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Research Article

A general inequality for warped product CR-submanifolds of Kähler manifolds

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Abstract

In this paper, warped product CR-submanifolds in Kähler manifolds and warped product contact CR-submanifolds in Sasakian, Kenmotsu and cosymplectic manifolds, are shown to possess a geometric property; namely $\mathcal{D}_T$-minimal. Taking benefit from this property, an optimal general inequality is established by means of the Gauss equation, we leave cosyplectic because it is an easy structure. Moreover, a rich geometry appears when the necessity and sufficiency are proved and discussed in the equality case. Applying this general inequality, the inequalities obtained by Munteanu are derived as particular cases. Up to now, the method used by Chen and Munteanu can not extended for general ambient manifolds, this is because many limitations in using Codazzi equation. Hence, Our method depends on the Gauss equation. The inequality is constructed to involve an intrinsic invariant (scalar curvature) controlled by an extrinsic one (the second fundamental form), which provides an answer for the well-know Chen's research problem (Problem 1.1). As further research directions, we have addressed a couple of open problems arose naturally during this work and depending on its results.

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1. Introduction

The notion of warped products has been playing some important roles in the theory of general relativity as they have been providing the best mathematical models of our universe for now. In the mathematical field these have been extensively studied for many years, and recently new types have also been introduced (see for example, [4, 7, 18, 26]). A great interest is also addressed to the CR-warped-product manifolds (see [2, 8, 9, 16, 17, 28, 29]), and the present work is aimed precisely at the latter, especially as submanifolds of Kähler manifolds, also of enormous importance, as well as in the mathematical field, also in string theory.

Another aspect to underline is that the often involved extrinsic and intrinsic Riemannian invariants have wide applications in other fields of science as well. Classically, among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. Among the main intrinsic invariants, sectional, Ricci and scalar curvatures are the well-known ones. So, based on Nash's Theorem, our research programs is to search for control of extrinsic quantities in relation to intrinsic quantities of Riemannian manifolds via Nash's Theorem and to search for their applications [10, 14]. Since it is an inevitable motivation, this was quite enough for Chen to address the following problem.

Problem 1.1. Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold.

Several famous results in differential geometry, such as isoperimetric inequality, Chern-Lashof's inequality, and Gauss-Bonnet's theorem among others, can be regarded as results in this respect. The current paper aims to continue this sequel of inequalities.

Combining special case inequalities in [12], we also have:

Theorem 1.2. *Let $M^n = N_T \times_f N_\perp$ be a CR-warped product submanifold in a complex space form $\tilde{M}^{2m}(c_{Ka})$. Then, we have the following*

$$\frac{1}{2}||h||^2 \geq 2n_1 n_2 \frac{c_{Ka}}{4} + n_2||\nabla \ln f||^2 - n_2\ \Delta(\ln f).$$

The current paper is organized to include eight sections. After the introduction, we present in section two, preliminaries, the basic definitions and formulas. In section three, we prove preparatory basic lemmas, which are necessary and useful for next sections. In the fourth section, it has been shown that warped product CR-submanifolds in Kähler and nearly Kähler manifolds possess a geometric property; namely $\mathcal{D}_T$-minimal submanifolds. Section five is devoted to present the statement and proof of the the main theorem in this article, here we consider warped product CR-submanifolds in complex space form to prove a general inequality involving the scalar curvature and the the squared norm of the second fundamental form. This inequality is derived using the Gauss equation, it generalizes all other inequalities which were derived by means of Codazzi equation. Moreover, it presents a new answer for Problem 1.1. Section six provides many geometric applications, part of them is obtaining the inequalities of [12] as particular case inequalities from our main inequality. In the seventh section, we extend this inequality to generalized complex space form as an ambient manifold. In the final section, we hypothesize two open problems arose naturally due to the results of this work.

2. Preliminaries

Let $\tilde{M}^m$ be a C^∞ real m-dimensional manifold*. The *curvature tensor* $\tilde{R}$ of $\tilde{\nabla}$ is a tensor field of type $(1,\ 3)$ given by

$$\tilde{R}(X,\ Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,\ Y]}Z, \tag{2.1}$$

*Throughout this work, we use the symbol ~ for ambient manifolds, in order to be distinguished from the terminology of submanifolds.

and the $(0,\ 4)$ tensor field defined by

$$\tilde{R}(X,\ Y,\ Z,\ W) = \tilde{g}(\tilde{R}(X,\ Y)Z,\ W) \tag{2.2}$$

is called the *Riemannian curvature tensor*, for any $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^m)$. It is well-known that the Riemannian curvature tensor is a local isometry invariant.

If we choose two linearly independent tangent vectors $X,\ Y \in T_x\tilde{M}^m$, then the *sectional curvature* of the 2-plane π spanned by X and Y is given in terms of the Riemannian curvature tensor $\tilde{R}$ by

$$\tilde{K}(X \wedge Y) = \frac{\tilde{g}(\tilde{R}(X,Y)Y,X)}{\tilde{g}(X,X)\tilde{g}(Y,Y) - (\tilde{g}(X,Y))^2}. \tag{2.3}$$

In case that the 2-plane π is spanned by orthogonal unit vectors X and Y from the tangent space $T_x\tilde{M}^m,\ x \in \tilde{M}^m$, the previous definition may be written as

$$\tilde{K}(\pi) = \tilde{K}_{\tilde{M}^m}(X \wedge Y) = \tilde{g}(\tilde{R}(X,\ Y)Y,\ X). \tag{2.4}$$

Next, consider a local field of orthonormal frames $\{e_1, \cdots, e_m\}$ on $\tilde{M}^m$.

In this context, we shall define another important Riemannian intrinsic invariant called the *scalar curvature* of $\tilde{M}^m$, and denoted by $\tilde{\tau}(T_x\tilde{M}^m)$, which, at some x in $\tilde{M}^m$, is given by

$$\tilde{\tau}(T_x\tilde{M}^m) = \sum_{1\le i<j\le m} \tilde{K}_{ij}, \tag{2.5}$$

where $\tilde{K}_{ij} = \tilde{K}(e_i \wedge e_j)$. It is clear that, equation (2.5) is congruent to

$$2\tilde{\tau}(T_x\tilde{M}^m) = \sum_{1\le i\neq j\le m} \tilde{K}_{ij}. \tag{2.6}$$

In particular, for a 2-dimensional Riemannian manifold, the scalar curvature is its *Gaussian curvature*.

