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COMBINATORIAL RESULTS OF COLLAPSE FOR ORDER-PRESERVING AND ORDER-DECREASING TRANSFORMATIONS

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Abstract. The full transformation semigroup $\mathcal{T}_n$ is defined to consist of all functions from $X_n = \{1, \ldots, n\}$ to itself, under the operation of composition. In [9], for any α in $\mathcal{T}_n$, Howie defined and denoted collapse by $c(\alpha) = \bigcup\limits_{t \in im(\alpha)} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}$. Let $\mathcal{O}_n$ be the semigroup of all order-preserving transformations and $\mathcal{C}_n$ be the semigroup of all order-preserving and decreasing transformations on X_n under its natural order, respectively. Let $E(\mathcal{O}_n)$ be the set of all idempotent elements of $\mathcal{O}_n$, $E(\mathcal{C}_n)$ and $N(\mathcal{C}_n)$ be the sets of all idempotent and nilpotent elements of $\mathcal{C}_n$, respectively. Let U be one of $\{\mathcal{C}_n, N(\mathcal{C}_n), E(\mathcal{C}_n), \mathcal{O}_n, E(\mathcal{O}_n)\}$. For $\alpha \in U$, we consider the set $im^c(\alpha) = \{t \in im(\alpha) : |t\alpha^{-1}| \geq 2\}$. For positive integers $2 \leq k \leq r \leq n$, we define

$$\begin{aligned} \mathcal{U}(k) &= \{\alpha \in \mathcal{U} : t \in im^c(\alpha) \text{ and } |t\alpha^{-1}| = k\}, \\ \mathcal{U}(k,r) &= \{\alpha \in \mathcal{U}(k) : |\bigcup_{t \in im^c(\alpha)} t\alpha^{-1}| = r\}. \end{aligned}$$

The main objective of this paper is to determine $|\mathcal{U}(k,r)|$, and so $|\mathcal{U}(k)|$ for some values r and k.

1. Introduction

For any non-empty finite set X, the set $\mathcal{T}_X$ of all transformations of X (i.e., all maps X to itself) is a semigroup under composition, and is called the *full transformation semigroup* on X. For any $n \in \mathbb{N}$, if $X = X_n = \{1, \ldots, n\}$, then $\mathcal{T}_X$ is denoted by $\mathcal{T}_n$. A transformation $\alpha \in \mathcal{T}_n$ is called *order-preserving*, if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$ and *decreasing* (*increasing*), if $x\alpha \leq x$ ($x\alpha \geq x$) for all $x \in X_n$. The subsemigroup of all order-preserving transformations in $\mathcal{T}_n$ is denoted by $\mathcal{O}_n$ and the order-decreasing transformations in $\mathcal{T}_n$ is denoted by $\mathcal{D}_n$.

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The subsemigroup of all order-preserving and decreasing (increasing) transformations in $\mathcal{T}_n$ is denoted by $\mathcal{C}_n$ $(\mathcal{C}_n^+)$ i.e., $\mathcal{C}_n = \mathcal{O}_n \cap \mathcal{D}_n$. Umar proved that $\mathcal{D}_n$ and $\mathcal{D}_n^+$ are isomorphic in [15, Corollary 2.7.]. Furthermore, Higgins proved that $\mathcal{C}_n$ and $\mathcal{C}_n^+$ are isomorphic semigroups in [8, Remarks, p. 290]. For any transformation α in $\mathcal{T}_n$, the *collapse*, the *fix*, the *image* and the *kernel* are denoted and definded, respectively, by

$$\begin{aligned} c(\alpha) &= \bigcup_{t \in im(\alpha)} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}, \quad (\alpha) = \{x \in X_n : x\alpha = x\}, \\ im(\alpha) &= \{x\alpha : x \in X_n\}, \text{ and } \ker(\alpha) = \{(x, y) : x\alpha = y\alpha \,\text{for all}\, x, y \in X_n\}. \end{aligned}$$

Given transformation α in $\mathcal{T}_n$ is called *collapsible*, if there exists $t \in im(\alpha)$ such that $|t\alpha^{-1}| \geq 2$.

