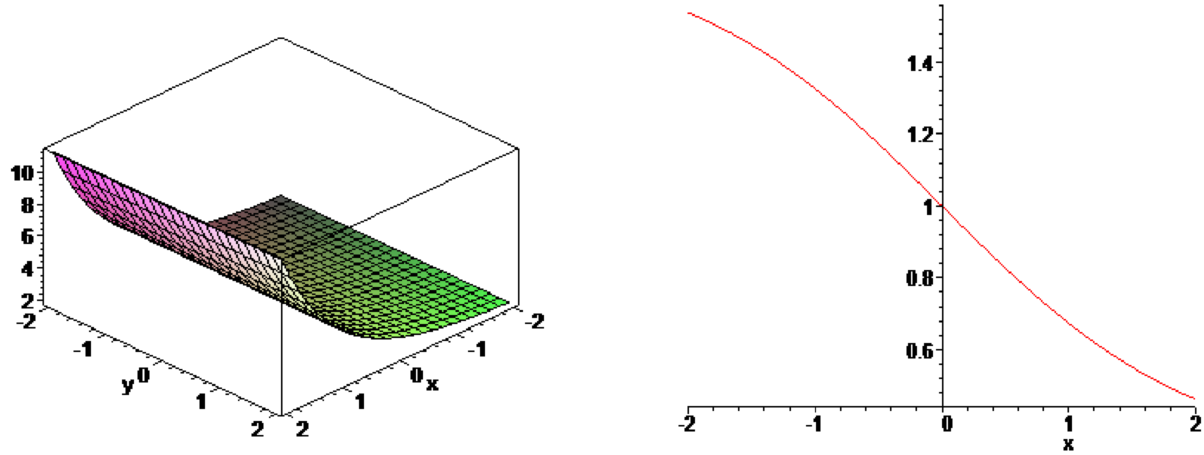


Fig. 12. 3D and 2D plots of travelling wave solutions (Case 4)

The plots indicate the wave solutions for $b_0 = 1$ in Equation (37).

<u>Case 5:</u>

$$a_{-1} = 0, b_0 = -a_0, a_1 = 0, w = 0, b_{-1} = \frac{3}{8}{a_0}^2 \tag{38}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{a_0}{\frac{3}{8}{a_0}^2 e^{-\xi} - a_0 + e^{\xi}} \tag{39}$$

where a_0 is a free parameter.

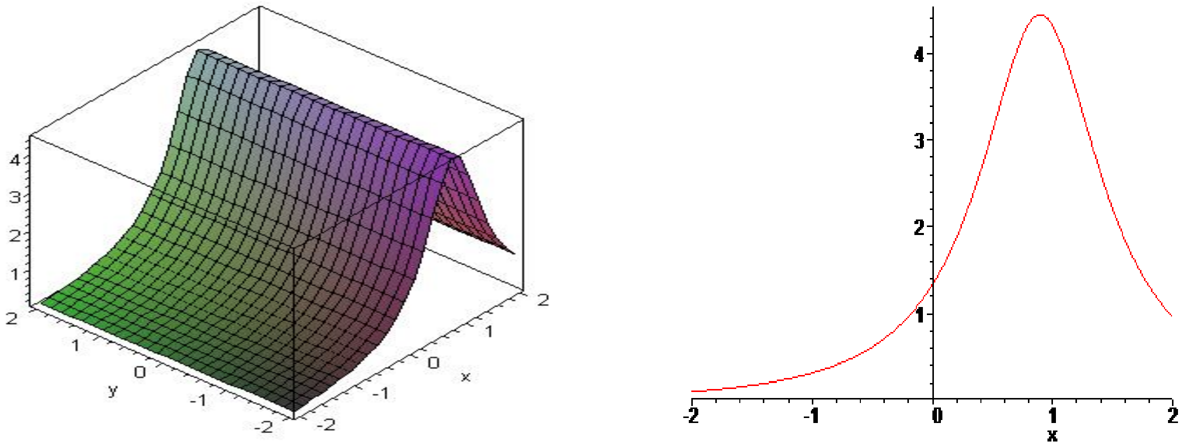


Fig. 13. 3D and 2D plots of travelling wave solutions (Case 5)

The plots indicate the wave solutions for $a_0 = 4$ in Equation (39).

