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Existence and controllability of fractional evolution inclusions with impulse and sectorial operator

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Abstract

Many evolutionary operations from diverse fields of engineering and physical sciences go through abrupt modifications of state at specific moments of time among periods of non-stop evolution. These operations are more conveniently modeled via impulsive differential equations and inclusions. In this work, firstly we address the existence of mild solutions for nonlocal fractional impulsive semilinear differential inclusions related to Caputo derivative in Banach spaces when the linear part is sectorial. Secondly, we determine the enough, conditions for the controllability of the studied control problem. We apply effectively fixed point theorems, contraction mapping, multivalued analysis and fractional calculus. Moreover, we enhance our results by introducing an illustrative examples.

Keywords: Controllability Contraction mapping Impulsive fractional differential inclusions Fixed point theorem Sectorial operators
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1. Introduction

Due to the widespread use of fractional calculus in applications, fractional equations and inclusions have appeared as an effective paradigm for the advent of recent ideas in mathematics, engineering and physics, further to their potential to describe the dynamic demeanor for the real life. The differential systems of fractional order are richer in summarizing problems than their corresponding ones in classical systems, as these models play a paramount function in describing memory and genetic traits of many substances and the start of applications in control theory, polymer biology, aerodynamics, electrical dynamics, nonlinear oscillation of earthquake and different other scientific disciplines. Such models are an efficient substitutional

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to non-linear differential equations and inclusions, and this has speeded up the researchers keenness to provide solution to these equations and inclusions and broadened the range of techniques that use fixed point methods. We suggest for readers the preceding studies [20, 23, 26, 29, 30, 34] and the references included within.

We add to this study the notion of impulsive differential systems, that interprets many operations of evolution, and which are subject to unexpected alterations in a numerous of the continuous evolution operations. It can be detected withinside the disciplines of engineering, physics, nanoscale electronics, biology, population dynamics, electromagnetic wave beam, aeronautics, and pharmaco kinetics, etc. For this reason producing a higher knowledge of a number of the real-global problems in scientific application which, in latest years, has been challenging investigations and a number of results can be found in preceding works [5, 6, 28, 31, 33] and the references contained therein.

Furthermore, the non-local Cauchy problems, which have strong backdrop coming from physical problems, were initiated by Byszewski [11]. Many researchers are concerned with non-local Cauchy problems for different types of differential equations and inclusions because the nonlocal initial conditions generalize classical terms and play an important role in engineering and physics. For nonlocal fractional differential equations and inclusions see [5, 6, 7, 13, 15, 26, 25].

In this article, we examine the following semilinear impulsive fractional evolution differential inclusion

$$\begin{cases} {}^cD_t^\alpha x(t) \in Ax(t) + \mathcal{F}(t, x(t)), & t \in J - \{t_1, t_2, \cdots, t_m\}, \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, ..., m, \\ x(0) = a_0 - \mathcal{G}(x), & a_0 \in E, \end{cases} \tag{1}$$

where E is a Banach space, ${}^cD^\alpha$ is the Caputo derivative , A is sectorial operator on Banach space E, $\mathcal{F} : J \times E \to P(E)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_i : E \to E$, $1 \le i \le m$, $\mathcal{G} : PC(J, E) \to E$ and $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^-) = \lim_{s \to t_i^-} x(s)$, $x(t_i^+) = \lim_{s \to t_i^+} x(s)$.

Recently,the topic of inclusions of evolution and equations involving sectorial or almost sectorial terms has been extensively studied (see [2, 7, 30, 31, 32, 34, 35, 37]). For instance, Shu et al. [31] introduced a different concept of mild solutions for (1) when $\mathcal{F}$ is a completely continuous single-valued function and A is a sectorial operator with $\{\mathcal{S}_\alpha(t) : t \ge 0\}$ and $\{\mathcal{T}_\alpha : t \ge 0\}$ are compact. Agarwal et al. [2] proved the results of (1) with the absence of impulses in case when the dimension of E is finite and A is a sectorial operator. They determined the dimension for mild solutions set. Zhang et al. [37] studied the existence and controllability of (1) when when $\mathcal{F}$ is Lipschitz continuous single-valued function and A is a sectorial operator.

In Section 3, we investigate the existence $PC-$mild solution of nonlocal fractional impulsive problem for abstract evolution inclusion (1). Our fundamental result, Theorem 3.1, extends Theorem 3.1 of [31] and Theorem 3.1 of [37] by discussing the problem in a new case. We study the problem when the function $\mathcal{F}$ is upper semicontinuous multifunction.

In Section 4, we discuss the nonlocal controllability of impulsive fractional evolution inclusion of the following form

$$\begin{cases} {}^cD_t^\alpha x(t) \in Ax(t) + \mathcal{F}(t, x(t)) + \mathcal{B}(u(t)), & t \in J - \{t_1, t_2, \cdots, t_m\}, \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, ..., m, \\ x(0) = a_0 - \mathcal{G}(x), & a_0 \in E, \end{cases} \tag{2}$$

where $\mathcal{B}$ is a linear bounded operator from X into E, $u \in L^2(J, X)$ is the control function.

The structure of the present article is as follows: Section 2 introduce some basic concepts from the definitions and assumptions necessary for the subsequent sections. Sections 3 and 4 are devoted to presenting the main existence and controllability results using multivalued analysis, fractional calculus and fixed point theorem. At last, we have provided numerical examples to validate our results.

2. Preliminaries

This section provides some basic concepts, definitions and preliminary information that will help for the development of this article.
Let $(E, \|\cdot\|)$ be a Banach space , $P_b(E) = \{A : A \subset E,\ A \neq \varnothing, A \textit{ is bounded }\}$, $P_{cl}(E) = \{A : A \subset E,\ A \neq \varnothing, A \textit{ is closed }\}$, $P_k(E) = \{A : A \subset E,\ A \neq \varnothing, A \textit{ is compact }\}$, $P_c(E) = \{A : A \subset E,\ A \neq \varnothing, A \textit{ is convex }\}$, $P_{ck}(E) = \{A : A \subset E,\ A \neq \varnothing, A \textit{ is convex and compact}\}$, for $i = 1, \cdots, m$ we have $J_0 = [0, t_1], J_i = (t_i, t_{i+1}]$, and
$PC(J, E) = \{x : J \to E,\ x_{|J_i} \in C(J_i, E),\ x(t_i^+) \textit{ and } x(t_i^-) \textit{ exist }\}$, with the norm $\|x\|_{PC} = \max\{\|x(t)\|, t \in J\}$, $L^1(J, E)$ the space of $E-$valued Bochner integrable with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\|dt$, $\mathcal{L}(E)$ space of all bounded linear operators on E.