Next, we recall two important differential operators of a differentiable function ψ on $\tilde{M}^m$; namely the *gradient* $\tilde{\nabla}\psi$ and the *Laplacian* $\Delta\psi$ of ψ, which are defined, respectively, as follows:

$$\tilde{g}(\tilde{\nabla}\psi, X) = X\psi \tag{2.7}$$

and

$$\Delta\psi = \sum_{i=1}^{m}((\tilde{\nabla}_{e_i}e_i)\psi - e_ie_i\psi), \tag{2.8}$$

for any vector field X tangent to $\tilde{M}^m$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}^m$. As a consequence, we have:

$$||\tilde{\nabla}\psi||^2 = \sum_{i=1}^{m}\big(e_i(\psi)\big)^2. \tag{2.9}$$

From the integration theory of manifolds, if $\tilde{M}^m$ is orientable compact, then we have:

$$\int_{\tilde{M}^m} \Delta f dV = 0, \tag{2.10}$$

where dV denotes to the volume element of $\tilde{M}^m$.

In an attempt to construct manifolds of negative curvatures, in [5] introduced the notion of *warped product manifolds* as follows:

Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_{N_1} and g_{N_2}, respectively, and $f > 0$ a C^∞ function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \mapsto N_1$ and $\pi_2 : N_1 \times N_2 \mapsto N_2$. Then, the *warped product* $\tilde{M}^m = N_1 \times_f N_2$ is the Riemannian manifold $N_1 \times N_2 = (N_1 \times N_2, \tilde{g})$ equipped with a Riemannian structure such that $\tilde{g} = g_{N_1} + f^2 g_{N_2}$.

A warped product manifold $\tilde{M}^m = N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. For a nontrivial warped product $N_1 \times_f N_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathcal{D}_2$ is obtained by tangent vectors of N_2 via the vertical lift.

Now, let $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_m\}$ be local fields of orthonormal frame of $\Gamma(T\tilde{M}^m)$ such that n_1, n_2 and m are the dimensions of N_1, N_2 and $\tilde{M}^m$, respectively. Then, for any Riemannian warped product $\tilde{M}^m = N_1 \times_f N_2$. It is well known that the sectional curvature and the warping function are related by [10, 14, 15]

$$\sum_{a=1}^{n_1} \sum_{A=n_1+1}^{m} \tilde{K}(e_a \wedge e_A) = \frac{n_2 \Delta f}{f}. \tag{2.11}$$

Now, we turn our attention to the differential geometry of the submanifold theory. The *Gauss* and *Weingarten formulas* are, respectively, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.12}$$

and

$$\tilde{\nabla}_X \zeta = -A_\zeta X + \nabla^\perp_X \zeta \tag{2.13}$$

for all $X, Y \in \Gamma(TM^n)$ and $\zeta \in \Gamma(T^\perp M^n)$, where $\tilde{\nabla}$ and ∇ denote respectively for the Levi-Civita and the *induced* Levi-Civita connections on $\tilde{M}^m$ and M^n, and $\Gamma(TM^n)$ is the module of differentiable sections of the vector bundle TM^n. $\nabla^\perp$ is the *normal connection* acting on the normal bundle $T^\perp M^n$.

Here, it is well-known that the *second fundamental form* h and the *shape operator* A_ζ of M^n are related by

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta) \tag{2.14}$$

for all $X, Y \in \Gamma(TM^n)$ and $\zeta \in \Gamma(T^\perp M^n)$, [3, 25].

Geometrically, M^n is called a *totally geodesic* submanifold in $\tilde{M}^m$ if h vanishes identically. Particularly, the *relative null space*, $\mathcal{N}_x$, of the submanifold M^n in the Riemannian manifold $\tilde{M}^m$ is defined at a point $x \in M^n$ by as

$$\mathcal{N}_x = \{X \in T_x M^n : h(X, Y) = 0 \quad \forall\, Y \in T_x M^n\}. \tag{2.15}$$

Likewise, we consider a local field of orthonormal frames * $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ on $\tilde{M}^m$, such that, restricted to M^n, $\{e_1, \cdots, e_n\}$ are tangent to M^n and $\{e_{n+1}, \cdots, e_m\}$ are normal to M^n. Then, the *mean curvature vector* $\vec{H}(x)$ is introduced as

$$\vec{H}(x) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{2.16}$$

On one hand, we say that M^n is a *minimal submanifold* of $\tilde{M}^m$ if $\vec{H} = 0$. On the other hand, one may deduce that M^n is totally umbilical in $\tilde{M}^m$ if and only if $h(X, Y) = g(X, Y)\vec{H}$, for any $X,\ Y \in \Gamma(TM^n)$. It is remarkable to note that the scalar curvature $\tau(x)$ of M^n at x is identical with the scalar curvature of the tangent space $T_x M^n$ of M^n at x; that is, $\tau(x) = \tau(T_x M^n)$ [10].

In this series, the well-known *equation* of *Gauss* is given by

$$\begin{aligned} R(X, Y, Z, W) &= \tilde{R}(X, Y, Z, W) \\ &+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{2.17}$$

for any vectors $X,\ Y,\ Z,\ W \in \Gamma(TM^n)$, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}^m$ and M^n, respectively.

*Throughout this work, $M^n = N_1 \times_f N_2$ denotes for the isometrically immersed warped product submanifold in $\tilde{M}^m$. The numbers $m,\ n,\ n_1$, and n_2 are the dimensions of $\tilde{M}^m$, M^n, N_1 and N_2, respectively.