An element e of a semigroup S is called *idempotent* if $e^2 = e$ and the set of all idempotents in S is denoted by $E(S)$. An element a of a finite semigroup S with a zero, denoted by 0, is called *nilpotent* if $a^m = 0$ for some positive integer m, and furthermore, if $a^{m-1} \neq 0$, then a is called an *m-nilpotent* element of S. Note that zero element is an 1-nilpotent element. The set of all nilpotent elements of S is denoted by $N(S)$. It was proven a finite semigroup S with zero is nilpotent when exactly the unique idempotent of S is the zero element (see, [6, Proposition 8.1.2]). The reader is referred to [5] and [11] for additional details in semigroup theory.

Recall that *Fibonacci sequence* $\{f_n\}$ is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$, where $f_1 = f_2 = 1$ (see [10]). As proved in [13, Theorem 2.1], $|\mathcal{C}_n| = |\mathcal{C}_n^+| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th *Catalan number* for $n \geq 1$ (see, [7]). That is why $\mathcal{C}_n$ is also called the *Catalan monoid*. In [13, Proposition 2.3] and [8, Theorem 3.19], it has been shown that $|N(C_n)| = |N(C_n^+)| = C_{n-1}$ and $|E(\mathcal{C}_n)| = |E(\mathcal{C}_n^+)| = 2^{n-1}$. Also, from [10, Theorem 2.1 and Theorem 2.3], we have that $|\mathcal{O}_n| = \binom{2n-1}{n-1}$ and $|E(\mathcal{O}_n)| = f_{2n}$.

As indicated in [5] if $\alpha \in \mathcal{C}_n$, we use

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix} \tag{1}$$

to notifty that $im(\alpha) = \{a_1 = 1 < a_2 < \ldots < a_r\}$ and $a_i\alpha^{-1} = A_i$ for each $1 \leq i \leq r$. Furthermore, $A_1, A_2, \ldots, A_r$ which are pairwise disjoint subsets of X_n are called *blocks* of α. It is clear that such an α is an idempotent if and only if $a_i \in A_i$ for all i. As defined in [4] a set $K \subseteq X_n$ is called *convex* if K is in the form $[i, i+t] = \{i, i+1, \ldots, i+t-1, i+t\}$. A partition $P = \{A_1, \ldots, A_r\}$ of X_n for $1 \leq r \leq n$ is called an *ordered partition*, and written $P = (A_1 < \cdots < A_r)$ if $x < y$ for all $x \in A_i$ and $y \in A_{i+1}$ $(1 \leq i \leq r-1)$. For a given $\alpha \in \mathcal{C}_n$ let $im(\alpha) = \{a_1 = 1 < a_2 < \ldots < a_r\}$ and $A_i = a_i\alpha^{-1}$ for every $1 \leq i \leq r$. Then, the set of kernel clasess of α, $X_n/\ker(\alpha) = \{A_1, \ldots, A_r\}$, is an ordered convex partition of X_n. Since $N(\mathcal{C}_n)$ is a nilpotent subsemigroup of $\mathcal{C}_n$, if $\alpha \in N(\mathcal{C}_n)$, then $1\alpha = 2\alpha = 1$, and that $|1\alpha^{-1}| \geq 2$.