Let us remember some definitions of fractional calculus. For more details [23, 29].

Definition 2.1. *([23]). Let $f \in L^1(J, E)$ and $\alpha > 0$. The fractional integral with the lower limit zero for f is given by*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s)ds, \quad x > 0,$$

provided that the right side is point-wise defined on $[0, \infty)$.

Definition 2.2. *([23]). Let $f \in C^n[0, \infty)$ and $\alpha > 0$. The Caputo derivative of the order α for f is given by*

$$\begin{aligned} {}^cD^\alpha f(x) &= \frac{1}{\Gamma(n - \alpha)} \int_0^x (x - s)^{(n-\alpha-1)} f^n(s)ds \\ &= I^{n-\alpha} f^n(x), \quad n = [\alpha] + 1, \end{aligned}$$

where $[\alpha]$ is the integer part of the real number α.

We also provide some primary definitions and results of multivalued functions.

A multivalued function $F : E \to P(E)$ is called upper semi-continuous ($u.s.c$) on E if $\forall\ x \in E$, $F(x)$ is nonempty subset of E and for each open set $B \subset E$ with $F(x) \subset B$, there is a neighborhood $\mathcal{V}$ of x with $F(\mathcal{V}(x)) \subset B$. F is closed if its graph $\Gamma_F = \{(x, y) \in E \times E : y \in F(x)\}$ is closed subset of the space $E \times E$. F completely continuous if for each bounded set $B \subset E$, $F(B)$ is relatively compact. If F is completely continuous with nonempty values, then F is $u.s.c.$ if and only if F has a closed graph. If there is $x \in E$ with $x \in F(x)$, then F has a fixed point. The fixed points set of F is denoted by $\mathrm{Fix}(F)$. For more details we refer to [18, 21, 22].

Definition 2.3. *Let $W = \{f_n : n \in \mathbb{N}\} \subset L^1(J, E)$. We said W is semi-compact if*

1. *It is integrably bounded i.e. there is $\vartheta \in L^1(J, \mathbb{R}^+)$ with $\|f_n(t)\| \leq \vartheta(t) a.e. t \in J$.*
2. *The set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in E a.e. $t \in J$.*

Lemma 2.4. *([22]). In $L^1(J, E)$ every semi-compact sequence is weakly compact.*

Lemma 2.5. *([24]). Let $G : J \times E \to P_{ck}(E)$ be Carathéodory multivalued map, for each $u \in E$ the set $S_G = S^1_{G,u} = \{f \in L^1(J, E) : f(t) \in G(t, u(t))\ \ a.e.\ t \in J\} \neq \emptyset$ and $\mathcal{F} : L^1(J, E) \to C(J, E)$ a continuous linear map. Then the operator*

$$\mathcal{F} \circ S_G : C(J, E) \to P_{ck}(C(J, E)),$$

$$u \to (\mathcal{F} \circ S_G)(u) = \mathcal{F}(S_{G,u})$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Definition 2.6. *Let $F : E \to P_{cl}(E)$. Then F is called* *(i)* $\gamma-$*Lipschitz if there is* $\gamma > 0$ *with*

$$H(F(x), F(y)) \leq \gamma d(x, y), \quad x, y \in E.$$

(ii) A contraction if it is $\gamma-$*Lipschitz with* $\gamma < 1$.

Definition 2.7. *Let* $A : D(A) \subset E \to E$ *be linear closed operator. We say that A is sectorial if* $\exists \quad \omega \in \mathbb{R}, \quad \theta \in [\frac{\pi}{2}, \pi]$ *and* $M > 0$, *with*

1. $\rho(A) \subset \Sigma_{(\theta,\omega)} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$.
2. $\|R(\lambda, A)\|_{\mathcal{L}(E)} \leq \dfrac{M}{|\lambda - \omega|}, \quad \lambda \in \Sigma_{(\theta,\omega)}$.

For more details on sectorial, we refer to [9, 16]. To study the existence of mild solutions and controllability of fractional inclusions when A is a sectorial of type (M, θ, ω), we need help of solutions operators $\mathcal{T}_\alpha(t)$, $\mathcal{S}_\alpha(t)$, where

$$\mathcal{T}_\alpha(t) = \frac{1}{2\pi i}\int_\Upsilon e^{\lambda t}\lambda^{\alpha-1}\ R(\lambda^\alpha, A)d\lambda,$$
$$\mathcal{S}_\alpha(t) = \frac{1}{2\pi i}\int_\Upsilon e^{\lambda t}\ R(\lambda^\alpha, A)d\lambda,$$

where Υ is a suitable path.

Lemma 2.8. *([31]). If* $A \in A^\alpha(\theta_0, \omega_0)$ *and* $\alpha \in (0, 1)$, *then* $\forall t > 0, \quad \omega > \omega_0$ *we have*

$$\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)} \leq Me^{\omega t}$$
$$\|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(E)} \leq Ce^{\omega t}(1 + t^{\alpha-1}).$$

Let $M_{\mathcal{T}_\alpha} = \sup_{0\leq t\leq b}\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)}, \quad M_{\mathcal{S}_\alpha} = \sup_{0\leq t\leq b} Ce^{\omega t}(1 + t^{\alpha-1})$. *Then we get*

$$\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(E)} \leq M_{\mathcal{T}_\alpha},$$
$$\|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(E)} \leq t^{\alpha-1}M_{\mathcal{S}_\alpha}.$$

Definition 2.9. *Let* $x \in PC(J, E)$, *we say that x is* $PC-$*mild solution for (1) if there exists an integrable selection* $f \in \mathcal{F}(\cdot, x(\cdot))$ *such that*

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i}\mathcal{T}_\alpha(t - t_k)I_k(x(t_k^-)) & \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_i, i = 1, \cdots, m. \end{cases}$$

Theorem 2.10. *(Covitz and Nadler [14]). Let* $\mathcal{P} : E \to P_{cl}(E)$ *be* $\gamma-$*contraction, then* $Fix(\mathcal{P}) \neq \emptyset$, *where* (E, d) *is a complete metric space.*

Theorem 2.11. *(Bohnenblust and Karlin [10]). Let E be Banach space and* $D \in P_{cl,c}(E)$. *If* $\mathcal{P} : D \to P_{cl,c}(E)$ *is u. s. c. with* $\mathcal{P}(D) \subseteq D$ *and* $\mathcal{P}(D)$ *is relatively compact, then* $\mathcal{P}$ *has a fixed point.*