From now on, we refer to the coefficients of the second fundamental form h of M^n with respect to the above local frame by the following notation:

$$h^r_{ij} = g(h(e_i, e_j), e_r), \tag{2.18}$$

where $i, j \in \{1, ..., n\}$, and $r \in \{n+1, ..., m\}$. First, by making use of (2.18), (2.17) and (2.4), we get the following:

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^{m} (g(h^r_{ii}\ e_r, h^r_{jj}\ e_r) - g(h^r_{ij}\ e_r, h^r_{ij}\ e_r)). \tag{2.19}$$

Equivalently,

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^{m} (h^r_{ii} h^r_{jj} - (h^r_{ij})^2), \tag{2.20}$$

where $\tilde{K}(e_i \wedge e_j)$ denotes the sectional curvature of the 2-plane spanned by e_i and e_j at x in the ambient manifold $\tilde{M}^m$. Secondly, by taking the summation in the above equation over the orthonormal frame of the tangent space of M^n, and due to (2.5), we immediately obtain:

$$2\tau(T_x M^n) = 2\tilde{\tau}(T_x M^n) + n^2 ||\vec{H}||^2 - ||h||^2, \tag{2.21}$$

where

$$\tilde{\tau}(T_x M^n) = \sum_{1 \le i < j \le n} \tilde{K}(e_i \wedge e_j) \tag{2.22}$$

denotes the scalar curvature of the n-plane $T_x M^n$ in the ambient manifold $\tilde{M}^m$.

For a warped product $M^n = N_1 \times_f N_2$, let $\varphi : M^n \to \tilde{M}^m$ be an isometric immersion of $N_1 \times_f N_2$ into an arbitrary Riemannian manifold $\tilde{M}^m$. As usual, let h be the second fundamental form of φ. We call the immersion φ *mixed totally geodesic* if $h(X, Z) = 0$ for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$, [10]. In particular, if we denote the restrictions of h to N_1 and N_2 respectively by h_1 and h_2, then for $i = 1$ and 2, we call h_i the *partial second fundamental form* of φ. Automatically, the *partial mean curvature vectors* $\vec{H}_1$ and $\vec{H}_2$ are defined by the following partial traces: *

$$\vec{H}_1 = \frac{1}{n_1} \sum_{a=1}^{n_1} h(e_a, e_a), \quad \vec{H}_2 = \frac{1}{n_2} \sum_{A=n_1+1}^{n_1+n_2} h(e_A, e_A) \tag{2.23}$$

for some orthonormal frame fields $\{e_1, \cdots, e_{n_1}\}$ and $\{e_{n_1+1}, \cdots, e_{n_1+n_2}\}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

This motivation for the following definition may not be evident at this moment, but it will emerge gradually as we prove its natural existence, then imposing it to have profoundly general results, [3, 10, 11, 13, 20, 23, 24].

Definition 2.1. An immersion $\varphi : N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ is called $\mathcal{D}_i$-totally geodesic if the partial second fundamental form h_i vanishes identically. If for all $X,\ Y \in \mathcal{D}_i$ we have $h(X,\ Y) = g(X,\ Y)\mathcal{K}$ for some normal vector $\mathcal{K}$, then φ is called $\mathcal{D}_i$-totally umbilical. It is called $\mathcal{D}_i$-minimal if the partial mean curvature vector $\vec{H}_i$ vanishes, for $i = 1$ or 2.

For an odd dimensional real C^∞ manifold $\tilde{M}^{2l+1}$, let ϕ, ξ, η and $\tilde{g}$ be respectively a $(1,\ 1)$ tensor field, a vector field, a 1-form and a Riemannian metric on $\tilde{M}^{2l+1}$ satisfying

$$\left.\begin{array}{ll} \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1 \\ \eta(X) = \tilde{g}(X, \xi), \quad \tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y), \end{array}\right\} \tag{2.24}$$

*Throughout this work, we use the following convention on the range of indices unless otherwise stated, the indices i, j run from 1 to n, the lowercase letters a, b from 1 to n_1, the uppercase letters A, B from n_1 to n and r from n to m.

for any $X,\ Y \in \Gamma(T\tilde{M}^{2l+1})$. Then we call $(\tilde{M}^{2l+1}, \phi, \xi, \eta, \tilde{g})$ an *almost contact metric manifold* and $(\phi, \xi, \eta, \tilde{g})$ an *almost contact metric structure* on $\tilde{M}^{2l+1}$, [6, 19].

A fundamental 2-form Φ is defined on $\tilde{M}^{2l+1}$ by $\Phi(X, Y) = \tilde{g}(\phi X, Y)$. An almost contact metric manifold $\tilde{M}^{2l+1}$ is called a contact metric manifold if $\Phi = \frac{1}{2}d\eta$. If the almost contact metric manifold $(\tilde{M}^{2l+1}, \phi, \xi, \eta, \tilde{g})$ satisfies $[\phi, \phi]+2d\eta\otimes\xi = 0$, then $(\tilde{M}^{2l+1}, \phi, \xi, \eta, \tilde{g})$ turns out to be a *normal almost contact manifold*, where the Nijenhuis tensor is defined as

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] \qquad \forall\ X,\ Y \in \Gamma(T\tilde{M}^{2l+1}).$$

For our purpose, we will distinguish four classes of almost contact metric structures; namely, Sasakian, Kenmotsu, cosymplectic and nearly trans-Sasakian structures. At first, an almost contact metric structure is is said to be *Sasakian* whenever it is both contact metric and normal, equivalently [27]

$$(\tilde{\nabla}_X\phi)Y = -\tilde{g}(X, Y)\xi + \eta(Y)X. \tag{2.25}$$

A 2-plane π in $T_x\tilde{M}^{2l+1}$ of an almost metric manifold $\tilde{M}^{2l+1}$ is called a ϕ-section if $\pi \perp \xi$ and $\phi(\pi) = \pi$. Accordingly, we say that $\tilde{M}^{2l+1}$ is of constant ϕ-sectional curvature if the sectional curvature $\tilde{K}(\pi)$ does not depend on the choice of the ϕ-section π of $T_x\tilde{M}^{2l+1}$ and the choice of a point $x \in \tilde{M}^{2l+1}$. Based on this preparatory concept, a Sasakian manifold $\tilde{M}^{2l+1}$ is said to be a *Sasakian space form* $\tilde{M}^{2l+1}(c_S)$, if the ϕ-sectional curvature is constant c_S along $\tilde{M}^{2l+1}$. Then the associated Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}^{2l+1}(c_S)$ is given by [19]:

$$\begin{aligned}\tilde{R}(X, Y; Z, W) = \frac{c_S + 3}{4}&\Big\{\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W)\Big\}\\ -\frac{c_S - 1}{4}&\Big\{\eta(Z)\Big(\eta(Y)\tilde{g}(X, W) - \eta(X)\tilde{g}(Y, W)\Big)\\ &+\Big(\tilde{g}(Y, Z)\eta(X) - \tilde{g}(X, Z)\eta(Y)\Big)\tilde{g}(\xi, W)\\ -\tilde{g}(\phi X, W)\tilde{g}(\phi Y, Z) &+ \tilde{g}(\phi X, Z)\tilde{g}(\phi Y, W) + 2\tilde{g}(\phi X, Y)\tilde{g}(\phi Z, W)\Big\},\end{aligned} \tag{2.26}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2l+1}(c_S))$.