Several authors studied certain problems concerning combinatorial aspects of semigroup theory during the years. The vast majority of papers have been written in this area such as [3, 9, 12, 13, 15, 16]. The rank (minimal size of a generating set) and idempotent rank (minimal size of an idempotent generating set) of several transformations semigroups have been studied in [9], [12] and [16] by using the combinatorial methods. A mapping $\alpha : dom(\alpha) \subseteq X_n \to im(\alpha) \subseteq X_n$ is called a *partial transformation*, and the set of all partial transformations is a semigroup under composition and denoted by $\mathcal{P}_n$. In the articles [1] and [14] the numbers $|\mathcal{T}_n(k,r)|$ and $|\mathcal{P}_n(k,r)|$ were calculated for $r = k = 2, 3$. Since then, $\mathcal{T}_n(k,r)$ were determined for $r = k$ for $2 \leq k \leq n$ in [2]. In the present paper, we calculate $|\mathcal{C}_n(k,k)|$, $|\mathcal{C}_n(k,2k)|$, $|\mathcal{C}_n(2,n)|$, $|N(\mathcal{C}_n)(k,k)|$, $|N(\mathcal{C}_n)(k,2k)|$, $|N(\mathcal{C}_n)(2,n)|$, $|E(\mathcal{C}_n)(k,r)|$, $|\mathcal{O}_n(k,k)|$ and $|E(\mathcal{O}_n)(k,k)|$. These invariants could be interesting and useful in the study of structure of semigroups.

2. Collapsible elements in $\mathcal{C}_n$

Let $\mathcal{U}(k,r) = \mathcal{C}_n(k,r)$ for positive integers $2 \leq k \leq r \leq n$. Then, it is obvious that $|\mathcal{C}_n(k,r)| = 0$ if k does not divide r, and further $|\mathcal{C}_n(n,n)| = 1$. Note that 1_n which denotes identity element of $\mathcal{C}_n$ and $\mathcal{O}_n$ is the only non-collapsible element of $\mathcal{C}_n$ and $\mathcal{O}_n$ then, the number of collapsible elements in $\mathcal{C}_n$ and $\mathcal{O}_n$ are $C_n - 1$ and $\binom{2n-1}{n-1} - 1$, respectively. The proof of the next combinatorial result is easy and is omitted.

Lemma 1. *For positive integers k and n where $1 \leq k \leq n$,*

$$\sum_{i=1}^{n-k+1} \binom{n-i}{n-k-i+1} = \binom{n}{k}.$$

□

Theorem 1. *For positive integers k and n where $2 \leq k \leq n$,*

$$|\mathcal{C}_n(k,k)| = \binom{n}{k}.$$

Proof. For a given $\alpha \in \mathcal{C}_n(k,k)$ it is clear that there exists $i \in im(\alpha)$ such that $|i\alpha^{-1}| = k$ and $\min(i\alpha^{-1}) = i$. So,

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & [i, k+i-1] & \{k+i\} & \cdots & \{n\} \\ 1 & 2 & \cdots & i-1 & i & (k+i)\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \leq i \leq n-k+1$. As can be seen the above form, we choose elements of $im(\alpha)$ from the set $[i+1, n]$ for the set $[k+i, n]$. There are $\binom{n-(i+1)+1}{n-(k+i)+1} = \binom{n-i}{n-k-i+1}$ ways to do that. This yields, there are $\binom{n-i}{n-k-i+1}$ elements in $\mathcal{C}_n(k,k)$ for a fixed i. Since $1 \leq i \leq n-k+1$, it follows directly from Lemma 1 that

$$|\mathcal{C}_n(k,k)| = \sum_{i=1}^{n-k+1} \binom{n-i}{n-k-i+1} = \binom{n}{k}.$$

□

Our next result computes $|\mathcal{C}_n(k,2k)|$.