3. Main Results

Theorem 3.1. *Let* $A \in A^\alpha(\theta_0, \omega_0)$ *for* $\theta_0 \in (0, \frac{\pi}{2}]$ *and* $\omega_0 \in \mathbb{R}$. *Suppose that the following conditions hold:*

H_1 : *The semigroup* $\{\mathcal{S}_\alpha(t) : t > 0\}$ *is compact.*

H_2 : $\mathcal{F} : J \times E \to P_{ck}(E)$, *for each* $x \in E$ $\mathcal{F}$ *is measurable to* t, *for each* $t \in J$, $\mathcal{F}$ *is upper semicontinuous to* x *and the set* $S^1_{\mathcal{F}(\cdot,x(\cdot))} = \{f \in L^1(J, E) : f(t) \in \mathcal{F}(t, x(t)), a.e.\}$ *is nonempty for each* $x \in PC(J, E)$.

H_3 : *There exists a function* $\vartheta \in L^1(J, \mathbb{R}^+)$ *with*

$$\|\mathcal{F}(t,x(t))\| = \sup\{\|f\| : f(t) \in \mathcal{F}(t,x(t)), t \in J\} \leq \vartheta(t).$$

H_4 : $\mathcal{G} : PC(J,E) \to E$ *is compact, continuous and* $\|\mathcal{G}(x)\| \leq N, \quad \forall\, x \in PC(J,E)$, *where* $N > 0$.

H_5 : *For each* $i = 1, 2, \cdots, m$, $I_i : E \to E$ *is continuous, compact and there is* $\zeta_i > 0$ *with* $\|I_i(x)\| \leq \zeta_i \|x\|$, $x \in E$.

Then (1) has $PC-$*mild solution provided that* $\exists\, r > 0$ *such that*

$$M_{\mathcal{T}_\alpha}(\|a_0\| + N + r\zeta) + M_{\mathcal{S}_\alpha}\frac{b^\alpha}{\alpha}\|\vartheta\|_{L^1(J,\mathbb{R}^+)} \leq r, \tag{3}$$

where $\zeta = \sum_{i=1}^m \zeta_i$

Proof. From H_2, for each $x \in PC(J,E)$, $S^1_{\mathcal{F}(\cdot,x(\cdot))} \neq \emptyset$. Thus, we are able to define the multivalued operator $\Omega : PC(J,E) \to 2^{PC(J,E)}$ as follows : if $x \in PC(J,E)$, then $y \in \Omega(x)$ if and only if

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) & \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_i, \end{cases} \tag{4}$$

where $i = 1, \cdots, m$ and $f \in S^1_{\mathcal{F}(\cdot,x(\cdot))}$. By using Theorem 2.11, we show that Ω has fixed point which is $PC-$mild solution for 1 . For easy of reading, we divide the proof into a sequence of steps.

Step 1: For each $x \in PC(J,E)$, $\Omega(x)$ is convex.

Let $y_1, y_2 \in \Omega(x)$ and $\lambda \in (0,1)$. If $t \in J_0$ then, from (4) we have

$$\lambda y_1(t) + (1-\lambda)y_2(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)(\lambda f_1(s) + (1-\lambda)\lambda f_2(s))ds,$$

where $f_1, f_2 \in S^1_{\mathcal{F}(\cdot,x(\cdot))}$. Because F has convex values, $S^1_{\mathcal{F}(\cdot,x(\cdot))}$ is convex. Then $(\lambda f_1(s) + (1-\lambda)f_2(s)) \in S^1_{\mathcal{F}(\cdot,x(\cdot))}$. Thus, $\lambda y_1(t) + (1-\lambda)y_2(t) \in \Omega(x)$. Similarly, for $t \in J_i$, $i = 1, \cdots, m$, we can prove that $\lambda y_1(t) + (1-\lambda)y_2(t) \in \Omega(x)$. Therefore, $\Omega(x)$ is convex for each $x \in PC(J,E)$.

Step 2: For any positive number r, let $\mathcal{E}_r = \{x \in PC(J,E) : \|x\|_{PC} \leq r\}$. Clearly, $\mathcal{E}_r$ is convex, closed and bounded subset of $PC(J,E)$. In this step we will prove that $\Omega(\mathcal{E}_r) \subseteq \mathcal{E}_r$.

Let $x \in \mathcal{E}_r$, $y \in \Omega(x)$, then by using Lemma 2.8, H_3 and H_4 for $t \in J_0$, we get

$$\begin{aligned} \|y(t)\| &= \|\mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds\| \\ &\leq M_{\mathcal{T}_\alpha}(\|a_0\| + N) + M_{\mathcal{S}_\alpha}\frac{b^\alpha}{\alpha}\|\vartheta\|_{L^1(J,\mathbb{R}^+)} \leq r. \end{aligned}$$

Similar if $t \in J_i$, $i = 1, \cdots, m$, from Lemma 2.8, H_3, H_4 and H_5 we obtain

$$\|y(t)\| \leq M_{\mathcal{T}_\alpha}(\|a_0\| + N + r\zeta) + M_{\mathcal{S}_\alpha}\frac{b^\alpha}{\alpha}\|\vartheta\|_{L^1(J,\mathbb{R}^+)} \leq r.$$

Hence, $\Omega(\mathcal{E}_r) \subseteq \mathcal{E}_r$.

Step 3: We show that $\Omega(\mathcal{E}_r)$ is equicontinuous in $PC(J,E)$.