An almost contact metric manifold $\tilde{M}^{2l+1}$ is called *Kenmotsu manifold* [19] if

$$(\tilde{\nabla}_X\phi)Y = \tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.27}$$

By analogy with Sasakian manifolds, a Kenmotsu manifold $\tilde{M}^{2l+1}$ is said to be a *Kenmotsu space form* $\tilde{M}^{2l+1}(c_{Ke})$, if the ϕ-sectional curvature is constant c_{Ke} along $\tilde{M}^{2l+1}$, whose Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}^{2l+1}(c_{Ke})$ is characterized by [1]:

$$\begin{aligned}\tilde{R}(X, Y; Z, W) = \frac{c_{Ke} - 3}{4}&\Big\{\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W)\Big\}\\ -\frac{c_{Ke} + 1}{4}&\Big\{\eta(Z)\Big(\eta(Y)\tilde{g}(X, W) - \eta(X)\tilde{g}(Y, W)\Big)\\ &+\Big(\tilde{g}(Y, Z)\eta(X) - \tilde{g}(X, Z)\eta(Y)\Big)\tilde{g}(\xi, W)\\ -\tilde{g}(\phi X, W)\tilde{g}(\phi Y, Z) &+ \tilde{g}(\phi X, Z)\tilde{g}(\phi Y, W) + 2\tilde{g}(\phi X, Y)\tilde{g}(\phi Z, W)\Big\},\end{aligned} \tag{2.28}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2l+1}(c_{Ke}))$. We notice that Kenmotsu manifolds are normal but not quasi-Sasakian and hence not Sasakian [6].

In the case of killing almost contact structure tensors, consider a normal almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ with both Φ and η are closed. Then, such $(\phi, \xi, \eta, \tilde{g})$ is called *cosymplectic*. Explicitly, cosymplectic manifolds are characterized by normality and the vanishing of Riemannian covariant derivative of ϕ, i.e.,

$$(\tilde{\nabla}_X \phi)Y = 0. \tag{2.29}$$

A cosymplectic manifold $\tilde{M}^{2l+1}$ is said to be a *cosymplectic space form* $\tilde{M}^{2l+1}(c_c)$, if the ϕ-sectional curvature is constant c_c along $\tilde{M}^{2l+1}$ with Riemannian curvature tensor $\tilde{R}$ expressed by [6]:

$$\begin{aligned}\tilde{R}(X,Y;Z,W) = \frac{c_c}{4}\Big\{&\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W)\\ &-\eta(Z)\Big(\eta(Y)\tilde{g}(X,W) - \eta(X)\tilde{g}(Y,W)\Big) - \Big(\tilde{g}(Y,Z)\eta(X) - \tilde{g}(X,Z)\eta(Y)\Big)\tilde{g}(\xi,W)\\ &+\tilde{g}(\phi X,W)\tilde{g}(\phi Y,Z) - \tilde{g}(\phi X,Z)\tilde{g}(\phi Y,W) - 2\tilde{g}(\phi X,Y)\tilde{g}(\phi Z,W)\Big\},\end{aligned} \tag{2.30}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2l+1}(c_c))$. Hereafter, we call the almost contact manifold $\tilde{M}^{2l+1}$ a *nearly cosymplectic* manifold if:

$$(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = 0. \tag{2.31}$$

A submanifold M^n of an almost contact metric manifold $\tilde{M}^{2l+1}$ is said to be a *contact CR-submanifold* if there exist on M^n differentiable distributions $\mathcal{D}_T$ and $\mathcal{D}_\perp$, satisfying the following

(i) $TM^n = \mathcal{D}_T \oplus \mathcal{D}_\perp \oplus \langle\xi\rangle$,
(ii) $\mathcal{D}_T$ is an invariant distribution, i.e., $\phi(\mathcal{D}_T) \subseteq \mathcal{D}_T$,
(iii) $\mathcal{D}_\perp$ is an anti-invariant distribution, i.e., $\phi(\mathcal{D}_\perp) \subseteq T^\perp M^n$.

Denote by ν the maximal ϕ-invariant subbundle of the normal bundle $T^\perp M^n$. Then it is well-known that the normal bundle $T^\perp M^n$ admits the following decomposition

$$T^\perp M^n = F\mathcal{D}_\perp \oplus \nu. \tag{2.32}$$

In almost contact manifolds $\tilde{M}^{2m+1}$, the warped product $N_T \times_f N_\perp$ is called a *CR-warped product* submanifold, if the submanifolds N_T and $N_\perp$ are integral manifolds of $\mathcal{D}_T$ and $\mathcal{D}_\perp$, respectively.

3. Basic lemmas

Now, we turn our attention to almost contact manifolds, we are going to explain the natural existence of $\mathcal{D}_i$-minimal warped product submanifolds in almost contact manifolds, for both $i = 1$ and $i = 2$. Observe that all almost contact manifolds considered in this thesis satisfy $(\tilde{\nabla}_\xi \phi)\xi = 0$. Hence, it is convenient to state:

Lemma 3.1. *Let M^n be a submanifold tangent to the characteristic vector field ξ in an almost contact manifold $\tilde{M}^{2l+1}$. If $(\tilde{\nabla}_\xi \phi)\xi = 0$ on $\tilde{M}^{2l+1}$, then $h(\xi,\xi) = 0$.*

Beginning with Sasakian manifolds, we call a warped product of type $M^n = N_T \times_f N_\perp$, a contact CR- warped product submanifold.