Proposition 1. *For positive integers k and n where $2 \leq k \leq n$,*

$$|\mathcal{C}_n(k,2k)| = \sum_{i=1}^{n-2k+1} \sum_{j=i+k}^{n-k+1} \sum_{l=j-k+1}^{j} \binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}.$$

Proof. Given $\alpha \in \mathcal{C}_n(k,2k)$, let $A_i = [i, k+i-1]$ and $A_j = [j, k+j-1]$ be any two blocks of α each of which contain k elements. So,

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & A_i & \{k+i\} & \cdots & A_j & \cdots & \{n\} \\ 1 & 2 & \cdots & i-1 & i & (k+i)\alpha & \cdots & j\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \leq i \leq n-2k+1$ and $i+k \leq j \leq n-k+1$. Let $j\alpha = l$ where $j-k+1 \leq l \leq j$. As can be seen above form, we choose elements of $im(\alpha)$ from the set $[i+1, l-1]$ for the set $[k+i, j-1]$ and from the set $[l+1, n]$ for the set $[k+j, n]$. However, this can be done $\binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}$ ways. This yields, there are $\binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}$ elements in $\mathcal{C}_n(k,2k)$ for fixed i, j and l. Since $1 \leq i \leq n-2k+1$, $i+k \leq j \leq n-k+1$ and $j-k+1 \leq l \leq j$, it follows quickly that

$$|\mathcal{C}_n(k,2k)| = \sum_{i=1}^{n-2k+1} \sum_{j=i+k}^{n-k+1} \sum_{l=j-k+1}^{j} \binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}.$$

□

Theorem 2. *For positive even integer $n \geq 2$,*

$$|\mathcal{C}_n(2,n)| = \frac{2}{(n+2)}\binom{n}{\frac{n}{2}}.$$

Proof. For any $\alpha \in \mathcal{C}_n(2,n)$, it is clear that n must be even, and so $|\mathcal{C}_n(n,2)| = 0$ if 2 does not divide n. Then, the result will clearly follow if we establish a bijection between $\mathcal{C}_n(2,n)$ and $\mathcal{C}_{\frac{n}{2}}$. Define a map $\theta : \mathcal{C}_n(2,n) \to \mathcal{C}_{\frac{n}{2}}$ by $(\alpha)\theta = \alpha'$ where

$$\begin{cases} (2i-1)\alpha = i\alpha' + i - 1, & i = 1, 2, \ldots, \frac{n}{2}; \\ (2i)\alpha = i\alpha' + i - 1, & i = 1, 2, \ldots, \frac{n}{2}, \end{cases}$$

that is,

$$\begin{cases} j\alpha = (\frac{j+1}{2})\alpha' + \frac{j-1}{2}, & j = 1, 3, \ldots, n-1; \\ j\alpha = \frac{j}{2}\alpha' + \frac{j-2}{2}, & j = 2, 4, \ldots, n. \end{cases}$$

This yields, θ is a well-defined bijection. Since $|\mathcal{C}_{\frac{n}{2}}| = C_{\frac{n}{2}}$, the proof is completed.

□

Example 1. *The function* $\theta : \mathcal{C}_6(2,6) \to \mathcal{C}_{\frac{6}{2}}$ *defined as in above is a bijection. Certainly,*

$$\begin{aligned}\mathcal{C}_6(2,6) \quad = \quad & \left\{ \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 4 \end{pmatrix}, \right. \\ & \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 5 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 3 & 4 \end{pmatrix}, \\ & \left. \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 3 & 5 \end{pmatrix} \right\} \text{ and}\end{aligned}$$

$$\begin{aligned}\mathcal{C}_3 \quad = \quad & \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\},\end{aligned}$$

as wanted. □

Let $\mathcal{U}(k,r) = N(\mathcal{C}_n)(k,r)$ for positive integers $2 \leq k \leq r \leq n$. Clearly, $|N(\mathcal{C}_n)(k,r)| = 0$ if k does not divide r, and also $|N(\mathcal{C}_n)(n,n)| = 1$ and $|N(\mathcal{C}_n(n-1,n-1)| = n-2$. Note that $\alpha \in N(\mathcal{C}_n)$, $1\alpha = 2\alpha = 1$ and $i\alpha \leq i-1$ for all $3 \leq i \leq n$, and so the number of collapsible emenets in $N(\mathcal{C}_n)$ is $|N(\mathcal{C}_n)| = C_{n-1}$.