Let $B = \Omega(\mathcal{E}_r)$. We show that for $i = 0, 1, \cdots, m,$, $B_{\overline{J}_i}$ is equicontinuous , where

$$B_{\overline{J}_i} = \{y^* \in C(\overline{J_i}, E) : \ y^*(t) = y(t), \ t \in J_i, \ y^*(t_i) = y(t_i^+), \ y \in B\}.$$

Let $y \in B$. Then there exist $x \in \mathcal{E}_r$ with $y \in \Omega(x)$. Form the definition of Ω, there is $f \in S^1_{\mathcal{F}(.,x(.))}$ such that

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) & \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_i, i = 1, \cdots, m. \end{cases}$$

We consider the following cases:

Case 1. When $i = 0$, let $t, t+\tau \in J_0 = [0, t_1]$, then

$$\|y^*(t+\tau) - y^*(t)\| = \|y(t+\tau) - y(t)\| \leq \|\mathcal{T}_\alpha(t+\tau)(a_0 - \mathcal{G}(x)) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x))\|$$

$$+\| \int_0^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds - \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds\|$$

$$\leq \Omega_1 + \Omega_2 + \Omega_3,$$

where
$\Omega_1 = \|\mathcal{T}_\alpha(t+\tau)(a_0 - \mathcal{G}(x)) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x))\|$,
$\Omega_2 = \| \int_0^t [\mathcal{S}_\alpha(t+\tau-s) - \mathcal{S}_\alpha(t-s)]f(s)ds\|$,
$\Omega_3 = \| \int_t^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds\|$.
We need to prove that $\Omega_i \to 0$ as $\tau \to 0$ $\forall$ $i = 1, 2, 3$.

$$\lim_{\tau \to 0} \Omega_1 \leq (\|\Omega_0\| + \sup_{x \in \mathcal{E}_r} \|\mathcal{G}(x)\|) \lim_{\tau \to 0} \|\mathcal{T}_\alpha(t+\tau) - \mathcal{T}_\alpha(t)\| = 0,$$

independently of $x \in \mathcal{E}_r$.
For Ω_2, from Lebesgue dominated convergence Theorem and the definition of $\mathcal{S}_\alpha(t)$, we obtain

$$\Omega_2 \leq \int_0^t \| \ [\mathcal{S}_\alpha(t+\tau-s) - \mathcal{S}_\alpha(t-s)]f(s)\|ds \to 0, \text{ as } \tau \to 0,$$

independently of $x \in \mathcal{E}_r$.
For Ω_3,

$$\lim_{\tau \to 0} Q_3 = \lim_{\tau \to 0} \| \int_t^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds\| \leq M_{\mathcal{S}_\alpha}\|\vartheta\| \lim_{\tau \to 0} \int_t^{t+\tau} (t-s)^{t+\tau-s}ds = 0,$$

independently of $x \in \mathcal{E}_r$.
Therefore,

$$\lim_{\tau \to 0} \|y(t+\tau) - y(t)\| = 0. \tag{5}$$

Case 2. When $t \in J_i, i \in \{1, 2, \cdots, m\}$. If $t, t+\tau \in J_i$, then by (4) we have

$$\begin{aligned} \|y^*(t+\tau) - y^*(t)\| &= \|y(t+\tau) - y(t)\| \\ &\leq \|\mathcal{T}_\alpha(t+\tau)(a_0 - \mathcal{G}(x)) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x))\| \\ &\quad + \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t+\tau-t_k)I_k(x(t_k^-)) - \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))\| \\ &\quad + \| \int_0^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds - \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds\|. \end{aligned}$$

We argue as in the first case, we get

$$\lim_{\tau \to 0} \|y(t+\tau) - y(t)\| = 0. \tag{6}$$

Case 3. When $t = t_i$, $i = 1, 2, \cdots, m$. If $\tau > 0$ and $\delta > 0$ such that $t_i + \tau \in J_i$ and $t_i < \delta < t_i + \tau \le t_{i+1}$, then

$$\|y^*(t_i + \tau) - y^*(t_i)\| = \lim_{\delta \to t_i^+} \|y(t_i + \tau) - y(\delta)\|.$$

From (4), we obtain

$$\begin{aligned}\|y(t_i + \tau) - y(\delta)\| \le & \|\mathcal{T}_\alpha(t_i + \tau)(ax_0 - \mathcal{G}(x)) - \mathcal{T}_\alpha(\delta)(a_0 - \mathcal{G}(x))\| \\ & + \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t_i + \tau - t_k) I_k(x(t_k^-)) - \mathcal{T}_\alpha(\delta - t_k) I_k(x(t_k^-))\| \\ & + \quad \| \int_0^{t_i+\tau} \mathcal{S}_\alpha(t_i + \tau - s) f(s) ds - \int_0^{\delta} \mathcal{S}_\alpha(\delta - s) f(s) ds\|.\end{aligned}$$

If we argue as in the first case, then we obtain

$$\lim_{\substack{\tau \to 0 \\ \delta \to t_i^+}} \|y(t_i + \tau) - y(\delta)\| = 0. \tag{7}$$

From (5), (6) and (7), $B_{\overline{J_i}}$ is equicontinuous $\forall\ i = 0, 1, \cdots, m$.

Step 4: We prove that $(\Omega \mathcal{E}_r)(t) = \{y(t) : y \in \Omega(\mathcal{E}_r)\}$ is relatively compact in E for each $t \in J$.

Let $0 < t \le s \le t_1$ and $\varepsilon \in (0, t)$

$$y_\varepsilon(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^{t-\varepsilon} \mathcal{S}_\alpha(t - s) f(s) ds,$$

where $f \in \mathcal{F}(\cdot, x(\cdot)))$. Since $\mathcal{G}$ is compact and for $t > 0$ $\mathcal{S}_\alpha(t)$ is compact, the set $Y_\varepsilon = \{y_\varepsilon(t) : y_\varepsilon \in \Omega(\mathcal{E}_r)\}$ is relatively compact in E. Moreover,

$$\|y(t) - y_\varepsilon(t)\| \le \| \int_{t-\varepsilon}^{t} \mathcal{S}_\alpha(t - s) f(s) ds\|. \tag{8}$$

Similarly, for $t \in (t_i, t_{i+1}]$, $i = 1, \cdots, m$. Let $t_i < t \le s \le t_{i+1}$ and $\varepsilon \in (0, t)$. For $x \in \mathcal{E}_r$, we define

$$y_\varepsilon(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(x(t_k^-)) + \int_0^{t-\varepsilon} \mathcal{S}_\alpha(t - s) f(s) ds,$$

where $f \in \mathcal{F}(\cdot, x(\cdot))$. Since $\mathcal{S}_\alpha(t)$ for $t > 0$, $\mathcal{G}$ and I_k, $k = 1, \cdots, m$ are compact , $Y_\varepsilon = \{y_\varepsilon(t) : y_\varepsilon \in \Omega(\mathcal{E}_r)\}$ is relatively compact in E. Furthermore,

$$\|y(t) - y_\varepsilon(t)\| \le \| \int_{t-\varepsilon}^{t} \mathcal{S}_\alpha(t - s) f(s) ds\|. \tag{9}$$

Obviously, (8) and (9) tend to zero when ε tend to 0. Then, a relatively compact set exists that can be arbitrary close to $(\Omega \mathcal{E}_r)(t) = \{y(t) : y \in \Omega(\mathcal{E}_r)\}$ for $t \in J$. Therefore, $(\Omega \mathcal{E}_r)(t)$ is relatively compact in E for $t \in J$. From Step 2 to 4 together with Arzelá-Ascoli theorem, Ω is completely continuous.