Corollary 3.2. *Let $M^n = N_T \times_f N_\perp$ be a contact CR- warped product submanifold in a Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, the following hold:*

(i) $h(X,\xi) = 0$;
(ii) $g(h(X,X), FZ) = 0$;
(iii) $g(h(X,X),\zeta) = -g(h(\phi X,\phi X),\zeta)$,

for every $X \in \Gamma(TN_T)$, $Z \in \Gamma(TN_\perp)$ and $\zeta \in \Gamma(\nu)$.

Proof. From (2.25) we obtain:

$$X-\eta(X)\xi=-\phi\nabla_X\xi-\phi h(X,\xi).$$

Applying ϕ on the above equation, taking into consideration $\eta(\nabla_X\xi)=0$, then it yields

$$\phi X=\nabla_X\xi+h(X,\xi).$$

By comparing the tangential and normal terms in the above equation we get (i). (ii) is well-known (for example, see [21, 22]). For the last part, we take an arbitrary $\zeta\in\Gamma(\nu)$, then by making use of (2.25) and (2.12), we obtain

$$\nabla_X\phi X+h(\phi X,X)-\phi\nabla_X X-\phi h(X,X)=-g(X,X)\xi+\eta(X)X,$$

taking the inner product with $\phi\zeta$ in the above equation, we deduce:

$$g(h(\phi X,X),\phi\zeta)-g(h(X,X),\zeta)=0, \tag{3.1}$$

interchanging X with ϕX in (3.1), gives

$$\begin{aligned} g(h(\phi X,\phi X),\zeta)&=g(h(\phi(\phi X),\phi X),\phi\zeta)=g(\tilde\nabla_{\phi X}\phi(\phi X),\phi\zeta)\\ &=-g(\tilde\nabla_{\phi X}X,\phi\zeta)+g(\tilde\nabla_{\phi X}(\eta(X)\xi),\phi\zeta)\\ &=-g(h(X,\phi X),\phi\zeta)+\eta(X)g(\tilde\nabla_{\phi X}\xi,\phi\zeta)\\ &=-g(h(X,\phi X),\phi\zeta)+\eta(X)g(h(\phi X,\xi),\phi\zeta). \end{aligned}$$

Making use of statement (i) in the above equation, we reach that

$$g(h(\phi X,\phi X),\zeta)=-g(h(X,\phi X),\phi\zeta). \tag{3.2}$$

From (3.1) and (3.2), we obtain statement (iii). $\square$

The following two direct, but significant, results are two other key lemma for this section that will be used later as well.

Lemma 3.3. *Let $\varphi: M^n=N_1\times_f N_2\longrightarrow\tilde M^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde M^m$. Then, we have:*

$$\begin{aligned}\tau(T_xM^n)=\frac{n_2\Delta f}{f}+\sum_{r=n+1}^{m}\Bigg\{\sum_{1\le a<b\le n_1}\Big(h^r_{aa}h^r_{bb}-(h^r_{ab})^2\Big)\\ +\sum_{n_1+1\le A<B\le n}\Big(h^r_{AA}h^r_{BB}-(h^r_{AB})^2\Big)\Bigg\}+\tilde\tau(T_xN_1)+\tilde\tau(T_xN_2),\end{aligned} \tag{3.3}$$

where $n_1,\ n_2,\ n$ and m are the dimensions of $N_1,\ N_2,\ M^n$ and $\tilde M^m$, respectively.

Proof. From the definition of the scalar curvature, we have:

$$\tau(T_xM^n)=\sum_{1\le i<j\le n}K_{ij}=\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}K_{aA}+\sum_{1\le a<b\le n_1}K_{ab}+\sum_{n_1+1\le A<B\le n}K_{AB}. \tag{3.4}$$

Now, we recall the following well-known relation

$$\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}K(e_a\wedge e_A)=\frac{n_2\Delta f}{f}, \tag{3.5}$$

where $\{e_1,\cdots,e_{n_1},e_{n_1+1},\cdots,e_n\}$ are local fields of orthonormal frame of $\Gamma(TM^n)$ such that $n_1,\ n_2$ and n are the dimensions of $N_1,\ N_2$ and M^n, respectively. Combining the above two equations, it yields

$$\tau(T_xM^n)=\frac{n_2\Delta f}{f}+\tau(T_xN_1)+\tau(T_xN_2). \tag{3.6}$$

It is direct to write

$$\tau(T_xN_1) = \sum_{r=n+1}^{m} \sum_{1\le a<b\le n_1} \left(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\right) + \tilde{\tau}(T_xN_1), \tag{3.7}$$

and

$$\tau(T_xN_2) = \sum_{r=n+1}^{m} \sum_{n_1+1\le A<B\le n} \left(h^r_{AA}h^r_{BB} - (h^r_{AB})^2\right) + \tilde{\tau}(T_xN_2). \tag{3.8}$$

By joining (3.6), (3.7) and (3.8) together, we get the result. □

Lemma 3.4. *Let φ be a $\mathfrak{D}_2$-minimal isometric immersion of a warped product submanifold $M^n = N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}^m$. If N_2 is totally umbilical in $\tilde{M}^m$, then φ is $\mathfrak{D}_2$-totally geodesic.*

Proof. Let $\check{h}$ and $\hat{h}$ denote the second fundamental forms of N_2 in M^n and $\tilde{M}^m$, respectively. Then for every vector fields Z and W tangent to N_2 we have

$$h(Z,W) = \hat{h}(Z,W) - \check{h}(Z,W), \tag{3.9}$$

and

$$\check{h}(Z,W) = -(g(Z,W)/f)\nabla(f). \tag{3.10}$$

Notice that, for every warped product the leaves are totally geodesic and the fibers are totally umbilical. Taking in consideration this fact and our hypothesis guarantees that N_2 is totally umbilical in both M^n and $\tilde{M}^m$. Considering this fact with the above two equations, we deduce that

$$h(Z,W) = g(Z,W)(\Psi + \nabla(\ln f)), \tag{3.11}$$

for some vector field $\Psi \in \Gamma(T\tilde{M}^m)$ such that Ψ is normal to $\Gamma(TN_2)$. Considering the local field of orthonormal frames as in the above proof. Then, taking the summation over the orthonormal frame fields of $\Gamma(TN_2)$ in the above equation, we get

$$\sum_{A,B=n_1+1}^{n} h(e_A,e_B) = \sum_{A,B=n_1+1}^{n} g(e_A,e_B)(\Psi + \nabla(\ln f)).$$