Lemma 2. *For positive integers k and n where $2 \leq k \leq n$,*

$$|N(\mathcal{C}_n)(k,k)| = \binom{n-2}{n-k}.$$

Proof. Given $\alpha \in N(\mathcal{C}_n)(k,k)$, since $1\alpha = 2\alpha = 1$ and $|1\alpha^{-1}| = k$, we have

$$\alpha = \begin{pmatrix} [1,k] & \{k+1\} & \{k+2\} & \cdots & \{n\} \\ 1 & (k+1)\alpha & (k+2)\alpha & \cdots & n\alpha \end{pmatrix}.$$

As can be seen above form, we choose elements of $im(\alpha)$ from the set $[2,n]$ for the set $[k+1,n-1]$. However, there are

$$|N(\mathcal{C}_n)(k,k)| = \binom{n-2}{n-k}$$

ways to do that, as required. □

Proposition 2. *For positive integers k and n where $2 \leq k \leq n$,*

$$|N(\mathcal{C}_n)(k,2k)| = \sum_{j=k+1}^{n-k+1} \sum_{l=2}^{j} \binom{l-2}{j-k-1} \binom{n-l}{n-k-j+1}.$$

Proof. Given $\alpha \in N(\mathcal{C}_n)(k,2k)$, let $A_1 = [1,k]$ and $A_j = [j, k+j-1]$ be any two blocks of α which contain k elements. This yields,

$$\alpha = \begin{pmatrix} A_1 & \{k+1\} & \cdots & A_j & \{n\} \\ 1 & (k+1)\alpha & \cdots & j\alpha & n\alpha \end{pmatrix},$$

where $k+1 \le j \le n-k+1$. Let $j\alpha = l$ where $2 \le l \le j$. As can be seen above form, we choose element of $im(\alpha)$ from the set $[2, l-1]$ for the set $[k+1, j-1]$ and from the set $[l+1, n]$ for the set $[k+j, n]$. However, this can be done $\binom{l-2}{j-k-1}\binom{n-l}{n-k-j-1}$ ways. This yields, there are $\binom{l-2}{j-k-1}\binom{n-l}{n-k-j-1}$ elements in $N(\mathcal{C}_n)(k,2k)$ for fixed j and l. Since $k+1 \le j \le n-k+1$ and $2 \le l \le j$, it follows quickly that

$$|N(\mathcal{C}_n)(k,2k)| = \sum_{j=k+1}^{n-k+1} \sum_{l=2}^{j} \binom{l-2}{j-k-1}\binom{n-l}{n-k-j+1}.$$

□

Theorem 3. *For positive even integer $n \ge 2$,*

$$|N(\mathcal{C}_n)(2,n)| = \frac{2}{n}\binom{n-2}{\frac{n-2}{2}}.$$

Proof. Let α be any element of $N(\mathcal{C}_n)(n,2)$. Then, it is clear that n must be even, and so $|N(\mathcal{C}_n)(2,n)| = 0$ if 2 does not divide n. If we construct a bijection between $N(\mathcal{C}_{\frac{n}{2}})$ and $|N(\mathcal{C}_n)(2,n)|$, then this completes the proof. Define a map $\theta : N(\mathcal{C}_n)(2,n) \to N(\mathcal{C}_{\frac{n}{2}})$ by $(\alpha)\theta = \alpha'$ where

$$\begin{cases} (2i-1)\alpha = i\alpha' + i - 1, & i = 1, 2, \ldots, \frac{n}{2}; \\ (2i)\alpha = i\alpha' + i - 1, & i = 1, 2, \ldots, \frac{n}{2}, \end{cases}$$

that is,

$$\begin{cases} j\alpha = (\frac{j+1}{2})\alpha' + \frac{j-1}{2}, & j = 1, 3, \ldots, n-1; \\ j\alpha = \frac{j}{2}\alpha' + \frac{j-2}{2}, & j = 2, 4, \ldots, n. \end{cases}$$