Step 5: We show that Ω has a closed graph.

Let $x_n \to \overline{x}$ as $n \to \infty$, $y_n \in \Omega(x_n)$ with $y_n \to \overline{y}$ as $n \to \infty$. We need to prove that $\overline{y} \in \Omega(\overline{x})$. Because $y_n \in \Omega(x_n)$, from (4) there exists $f_n \in S^1_{F(\cdot, x_n(\cdot))}$ such that

$$y_n(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_n)) + \int_0^t \mathcal{S}_\alpha(t-s) f_n(s) ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_n)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x_n(t_k^-)) & \\ + \int_0^t \mathcal{S}_\alpha(t-s) f_n(s) ds, & t \in J_i. \end{cases}$$

For $t \in J_0$, we show that $\exists\ \overline{f} \in S^1_{\mathcal{F}(\cdot,\overline{x}(\cdot))}$ with

$$\overline{y}(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) + \int_0^t \mathcal{S}_\alpha(t-s)\overline{f}(s) ds.$$

Let $\mathfrak{F} : L^1(J_0, E) \to C(J_0, E)$ defined by

$$\mathfrak{F}(f)(t) = \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds.$$

Obviously, $\mathfrak{F}$ is continuous linear operator. Moreover,

$$\begin{aligned} \|\mathfrak{F} f\|_{PC} &\le M_{\mathcal{S}_\alpha} \int_0^t (t-s)^{\alpha-1} f(s) ds \\ &\le M_{\mathcal{S}_\alpha} \frac{b^\alpha}{\alpha} \|\vartheta\|_{L^1(J_0,E)}. \end{aligned}$$

Obviously,
$\|[y_n(t) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_n))] - [\overline{y}(t) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x}))]\| \to 0$ as $n \to \infty$.
From Lemma 2.5, $\mathfrak{F} \circ S^1_{\mathcal{F}}$ is closed graph operator. Also, from the definition of $\mathfrak{F}$, we get $\forall t \in J_0$,

$$y_n - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_n)) \in \mathfrak{F}(S^1_{\mathcal{F}(\cdot,x_n(\cdot))}).$$

Since $x_n \to \overline{x}$ as $n \to \infty$, from Lemma 2.5 we get

$$\overline{y} - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) = \int_0^t \mathcal{S}_\alpha(t-s)\overline{f}(s) ds,$$

for some $\overline{f} \in S^1_{\mathcal{F}(\cdot,\overline{x}(\cdot))}$. Therefore,

$$\overline{y}(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) + \int_0^t \mathcal{S}_\alpha(t-s)\overline{f}(s) ds.$$

Similarly, if $t \in J_i$, $i = 1, \cdots, m$, we show that $\exists\ \overline{f} \in S^1_{\mathcal{F}(\cdot,\overline{x}(\cdot))}$ such that

$$\overline{y}(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\overline{x}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s)\overline{f}(s) ds.$$

Let $\mathfrak{F} : L^1(J_i, E) \to C(J_i, E)$ defined by

$$\mathfrak{F}(f)(t) = \int_0^t \mathcal{S}_\alpha(t-s) f(s) ds.$$

Clearly,
$\|[y_n(t) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_n)) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x_n(t_k^-))] - [\overline{y}(t) - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(\overline{x}(t_k^-))]\| \to 0$ as $n \to \infty$.

From Lemma 2.5, $\mathfrak{F} \circ S^1_{\mathcal{F}}$ is closed graph operator. Also, from the definition of $\mathfrak{F}$, we obtain

$$y_n - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_n)) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x_n(t_k^-)) \in \mathfrak{F}(S^1_{\mathcal{F}(\cdot,x_n(\cdot))}).$$

Since $x_n \to \overline{x}$ as $n \to \infty$, from Lemma 2.5 we have

$$\overline{y} - \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) - \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(\overline{x}(t_k^-)) = \int_0^t \mathcal{S}_\alpha(t-s)\overline{f}(s)ds,$$

for some $\overline{f} \in S^1_{\mathcal{F}(\cdot,\overline{x}(\cdot))}$. Therefore,

$$\overline{y}(t) = \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(\overline{x})) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(\overline{x}(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s)\overline{f}(s)ds.$$

This means $\overline{y} \in \Omega(\overline{x})$. Hence, Ω has closed graph.
Therefore, by Theorem 2.11, Ω has fixed point which is $PC-$mild solution of problem (1).

□

4. Controllability results

Our aims in this section to investigate the controllability of the system (2).

Definition 4.1. *The function $x \in PC(J,E)$ is called $PC-$mild solution for (2) if there exists an integrable selection $f \in \mathcal{F}(\cdot, x(\cdot))$ with*

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds & \\ + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_f(s))ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) & \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_f(s)))ds, & t \in J_i, 1 \le i \le m. \end{cases}$$

Definition 4.2. *We say that the system (2) is nonlocally controllable on $[0,b]$ if $\forall\, a_0, a_1 \in E$, $\exists\, u \in L^2(J,X)$ with a solution of (2) satisfies $x(0) = a_0 - \mathcal{G}(x)$ and $x(b) = a_1 - \mathcal{G}(x)$, where u is called control function.*

Let E be a real separable Banach space and X a real Banach space.

Theorem 4.3. *We assume the following conditions:*

C_1 : *A is a sectorial operator.*
C_2 : *Let $\mathcal{F} : J \times E \to P_{ck}(E)$, with $t \to \mathcal{F}(t,x)$, is measurable for each $x \in E$.*
C_3 : *There exists a function $\varsigma \in L^1(J, \mathbb{R}^+)$ with*
  (a) *$H(\mathcal{F}(t,x), \mathcal{F}(t,y)) \le \varsigma(t)\|x-y\|$, for a. e. $t \in J$, $x, y \in E$.*
  (b) *$\|\mathcal{F}(t,x)\| = \sup\{\|f\| : f(t) \in \mathcal{F}(t,x(t))\} \le \varsigma(t)$, a. e. $t \in J$ and $x \in E$.*
C_4 : *$\mathcal{G} : PC(J,E) \to E$, there is $\mathcal{R} > 0$ with*

$$\|\mathcal{G}(x) - \mathcal{G}(y)\| \le \mathcal{R}\|x-y\|, \forall\, x, y \in PC(J,E).$$

C_5 : *For each $i = 1, \cdots, m$, $I_i : E \to E$, there is $\kappa_i > 0$ with*

$$\|I_i(x) - I_i(y)\| \le \kappa_i\|x-y\|, \quad \forall\, x, y \in E.$$

C_6 *: The linear bounded operator* $W : L^2(J,X) \to E$ *given by*

$$W(u) = \int_0^b \mathcal{S}_\alpha(b-s)\mathcal{B}(u(s))ds$$

has an invertible operator $W^{-1} : E \to {}^2(J,X)/Ker(W)$ *and there are two positive constants* β_1, β_2 *with* $\|\mathcal{B}\| \le \beta_1$ *and* $\|W^{-1}\| \le \beta_2$.