Taking into account $\mathfrak{D}_2$-minimality of φ, the left hand side of the above equation vanishes and we get

$$0 = n_2\ (\Psi + \nabla(\ln f)).$$

Since N_2 is not empty, we obtain

$$\Psi = -\nabla(\ln f).$$

Making use of the above equation in (3.11), we obtain

$$h(Z,W) = 0,$$

for every vector fields $Z, W \in \Gamma(TN_2)$. Meaning that; φ is $\mathfrak{D}_2$-totally geodesic. This completes the proof. □

4. $\mathcal{D}_T$-minimality of warped product CR-submanifolds in Kähler manifolds

Recently, it was proven that $\mathcal{D}_T$-minimality is possessed by a wide class of warped product submanifolds, some of these warped product submanifolds were shown to have this geometric property in [23, 24].

In the sense of Definition 2.1, we are going to show the natural existence of $\mathcal{D}_T$-minimal warped product CR-submanifolds in both Kähler and nearly Kähler manifolds.

Secondly, we provide the next key result which will be referred to frequently during this section.

Lemma 4.1. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold in Sasakian manifolds $\tilde{M}^{2l+1}$ such that ξ is tangent to N_T. Then, M^n is $\mathcal{D}_1$-minimal warped product, where $\mathcal{D}_1 = \mathcal{D}_T \oplus \langle \xi \rangle$.*

Proof. Consider the following local field of orthonormal frames of the Kähler manifold $\tilde{M}^{2m}$: $\{\xi, e_1, \cdots, e_s, e_{s+1} = \phi e_1, \cdots, e_{n_1} = e_{2s} = \phi e_s, e_{n_1+1} = e_1^\star, \cdots, e_{n_1+n_2} = e_n = e_q^\star, e_{n+1} = \phi e_1^\star, \cdots, e_{n+n_2} = \phi e_q^\star, e_{n+n_2+1} = \bar{e}_1, \cdots, e_{2m} = \bar{e}_{2l=\gamma}\}$ such that $\{e_1, \cdots, e_s, e_{s+1} = \phi e_1, \cdots, e_{n_1} = e_{2s} = \phi e_s\}$, $\{e_{n_1+1} = e_1^\star, \cdots, e_{n_1+n_2} = e_n = e_q^\star\}$, $\{e_{n+1} = \phi e_1^\star, \cdots, e_{n+n_2} = \phi e_q^\star\}$ and $\{e_{n+n_2+1} = \bar{e}_1, \cdots, e_{2m} = \bar{e}_{2l=\gamma}\}$ are the local fields of orthonormal frames of $\Gamma(TN_T)$, $\Gamma(TN_\perp)$, $\Gamma(JTN_\perp)$ and $\Gamma(\nu)$, respectively.

Using the terminology in (2.18), it is straightforward to have

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1}^{2m} (h_{11}^r + \cdots + h_{n_1 n_1}^r).$$

In view of (2.32), the right hand side summation can be decomposed as

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1}^{2m-\gamma} (h_{11}^r + \cdots + h_{n_1 n_1}^r) + \sum_{r=n+1+q}^{2m} (h_{11}^r + \cdots + h_{n_1 n_1}^r).$$

Taking into account part (i) of Corollary 3.2, the first summation on the right hand side of the above equation vanishes, whereas we expand the other summation in view of the above orthonormal frames to get

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1+q}^{2m} (h_{11}^r + \cdots + h_{ss}^r + h_{s+1s+1}^r + \cdots + h_{2s2s}^r).$$

Equivalently,

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1+q}^{2m} \Big(g(h(e_1, e_1), e_r) + \cdots + g(h(e_s, e_s), e_r)$$

$$+ g(h(Je_1, Je_1), e_r) + \cdots + g(h(Je_s, Je_s), e_r) \Big).$$

Now, if we apply part (ii) of Corollary 3.2 on the above equation, then it automatically gives

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1+q}^{2m} \Big(g(h(e_1, e_1), e_r) + \cdots + g(h(e_s, e_s), e_r)$$

$$- g(h(e_1, e_1), e_r) - \cdots - g(h(e_s, e_s), e_r) \Big)$$

$$= 0.$$

Clearly, this proves the vanishing of the coefficients h_{aa}^r under summation, for $a \in \{1, \cdots, n_1\}$ and $r \in \{n+1, \cdots, 2m\}$. Therefore, the partial mean curvature vector $\vec{H}$ defined in (2.23) does vanish. Hence, in the sense of Definition 2.1, we get the assertion. □

Remark 4.2. Putting $\mathcal{D}_1 = \mathcal{D}_T$, then by following the above scheme typically one can show that warped product submanifolds of the type $M^n = N_T \times_f N_\perp$, are $\mathcal{D}_1$-minimal in nearly Kähler manifolds.

5. A general inequality for warped product CR-submanifolds in Kähler manifolds

By making use of the Gauss equation, we construct a new general inequality for $\mathcal{D}_T$-minimal warped product CR-submanifolds in arbitrary Kähler manifolds. This inequality generalizes all inequalities in [12].

Now, we present the main theorem of this article.

Theorem 5.1. *Let $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^m$ be an isometric immersion of a warped product CR-submanifold M^n into a Kähler manifold $\tilde{M}^m$. Then, we have:*

(i) $\frac{1}{2}||h||^2 \geq \tilde{\tau}(T_xM^n) - \tilde{\tau}(T_xN_T) - \tilde{\tau}(T_xN_\perp) - \frac{n_2\Delta f}{f}$.