Now it is easy to check that θ is a well-defined bijection. Since $|N(\mathcal{C}_{\frac{n}{2}})| = C_{\frac{n}{2}-1}$, the proof is complete. □

Example 2. *The function $\theta : N(\mathcal{C}_8)(2,8) \to N(\mathcal{C}_{\frac{8}{2}})$ defined as in above is a bijection. Indeed, $= N(\mathcal{C}_8)(2,8) =$*

$$\left\{ \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 3 & 5 \end{pmatrix}, \right.$$
$$\begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 3 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 4 & 5 \end{pmatrix},$$
$$\left. \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 4 & 6 \end{pmatrix} \right\} \text{ and}$$

$$N(\mathcal{C}_4) = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}\right\},$$

as required. □

Let $\mathcal{U}(k,r) = E(\mathcal{C}_n)(k,r)$ for positive integers $2 \leq k \leq r \leq n$. Clearly, $|E(\mathcal{C}_n)(k,r)| = 0$ if k does not divide r, and also $|E(\mathcal{C}_n)(n,n)| = 1$. Note that the number of collapsible elements in $E(\mathcal{C}_n)$ is $2^{n-1} - 1$.

Theorem 4. *For positive integers k, r and n where $2 \leq k \leq r \leq n$ and $r = kt$,*

$$|E(\mathcal{C}_n)(k,r)| = \binom{n+t-r}{t}.$$

Proof. If $\alpha \in E(\mathcal{C}_n)(k,r)$ and $r = kt$, then $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_{n+t-r} \\ 1 & a_2 & \cdots & a_{n+t-r} \end{pmatrix}$, where $a_i \in A_i$ for all $1 \leq i \leq n+t-r$. Since $r = kt$, ordered partition of α contains $n+t-r$ blocks such that t blocks contain k elements and $n-kt$ blocks contain one element. Without loss of generality assume that each of the sets $A_1, A_2, \ldots, A_t$ contains k elements and each of the sets $A_{t+1}, A_{t+2}, \ldots, A_{n+t-r}$ contains one element. Since α is an idempotent, it is clear that α is the only element in $E(\mathcal{C}_n)(k,r)$ with this ordered partition. Hence, all elements of $E(\mathcal{C}_n)(k,r)$ are entirely determined by choosing t blocks which contain k elements. Since we choose t blocks $\binom{n+t-r}{t}$ ways, this completes the proof. □

The next result is clear from the definition of $\mathcal{U}(k)$ and $\mathcal{U}(k,r)$:

$$|\mathcal{U}(k)| = \sum_{i=1}^{t} |\mathcal{U}(k,ik)|,$$

where $t = \frac{n}{k}$.

Example 3. *We obtain $|E(\mathcal{C}_6)(2,4)| = \binom{6+2-4}{2} = 6$ by Theorem 4. Since $n = 6, r = 4, k = 2, t = 2$, each element in $E(\mathcal{C}_6)(2,4)$ have $6+2-4$ blocks such that 2 blocks contain 2 elements and 2 blocks are singletons. Indeed, $E(\mathcal{C}_6)(2,4) =$*

$$\left\{\begin{pmatrix} \{1,2\} & \{3,4\} & \{5\} & \{6\} \\ 1 & 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4,5\} & \{6\} \\ 1 & 3 & 4 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} & \{5,6\} \\ 1 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2,3\} & \{4,5\} & \{6\} \\ 1 & 2 & 4 & 6 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2,3\} & \{4\} & \{5,6\} \\ 1 & 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 3 & 5 \end{pmatrix}\right\}.$$

Furthermore, $|E(\mathcal{C}_6)(2)| = \sum_{i=1}^{3} |E(\mathcal{C}_6)(2,i2)| = |E(\mathcal{C}_6)(2,2)| + |E(\mathcal{C}_6)(2,4)| + |E(\mathcal{C}_6)(2,6)| = \binom{6+1-2}{1} + \binom{6+2-4}{2} + \binom{6+3-6}{3} = 12.$ □