Then (2) is controllable on $[0,b]$ *provided that*

$$\left[M_{\mathcal{T}_\alpha}(\mathcal{R}+\kappa) + \eta + \beta_1\beta_2 M_{\mathcal{S}_\alpha} \frac{b^{\frac{2\alpha-1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} \left(M_{\mathcal{T}_\alpha}\mathcal{R} + \mathcal{R} + \eta \right) \right] < 1. \tag{10}$$

Where $\kappa = \sum_{i=1}^m \kappa_i$, $\eta = M_{\mathcal{S}_\alpha}\frac{b^\alpha}{\alpha}\|\varsigma\|_{L^1}$

Proof. From C_2, C_3 and Theorem 8.2.8 in [8], we get $\forall\ x \in PC(J,E)$ the set

$$S^1_{\mathcal{F}(\cdot,x(\cdot))} = \{f \in L^1(,E) : f \in \mathcal{F}(t,x(t)) \ \ a.e.\} \neq \emptyset.$$

Then, we can define the multifunction $\mathcal{P} : PC(J,E) \to 2^{PC(J,E)}$ as follow: $y \in \mathcal{P}(x)$ if and only if

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_f(s))ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_f(s))ds, & t \in J_i, \end{cases} \tag{11}$$

where $1 \le i \le m$, and $f \in S^1_{\mathcal{F}(\cdot,x(\cdot))}$. Let $x \in PC(J,E)$ and $f \in S^1_{\mathcal{F}(\cdot,x(\cdot))}$, by using C_6 we may define the control function u_f as

$$\begin{aligned} u_f(t) = \quad & W^{-1}\{a_1 - \mathcal{G}(x) - \mathcal{T}_\alpha(b)(a_0 - \mathcal{G}(x)) - \textstyle\sum_{k=1}^{k=m} \mathcal{T}_\alpha(b-t_k)I_k(x(t_k^-)) \\ & \qquad - \textstyle\int_0^b \mathcal{S}_\alpha(b-s)f(s)ds\}(t). \end{aligned} \tag{12}$$

By using (11), we prove that $\mathcal{P}$ has fixed point and it is $PC-$mild solution of the system (2).

As the first, we show that the system (2) is nonlocally controllable on $[0,b]$. Let x be a fixed point for $\mathcal{P}$, then from (11) and (12) we get

$$\begin{aligned} x(b) = {} & \mathcal{T}_\alpha(b)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=m} \mathcal{T}_\alpha(b-t_k)I_k(x(t_k^-)) + + \int_0^b \mathcal{S}_\alpha(b-s)f(s)ds \\ & + \int_0^b \mathcal{S}_\alpha(b-s)\mathcal{B}(u_f(s))ds \\ = {} & \mathcal{T}_\alpha(b)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=m} \mathcal{T}_\alpha(b-t_k)I_k(x(t_k^-)) \\ & + \int_0^b \mathcal{S}_\alpha(b-s)f(s)ds + W(u_f) \\ = {} & \mathcal{T}_\alpha(b)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=m} \mathcal{T}_\alpha(b-t_k)I_k(x(t_k^-)) + \int_0^b \mathcal{S}_\alpha(b-s)f(s)ds \\ & + a_1 - \mathcal{G}(x) - \mathcal{T}_\alpha(b)(a_0 - \mathcal{G}(x)) - \sum_{k=1}^{k=m} \mathcal{T}_\alpha(b-t_k)I_k(x(t_k^-)) \\ & - \int_0^b \mathcal{S}_\alpha(b-s)f(s)ds \\ = {} & a_1 - \mathcal{G}(x). \end{aligned}$$

This means that u_f steers the system (2) from a_0 to a_1 in a finite time b.

Now, let us prove that $\mathcal{P}$ fulfills all hypotheses of Theorem 2.10. The proof will present in two steps:
Step1. We prove that $\mathcal{P}(x)$ is closed for every $x \in PC(J,E)$.
Let $\{\omega\}_{n\geq 1}$ be a sequence in $\mathcal{P}(x)$ such that $\omega_n \to \omega$ in $PC(J,E)$. We want to prove that $\omega \in \mathcal{P}(x)$. From (11), there exists a sequence $\{f_n : n \geq 1\}$ in $S^1_{\mathcal{F}(\cdot,x(\cdot))}$ with

$$\omega_n(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f_n(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_n}(s)))ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f_n(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_n}(s)))ds, & t \in J_i. \end{cases} \tag{13}$$

By (C_3), for every $n \geq 1$ and $a.e.$ $t \in J$ we have $\|f_n(t)\| \leq \varsigma(t)$. Thus, $\{f_n : n \geq 1\}$ is integrable bounded. Also, $\{f_n(t) : n \geq 1\}$ is relatively compact in E for $a.e.$ $t \in J$ because $\{f_n(t) : n \geq 1\} \subset \mathcal{F}(t, x(t))$. Then, the set $\{f_n : n \geq 1\}$ semicompact. By Lemma 2.4, in $L^1(J,E)$ $\{f_n : n \geq 1\}$ is weakly compact. We may assume that $\{f_n : n \geq 1\}$ converges weakly to $f \in L^1(J,E)$. There is a sequence $\{\mathcal{V}_n\}_{n=1}^\infty \subseteq \overline{conv}\{f_n : n \geq 1\}$ such that $\mathcal{V}_n$ converges strongly to f (Mazur's Lemma). As $\mathcal{F}$ has convex and compact values, $S^1_{\mathcal{F}(\cdot,x(\cdot))}$ is convex and compact set. Therefore, $\{\mathcal{V}_n\}_{n=1}^\infty \subseteq S^1_{\mathcal{F}(\cdot,x(\cdot))}$ and $f \in S^1_{\mathcal{F}(\cdot,x(\cdot))}$. Also, by using Lemma 2.8 we have $\forall\, n \geq 1, \quad t, s \in J, s \in (0,t]$,

$$\|\mathcal{S}_\alpha(t-s)f_n(s)\| \leq |t-s|^{\alpha-1}M_{\mathcal{S}_\alpha}\varsigma(s) \in L^1(J, \mathbb{R}^+),$$
$$\|\mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_n}(s)))\| \leq \beta_1|t-s|^{\alpha-1}M_{\mathcal{S}_\alpha}\varsigma(s) \in L^1(J, \mathbb{R}^+).$$