(ii) *The equality in (i) holds identically if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$, respectively.*

Proof. Via (2.21), we first have:

$$||h||^2 = -2\tau(T_xM^n) + 2\tilde{\tau}(T_xM^n) + n^2||\vec{H}||^2.$$

In view of Lemma 3.3, the above equation takes the following form

$$\begin{aligned}||h||^2 &= 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_T) - 2\tilde{\tau}(T_xN_\perp) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2\\ &-2\left(\sum_{r=n+1}^{m}\sum_{1\leq a<b\leq n_1}(h^r_{aa}h^r_{bb} - (h^r_{ab})^2)\right)\\ &-2\left(\sum_{r=n+1}^{m}\sum_{n_1+1\leq A<B\leq n}(h^r_{AA}h^r_{BB} - (h^r_{AB})^2)\right).\end{aligned}$$

This is equivalent to

$$\begin{aligned}||h||^2 &= 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_T) - 2\tilde{\tau}(T_xN_\perp) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2\\ &-\left(\sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}(h^r_{aa}h^r_{bb} - (h^r_{ab})^2)\right)\\ &-\left(\sum_{r=n+1}^{m}\sum_{n_1+1\leq A\neq B\leq n}(h^r_{AA}h^r_{BB} - (h^r_{AB})^2)\right).\end{aligned}\tag{5.1}$$

Since φ is $\mathcal{D}_T$-minimal immersion, then

$$-\left(\sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}(h^r_{aa}h^r_{bb} - (h^r_{ab})^2)\right) =$$

$$\sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}(h^r_{ab})^2 - \sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}h^r_{aa}h^r_{bb} =$$

$$\overbrace{\sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}(h^r_{ab})^2 + \left(\sum_{r=n+1}^{m}((h^r_{11})^2 + \cdots + (h^r_{n_1n_1})^2)\right)}$$

$$\overbrace{-\left(\sum_{r=n+1}^{m}((h^r_{11})^2 + \cdots + (h^r_{n_1n_1})^2)\right) - \sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}h^r_{aa}h^r_{bb}}.$$

By means of the binomial theorem, we deduce that

$$\overbrace{\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}(h^r_{ab})^2+\left(\sum_{r=n+1}^{m}((h^r_{11})^2+\cdots+(h^r_{n_1n_1})^2)\right)}=\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2,$$

and

$$\overbrace{-\left(\sum_{r=n+1}^{m}((h^r_{11})^2+\cdots+(h^r_{n_1n_1})^2)\right)-\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}h^r_{aa}h^r_{bb}}=$$
$$-\sum_{r=n+1}^{m}(h^r_{11}+\cdots+h^r_{n_1n_1})^2.$$

Next, by combining the last three equations together we obtain:

$$-\left(\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}(h^r_{aa}h^r_{bb}-(h^r_{ab})^2)\right)=\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2-\sum_{r=n+1}^{m}(h^r_{11}+\cdots+h^r_{n_1n_1})^2. \tag{5.2}$$

By Definition 2.1, the second term in the right hand side vanishes whenever φ is $\mathcal{D}_T$-minimal, consequently (5.2) reduces to

$$-\left(\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}(h^r_{aa}h^r_{bb}-(h^r_{ab})^2)\right)=\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2. \tag{5.3}$$

Combining (5.3) and (5), it yields to

$$\begin{aligned}||h||^2 &= 2\tilde\tau(T_xM^n)-2\tilde\tau(T_xN_T)-2\tilde\tau(T_xN_\perp)-2\frac{n_2\Delta f}{f}+n^2||\vec H||^2\\ &\quad+\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2\\ &\quad-\left(\sum_{r=n+1}^{m}\sum_{n_1+1\le A\ne B\le n}(h^r_{AA}h^r_{BB}-(h^r_{AB})^2)\right).\end{aligned}$$

Equivalently,

$$\begin{aligned}||h||^2 &\ge 2\tilde\tau(T_xM^n)-2\tilde\tau(T_xN_T)-2\tilde\tau(T_xN_\perp)-2\frac{n_2\Delta f}{f}+n^2||\vec H||^2\\ &\quad-\left(\sum_{r=n+1}^{m}\sum_{n_1+1\le A\ne B\le n}(h^r_{AA}h^r_{BB}-(h^r_{AB})^2)\right).\end{aligned}$$

Again, by adding and subtracting similar term technique, the above inequality becomes:

$$\begin{aligned}||h||^2 &\ge 2\tilde\tau(T_xM^n)-2\tilde\tau(T_xN_T)-2\tilde\tau(T_xN_\perp)-2\frac{n_2\Delta f}{f}+n^2||\vec H||^2\\ &\quad-\sum_{r=n+1}^{m}\left((h^r_{n_1+1n_1+1})^2+\cdots+(h^r_{nn})^2+\sum_{n_1+1\le A\ne B\le n}h^r_{AA}h^r_{BB}\right)\\ &\quad+\sum_{r=n+1}^{m}\left((h^r_{n_1+1n_1+1})^2+\cdots+(h^r_{nn})^2+\sum_{n_1+1\le A\ne B\le n}(h^r_{AB})^2\right).\end{aligned}$$

Applying the binomial theorem on the last two terms of the above equation, we derive that:

$$||h||^2\ge 2\tilde\tau(T_xM^n)-2\tilde\tau(T_xN_T)-2\tilde\tau(T_xN_\perp)-2\frac{n_2\Delta f}{f}+n^2||\vec H||^2$$

$$- \sum_{r=n+1}^{m} (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2 + \sum_{r=n+1}^{m} \sum_{A,B=n_1+1}^{n} (h^r_{AB})^2.$$

Consequently,

$$||h||^2 \geq 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_T) - 2\tilde{\tau}(T_xN_\perp) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2 - \sum_{r=n+1}^{m} (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2.$$

We know that the last term in the right hand side of the above inequality is equal to $-n^2||\vec{H}||^2$ for $\mathcal{D}_T$-minimal warped product CR-submanifolds. By this fact, the inequality of statement (i) follows immediately from the above inequality.

Now, the equality sign of the inequality in (i) holds if and only if

$$(a)\ h(\mathcal{D}_T, \mathcal{D}_T) = 0, \qquad (b)\ h(\mathcal{D}_\perp, \mathcal{D}_\perp) = 0.$$

Hence, we need to show that (a) and (b) hold if and only if N_T, $N_\perp$ and M^n are respectively totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$.

First, assume that (a) and (b) are satisfied. Since $M^n = N_T \times_f N_\perp$ is a warped product, then N_T and $N_\perp$ are totally geodesic and totally umbilical in M^n, respectively. Therefore, part (a) above implies that the first factor is a totally geodesic submanifold in $\tilde{M}^m$. The second factor is totally umbilical in $\tilde{M}^m$ because of part (b). Moreover, (b) and (a) together imply that M^n is minimal in $\tilde{M}^m$.