3. Collapsible elements in $\mathcal{O}_n$

Let $U(k,r) = \mathcal{O}_n(k,r)$ for positive integers $2 \le k \le r \le n$. Then, it is clear that $|(\mathcal{O}_n)(r,k)| = 0$ if k does not divide r, and also $|(\mathcal{O}_n)(n,n)| = n$. By convention, we take $\binom{0}{0} = 1$ in the following theorem.

Theorem 5. *For positive integers k and n where $2 \le k \le n$,*

$$|\mathcal{O}_n(k,k)| = \sum_{i=1}^{n-k+1} \sum_{j=i}^{k+i-1} \binom{j-1}{i-1} \binom{n-j}{n-k-i+1}.$$

Proof. For any $\alpha \in \mathcal{O}_n(k,k)$, let

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & [i,k+i-1] & \{k+i\} & \cdots & \{n\} \\ 1\alpha & 2\alpha & \cdots & (i-1)\alpha & i\alpha & (k+i)\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \le i \le n-k+1$. As can be seen above form, the set of all value of $i\alpha$ is the set $[i,k+i-1]$ and for all distinct $m,r \in X_n \setminus [i,k+i-1]$, it is clear that $m\alpha \ne r\alpha$. Let $i\alpha = j$ where $i \le j \le k+i-1$. Then, we choose elements of $im(\alpha)$ for the left and right sides of $i\alpha = j$. For the left side, we choose elements from the set $[1,j-1]$ for the set $[1,i-1]$. There are $\binom{j-1}{i-1}$ ways to do that. For the right side, we choose the elements from the set $[j+1,n]$ for the set $[k+i,n]$. There are $\binom{n-j}{n-k-i+1}$ ways to do that. This yields, there are $\binom{j-1}{i-1}\binom{n-j}{n-k-i+1}$ elements in $\mathcal{O}_n(k,k)$ for fixed i and j. Since $1 \le i \le n-k+1$ and $i \le j \le k+i-1$, it follows that

$$|\mathcal{O}_n(k,k)| = \sum_{i=1}^{n-k+1} \sum_{j=i}^{k+i-1} \binom{j-1}{i-1} \binom{n-j}{n-k-i+1}.$$

□

Let $\mathcal{U}(k,r) = E(\mathcal{O}_n)(k,r)$ for positive integers $2 \le k \le r \le n$. Clearly, $|E(\mathcal{O}_n)(k,r)| = 0$ if k does not divide r. Notice that the number of collapsible elements in $E(\mathcal{O}_n)$ is $f_{2n} - 1$.

Lemma 3. *For positive integers k and n where $2 \le k \le n$,*

$$|E(\mathcal{O}_n)(k,k)| = k(n-k+1).$$

Proof. For any $\alpha \in \mathcal{O}_n(k,k)$, let

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & [i,k+i-1] & \{k+i\} & \cdots & \{n\} \\ 1\alpha & 2\alpha & \cdots & (i-1)\alpha & i\alpha & (k+i)\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \le i \le n-k+1$. As can be seen above form, the set of all value of $i\alpha$ is the set $[i,k+i-1]$. Moreover, since α is an idempotent, $m\alpha = m$ for all $m \in X_n \setminus [i,k+i-1]$. Let $i\alpha = j$ where $i \le j \le k+i-1$. Then, it is easy to see that α is the only element in $E(\mathcal{O}_n)(k,k)$ for fixed i and j. Since $i \leq j \leq k+i-1$, there are k elements in $E(\mathcal{O}_n)(k,k)$ for fixed i. Since $1 \leq i \leq n-k+1$, it follows that

$$|E(\mathcal{O}_n)(k,k)| = k(n-k+1).$$

$\square$

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