Case 6:

$$a_{-1} = {a_0}^2 - \frac{1}{2}\left(1 + \frac{1}{2} I\sqrt{2}\right){a_0}^2 + \left(1 + \frac{1}{2} I\sqrt{2}\right) b_0 a_0 - \frac{3}{4}\left(1 + \frac{1}{2} I\sqrt{2}\right){b_0}^2 - 2b_0 a_0 + \frac{3}{2}{b_0}^2$$

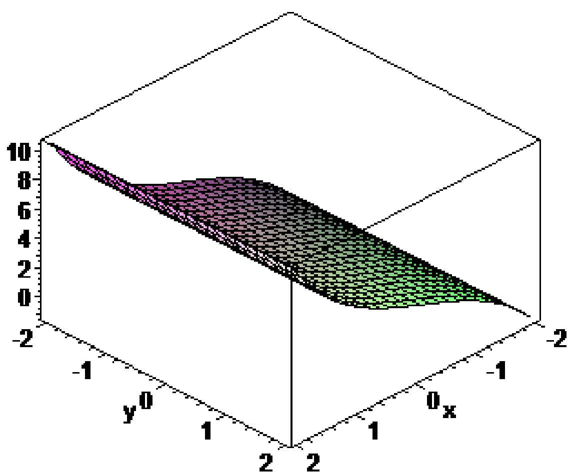
$$a_{-1} = {a_0}^2 - \frac{1}{2}\left(1 - \frac{1}{2} I\sqrt{2}\right){a_0}^2 + \left(1 - \frac{1}{2} I\sqrt{2}\right) b_0 a_0 - \frac{3}{4}\left(1 - \frac{1}{2} I\sqrt{2}\right){b_0}^2 - 2b_0 a_0 + \frac{3}{2}{b_0}^2 \tag{40}$$

$$b_{-1} = -b_0 a_0 + \frac{1}{2}{a_0}^2 + \frac{3}{4}{b_0}^2, w = \frac{3}{2} I\sqrt{2}, w = -\frac{3}{2} I\sqrt{2}, a_1 = 1 - \frac{1}{2} I\sqrt{2}, a_1 = 1 + \frac{1}{2} I\sqrt{2}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{\left({a_0}^2 - \frac{1}{2}\left(1 + \frac{1}{2} I\sqrt{2}\right){a_0}^2 + \left(1 + \frac{1}{2} I\sqrt{2}\right) b_0 a_0 - \frac{3}{4}\left(1 + \frac{1}{2} I\sqrt{2}\right){b_0}^2 - 2b_0 a_0 + \frac{3}{2}{b_0}^2\right) e^{-\xi} + a_0 + (1 - \frac{1}{2} I\sqrt{2}) e^{\xi}}{\left(-b_0 a_0 + \frac{1}{2}{a_0}^2 + \frac{3}{4}{b_0}^2\right) e^{-\xi} + b_0 + e^{\xi}} \tag{41}$$

where b_0 and a_0 are free parameters.



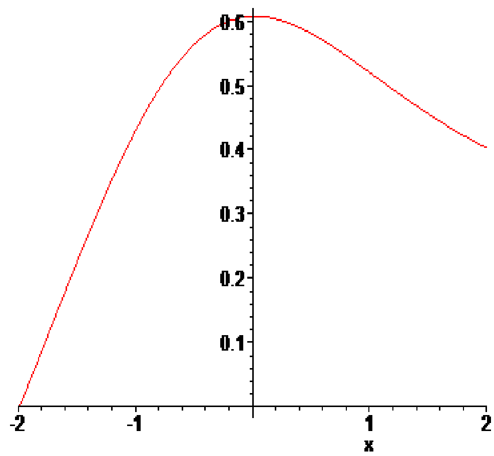


Fig. 14. 3D and 2D plots of travelling wave solutions (Case 6)

The plots indicate the wave solutions for $a_0 = 1, b_0 = 1$ in Equation (41)

Case 7:

$$a_0 = \frac{b_0({a_1}^2 - a_1 + 1)}{-1 + a_1}, w = \sqrt{-6\, a_1 + 3{a_1}^2}, w = -\sqrt{-6\, a_1 + 3{a_1}^2} \tag{42}$$

$$b_{-1} = \frac{1}{8}\frac{{b_0}^2(2{a_1}^2 - 4a_1 + 3)}{(-1 + a_1)^2}, a_{-1} = \frac{1}{8}\frac{{b_0}^2(2{a_1}^2 - 4a_1 + 3)a_1}{(-1 + a_1)^2}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{\left(\frac{1}{8}\frac{{b_0}^2(2{a_1}^2 - 4a_1 + 3)a_1}{(-1 + a_1)^2}\right) e^{-\xi} + \frac{b_0({a_1}^2 - a_1 + 1)}{-1 + a_1} + a_1 e^{\xi}}{\left(\frac{1}{8}\frac{{b_0}^2(2{a_1}^2 - 4a_1 + 3)}{(-1 + a_1)^2}\right) e^{-\xi} + b_0 + e^{\xi}} \tag{43}$$

where b_0 and a_1 are free parameters.