For the converse, (a) is clear. To obtain (b), we first notice that minimality and $\mathcal{D}_T$-minimality of M^n in $\tilde{M}^m$ yield to $\mathcal{D}_\perp$-minimality of M^n in $\tilde{M}^m$. Hence, Lemma 3.4 proves (b). This gives the assertion. □

6. Special inequalities and applications

As a first application, we embark on by deriving the three theorems of [12] from Theorem 5.1 to be particular case theorems. For this, consider the warped product CR-submanifolds of type $N_T \times_f N_\perp$ in complex space forms. Since the ambient manifold $\tilde{M}^m$ of Theorem 5.1 is an arbitrary Kähler manifold, we can consider $\tilde{M}^m$ to be a complex space form $\tilde{M}^{2m}(c_{Ka})$. Hence, for every CR-warped product $M^n = N_T \times_f N_\perp$ in $\tilde{M}^{2m}(c_{Ka})$, we just use the curvature tensor of complex space forms ([12]) to compute the following:

$$\begin{aligned} 2\Big(\tilde{\tau}(T_xM^n) - \tilde{\tau}(T_xN_1) - \tilde{\tau}(T_xN_2)\Big) &= \frac{c_{Ka}}{4}\Big(n(n-1) + 3n_1 - n_1(n_1-1) - 3n_1 - n_2(n_2-1)\Big) \\ &= \frac{c_{Ka}n_1n_2}{2}. \end{aligned}$$

Substituting the above expression in Theorem 5.1, because CR-warped product submanifolds of Kaehler manifolds are $\mathfrak{D}_1$-minimal, we obtain the following theorem as special case.

Theorem 6.1. *Let $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^m$ be an isometric immersion of a warped product CR-submanifold M^n into a complex space form $\tilde{M}^m$. Then, we have:*

(i) $||h||^2 \geq 2n_2\Big(||\nabla(\ln f)||^2 - \Delta(\ln f) + \frac{c_S+3}{2}s + 1\Big).$

(ii) *The equality in (i) holds identically if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$, respectively.*

Remark 6.2. Inequalities of Theorems 4.1, 5.1 and 6.1 in [12] are special cases of Theorem 5.1, where the ambient manifold is a complex Euclidean, a complex projective and a complex hyperbolic space, respectively.

As another application of Theorem 5.1, we have:

Corollary 6.3. *Let $M^n = N_T \times_f N_\perp$ be a warped product CR-submanifold in a Kähler manifold $\tilde{M}^m$ and suppose N_T is compact. Denote by dv_T and $vol(N_T)$ the volume element and the volume on N_T. Let λ_T be the first non zero eigenvalue of the Laplacian on N_T. Then*

$$\frac{1}{2}\int_{N_T} ||h||^2 dv_T \geq n_1\Big(\tilde{\tau}(T_xM) - \tilde{\tau}(T_xN_T) - \tilde{\tau}(T_xN_\perp)\Big)vol(N_T) + n_1\lambda_T\int_{N_T}(\ln f)^2 dv_T.$$

Proof. From the minimum principle we have

$$\int_{N_T} ||\nabla \ln f||^2 dv_T \geq \lambda_T \int_{N_T} (\ln f)^2 dv_T.$$

Now we have to integrate on N_T the inequality of Theorem 5.1 which is satisfied by the norm of h, and then we obtain immediately the result. □

Above integration over N_T can be generalized to integration of a general measurable manifold with invariance properties. For this we will state the following:

Theorem 6.4. *Let M_μ be a measurable manifold with a measure μ defined on it. Moreover, let $g : \mu \to \mu'$ be an invariance transformation from measure μ to measure μ'. Then, we can express the integral $\int_{M_\mu} X$ over a quantity X as the limit $\int_{M_\mu} = lim_{g\mapsto id}\sum_{x\in M_\mu}\mu(x)X(gx)$ where x is an element of the manifold, here, the covering basis of it and id is the identity operator.*

Proof. Consider two values of a quantity X, namely $X(gx)$ and $X(x)$ for any manifold covering x. The transformation g will now tend to the identity transform.
Thus, $X(gx) - X(x)$ will be infinitesimal in the case when the function is smooth. In non-smooth case, the transformation g will shift the covering x from the singularity apart by appropriate choice of it. Since the manifold M_μ is measurable, we can define a measure on it and can also compute a measure-weighted sum over X. □

7. An extension of the inequality to warped product CR-submanifolds in nearly Kähler manifolds

Theorem 7.1. *Let $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^m$ be an isometric immersion of a warped product CR-submanifold M^n into a nearly Kähler manifold $\tilde{M}^m$. Then, we have:*

(i) $||h||^2 \geq 2n_2\Big(\frac{c_{K_e}-3}{2}s - \Delta(\ln f)\Big)$.

(ii) *The equality in (i) holds identically if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$, respectively.*

Following a similar analogue of the previous section, we can use the above theorem to obtain a special inequality of generalized complex space forms.

Theorem 7.2. *Let $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^m$ be an isometric immersion of a warped product CR-submanifold M^n into a generalized complex space form $\tilde{M}^m$. Then, we have:*

(i) $||h||^2 \geq 2n_2\Big(||\nabla(\ln f)||^2 - \Delta(\ln f) + n_1\frac{c_{RK}+3\gamma}{4}\Big)$.

(ii) *The equality in (i) holds identically if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$, respectively.*

It is clear that the above theorem generalizes Theorem 6.1. To see that, just let γ vanish.

8. Research problems based on the main inequality: Theorem 5.1

Due to the results of this paper, we hypothesize a pair of open problems, the first is about proving this inequality whereas the second is to classify warped products CR-submanifolds.

Firstly, since warped product CR-submanifolds do exist if the ambient manifold is locally conformal Kähler space form, we suggest the following:

Problem 8.1. Prove the above inequality for warped product CR-submanifolds in locally conformal Kähler space forms.

Secondly, we asked:

Problem 8.2. Can we classify warped product CR-submanifolds satisfying the equality cases of this inequality in locally conformal Kähler space forms?

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