So, by the Lebesgue dominated convergence Theorem, for $1 \leq i \leq m$, we obtain

$$\lim_{n\to\infty} \omega_n(t) = \omega(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_f(s))ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_f(s))ds, & t \in J_i. \end{cases}$$

This means, $\omega \in \mathcal{P}(x)$. Therefore, $\forall\, x \in PC(J,E)$, $\mathcal{P}(x)$ is closed.
Step2. We show that $\mathcal{P}$ is $\gamma-$contraction.
Let $x_1, x_2 \in PC(J,E)$. We prove that $\exists\, \gamma \in (0,1)$ with

$$H(\mathcal{P}(x_1), \mathcal{P}(x_2)) \leq \gamma\|x_1 - x_2\|_{PC}, \quad \forall\, x_1, x_2 \in PC(J,E).$$

Let $z_1 \in \mathcal{P}(x_1)$, then there exists $f_1 \in S^1_{\mathcal{F}(\cdot,x_1(\cdot))}$ such that

$$z_1(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_1)) + \int_0^t \mathcal{S}_\alpha(t-s)f_1(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_1}(s)))ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_1)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x_1(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f_1(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_1}(s)))ds, & t \in J_i. \end{cases}$$

From C_3 (a) we have

$$H(\mathcal{F}(t,x_1), \mathcal{F}(t,x_2)) \leq \varsigma(t)\|x_1(t) - x_2(t)\|_{PC}.$$

Thus, $\exists\, \mu \in \mathcal{F}(t, x_2(t))$ such that

$$\|f_1(t) - \mu\| \leq \varsigma(t)\|x_1(t) - x_2(t)\|, \quad t \in J.$$

Let $\mathcal{M} : J \to 2^E$ defined as:

$$\mathcal{M}(t) = \{\mu \in E : \|f_1(t) - \mu\| \leq \varsigma(t)\|x_1(t) - x_2(t)\|\}.$$

Since f_1, ς, x_1, x_2 are measurable, by using proposition 3.4 in [12], we get $t \to \mathcal{M}(t) \cap F(t, x_2(t))$ is measurable. Moreover, its values are nonempty and compact. So, from Theorem 1.3.1 in [19], there exist $f_2 \in \mathcal{F}(t, x_2(t))$ such that

$$\|f_1(t) - f_2(t)\| \leq \varsigma(t)\|x_1(t) - x_2(t)\|, \qquad \forall\ t \in J.$$

Now, define

$$z_2(t) = \begin{cases} \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_2) + \int_0^t \mathcal{S}_\alpha(t-s) f_2(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_2}(s))ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(a_0 - \mathcal{G}(x_2) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x_2(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s) f_2(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathcal{B}(u_{f_2}(s))ds, & t \in J_i. \end{cases}$$

Clearly, $Z_2 \in \mathcal{P}(x_2)$. Let $t \in J_0$, by C_3, C_4, (11) and (12) we have

$$\begin{aligned} \|z_1(t) - z_2(t)\| &\leq M_{\mathcal{T}_\alpha}\|\mathcal{G}(x_1) - \mathcal{G}(x_2)\| + M_{\mathcal{S}_\alpha}\int_0^t (t-s)^{\alpha-1}\|f_1(s) - f_2(s)\|ds \\ &\quad + M_{\mathcal{S}_\alpha}\int_0^t (t-s)^{\alpha-1}\|\mathcal{B}(u_{f_1}(s)) - \mathcal{B}(u_{f_2}(s))\|ds \\ &\leq M_{\mathcal{T}_\alpha}\mathcal{R}\|x_1 - x_2\|_{PC} + M_{\mathcal{S}_\alpha}\int_0^t (t-s)^{\alpha-1}\varsigma(s)\|x_1 - x_2\|_{PC}ds \\ &\quad + M_{\mathcal{S}_\alpha}\int_0^t (t-s)^{\alpha-1}\|\mathcal{B}\|\ \|u_{f_1}(s) - u_{f_2}(s)\|ds \\ &\leq M_{\mathcal{T}_\alpha}\mathcal{R}\|x_1 - x_2\|_{PC} + M_{\mathcal{S}_\alpha}\int_0^t (t-s)^{\alpha-1}\varsigma(s)\|x_1 - x_2\|_{PC}ds \\ &\quad + \beta_1 M_{\mathcal{S}_\alpha}\left[\int_0^t (t-s)^{2(\alpha-1)}ds\right]^{\frac{1}{2}}\|u_{f_1} - u_{f_2}\|_{L^2(J,X)} \end{aligned}$$

.

To simplify the calculation , we find

$$\begin{aligned} \|u_{f_1} - u_{f_2}\|_{L^2(J,X)} &\leq \|W^{-1}\|\ [\ \|\mathcal{G}(x_1) - \mathcal{G}(x_2)\| + \|\mathcal{T}_\alpha(b)\|\|\mathcal{G}(x_1) - \mathcal{G}(x_2)\| \\ &\quad + \|\int_0^b \mathcal{S}_\alpha(b-s)(f_1(s) - f_2(s))ds\|] \\ &\leq \beta_2\mathcal{R}\|x_1 - x_2\|_{PC} + \mathcal{R}\beta_2 M_{\mathcal{T}_\alpha}\|x_1 - x_2\|_{PC} \\ &\quad + \beta_2 M_{\mathcal{S}_\alpha}\int_0^b (b-s)^{\alpha-1}\|f_1(s) - f_2(s)\|ds \\ &\leq \beta_2\|x_1 - x_2\|_{PC}\left[\mathcal{R} + \mathcal{R}M_{\mathcal{T}_\alpha} + \eta\right]. \end{aligned} \tag{14}$$