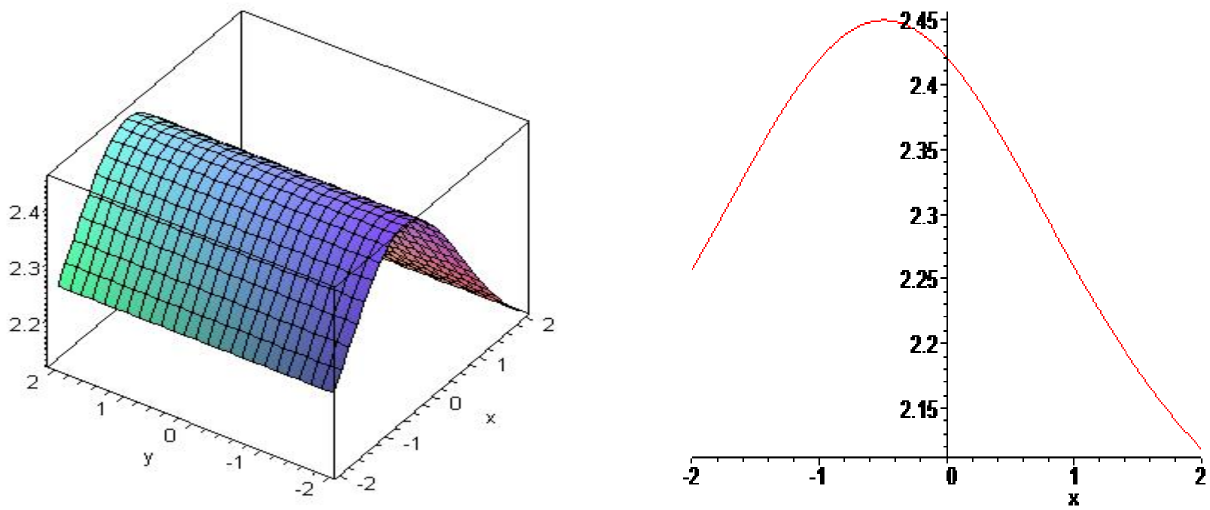


Fig.15. 3D and 2D plots of travelling wave solutions (Case 7)

The plots indicate the wave solutions for $a_1 = 2, b_0 = 1$ in Equation (43).

<u>Case 8:</u>

$$a_{-1} = \frac{1}{8}{a_0}^2, b_0 = 0, \qquad a_1 = 1, w = I\sqrt{3}\,, w = -I\sqrt{3},\ b_{-1} = \frac{1}{8}{a_0}^2 \tag{44}$$

Replace these outcomes into Equation (6), we produce a subsequent exact solution,

$$u(\xi) = \frac{-{a_0}^2 e^{-\xi} + 8a_0 + 8e^{\xi}}{{a_0}^2 e^{-\xi} + 8e^{\xi}} \tag{45}$$

where a_0 is a free parameter.

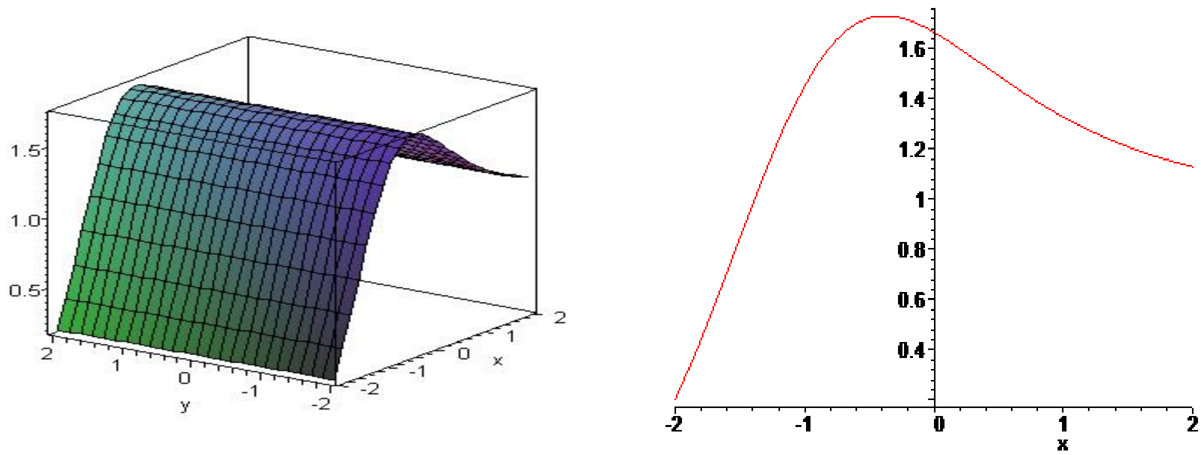


Fig. 16. 3D and 2D plots of travelling wave solutions (Case 8)

The plots indicate the wave solutions for $a_0 = 1$ in Equation (45).

Remark: With the aid of Maple, we have verified all solutions in Section 3 by putting them back into the originals Equations (10) and (11).

4. Conclusion

In this paper, we have been obtained the new exact solution of the Conformable Time Fractional Bad and Good Modified Boussinesq Equations. We converted the Conformable Time Fractional Bad and Good Modified Boussinesq Equations into an ordinary differential equation with the help of a travelling wave transformation. We obtained new exact solutions by using the Exp-function method, which is different from previous literature works. These results show that the Exp-function method is a powerful and effective method to obtain the exact solutions of nonlinear evolution equations born in mathematical physics and non-linear dynamic systems.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

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