Then by using (14), we get

$$\begin{aligned} \|z_1(t) - z_2(t)\| &\leq M_{\mathcal{T}_\alpha}\mathcal{R}\|x_1 - x_2\|_{PC} + \eta\ \|x_1 - x_2\|_{PC} \\ &\quad + \beta_1 M_{\mathcal{S}_\alpha}\frac{b^{\frac{2\alpha-1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\beta_2\|x_1 - x_2\|_{PC}\left[\mathcal{R} + \mathcal{R}M_{\mathcal{T}_\alpha} + \eta\right] \\ &\leq \left[M_{\mathcal{T}_\alpha}\mathcal{R} + \eta + \beta_1\beta_2 M_{\mathcal{S}_\alpha}\frac{b^{\frac{2\alpha-1}{2}}}{(2\alpha-1)^{\frac{1}{2}}}\left(\mathcal{R} + \mathcal{R}M_{\mathcal{T}_\alpha} + \eta\right)\right]\|x_1 - x_2\|_{PC} \\ &\leq \gamma\|x_1 - x_2\|_{PC}. \end{aligned} \tag{15}$$

Similarly, if $t \in J_i$, for any $i = 1, \cdots, m$, by using C_3, C_4, C_5 and (14) we obtain

$$\begin{aligned} \|z_1(t) - z_2(t)\| &\leq \left[M_{\mathcal{T}_\alpha}(\mathcal{R} + \kappa) + \eta + \beta_1\beta_2 M_{\mathcal{S}_\alpha} \frac{b^{\frac{2\alpha-1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} ((M_{\mathcal{T}_\alpha}\mathcal{R} + \mathcal{R}) + \eta) \right] \|x_1 - x_2\|_{PC} \\ &\leq \gamma \|x_1 - x_2\|_{PC}. \end{aligned} \tag{16}$$

From (15) and (16) we get

$$H(\mathcal{P}(x_1), \mathcal{P}(x_2)) \leq \gamma \|x_1 - x_2\|_{PC}, \quad \forall x_1, x_2 \in PC(J, E).$$

From (10), we have

$$\gamma = \left[M_{\mathcal{T}_\alpha}(\mathcal{R} + \kappa) + \eta + \beta_1\beta_2 M_{\mathcal{S}_\alpha} \frac{b^{\frac{2\alpha-1}{2}}}{(2\alpha-1)^{\frac{1}{2}}} ((M_{\mathcal{T}_\alpha}\mathcal{R} + \mathcal{R}) + \eta) \right] < 1.$$

Thus, $\mathcal{P}$ is $\gamma-$contraction. Therefore, by Theorem 2.10 $\mathcal{P}$ has a fixed point which is $PC-$mild solution for the system (2). Consequently, the system (2) is controllable on the interval J. □

5. Examples

To illustrate the application of our main results, let us examine the following examples:

5.1. Example

Let $J = [0, 1]$, $E = \mathbb{R}$

$$\begin{cases} {}^cD_t^{\frac{1}{2}} x(t) \in x(t) + \mathcal{F}(t, x(t)) + u(t)), & t \in [0, 1], \\ x(0) = 0, \end{cases} \tag{17}$$

where $A = \mathcal{B} = 1$. It is clear that C_1 and C_6 are satisfied.
The mild solution of (17) will be

$$x(t) = \int_0^t \mathcal{S}_\alpha(t-s)(f(s) + u_f(s))ds.$$

We will define u_f as

$$u_f(t) = W^{-1} \left[-\int_0^1 \mathcal{S}_\alpha(t-s) f(s) ds \right](t).$$

Let $\mathcal{P} : \mathbb{R} \to P(\mathbb{R})$ with

$$\mathcal{P}(x)(t) = \int_0^t \mathcal{S}_\alpha(t-s)(f(s) + u_f(s))ds.$$

$\mathcal{F} : [0, 1] \times \mathbb{R} \to P(\mathcal{R})$ with

$$\mathcal{F}(t, x(t)) = \left(\frac{t}{\sqrt{(t+1)}}, \frac{\cot^{-1} x}{2\sqrt{t^2+1}} + \frac{\sin t}{\sqrt{t+1}} \right).$$

One can easily prove that $\mathcal{F}$ satisfies C_2. The condition C_3 is satisfied because

$$\|\mathcal{F}(t, x)\| \leq \frac{\pi}{2\sqrt{t^2+1}} + \frac{\sin t}{\sqrt{t+1}} = \varsigma(t).$$

Also,

$$\|\mathcal{P}(x) - \mathcal{P}(y)\| \leq \frac{1}{2}\|x - y\|, \quad \forall\, x, y \in \mathbb{R}.$$

So, $\gamma \leq \frac{1}{2} < 1$. Moreover, C_4 and C_5 are valid because $\mathcal{G} = 0$, $I_i(x) = 0$. Therefore, by Theorem 4.3 the system (17) is controllable on $[0, 1]$.

5.2. Example

Let $J = [0, 5]$.

$$\begin{cases} {}^cD^\alpha x(t) = \dfrac{|x(t)|}{(1+e^t)(1+|x(t)|}, & t \in [0,5], \\ x(0) = 0, \\ \Delta x(t_i) = \frac{x}{2^i}, & i = 1, \cdots, m, \end{cases} \tag{18}$$

Not that $A = \mathcal{B} = 0$ and thus C_1 and C_6 are satisfied.
Let $\mathcal{F} : J \times [0, \infty) \to [0, \infty)$. Clearly, for each $x \in [0, \infty), t \in J$ we have

$$\|\mathcal{F}(t, x)\| = \frac{1}{1+e^t}\frac{|x|}{|(1+x)|} \leq \frac{1}{4}|x| = \varsigma(t).$$

So, C_2 and C_3 are satisfied.
For every $i = 1, \cdots, m$, $I_i : [0, \infty) \to [0, \infty)$ define by

$$I_i(x(t_i)) = \frac{x}{2^i}.$$

Note that C_5 is valid since

$$\|I_i(x(t_i)) - I_i(y(t_i))\| = \frac{1}{2^i}\|x - y\|.$$

Also, C_4 is valid because $\mathcal{G} = 0$. Thus, all assumptions of Theorem 4.3 are satisfied. So, the system (18) is controllable on [0,5].

Conclusion

We investigated the existence of mild solutions for nonlocal fractional impulsive semilinear differential inclusions related to Caputo derivative in Banach spaces when the linear part is sectorial. Secondly, we determined the enough, conditions for the controllability of the studied control problem. We applied effectively fixed point theorems, contraction mapping, multivalued analysis and fractional calculus. Moreover, we enhanced our results by introducing an illustrative examples.

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