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Research Article

Starlike functions associated with an epicycloid

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Abstract

For a natural number $n \geqslant 2$, the function $\phi_{n\mathcal{L}}(z) = 1 + nz/(n+1) + z^n/(n+1)$ maps the open unit disk onto a domain bounded by an epicycloid with $(n-1)$ cusps. A class of starlike functions associated with $\phi_{n\mathcal{L}}$ is defined in the unit disk and its sharp bounds on initial coefficients, various inclusion relations and radii problems related to the other subclasses of starlike functions are investigated. As an application, the corresponding results are determined in the limiting case for the class of normalized analytic functions f satisfying $|zf'(z)/f(z) - 1| < 1$ in the unit disk.

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1. Introduction

An epicycloid [16] is a plane curve produced by tracing the path of a chosen point on the circumference of a circle of radius r which rolls without slipping around a fixed circle of radius s. The parametric equation of an epicycloid is

$$x(t) = m\cos t - r\cos\left(\frac{mt}{r}\right), \quad y(t) = m\sin t - r\sin\left(\frac{mt}{r}\right), \qquad -\pi \leqslant t \leqslant \pi,$$

where $m = s + r$. If m/r is an integer, then the curve has $(m/r) - 1$ cusps. Some of the epicycloids have been given special names. For $s = r$, the curve obtained is called a cardioid and has one cusp; for $s = 2r$, it is a nephroid with two cusps and for $s = 5r$, the curve formed is called ranunculoid, a five-cusped epicycloid. Note that a parametric curve $(x(t), y(t))$ has a cusp [8] at the point t_0 if $x'(t_0)$ and $y'(t_0)$ are both zero but either $x''(t_0)$ or $y''(t_0)$ is not equal to zero. Many curves have been widely studied in the literature having no cusp, one cusp, two cusps and three cusps. For instance, the boundary of image domains of the functions e^z, $1 + \sin z$ and $2/(1 + e^{-z})$ [4, 6, 23], under the unit disk, has no cusp. The right-half of lemniscate of Bernoulli, the left-half of the shifted lemniscate of Bernoulli and cardioid shaped domains studied in [7, 13, 15, 22, 27, 30] contain one cusp

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on the real axis. Nephroid [33] has two cusps on the real axis whereas lune [25] and a petal-like domain studied in [14] contain two cusps at the angles $\pi/2$ and $3\pi/2$. Gandhi [5] studied the class of functions for which boundary of the image domain contains three cusps, one on real axis and rest two at the angles $\pi/3$ and $5\pi/3$. Taking into account this geometrical notion of cusp, a more general domain is considered whose boundary has the following parametric form:

$$x(t) = 1 + \frac{n}{n+1}\cos t + \frac{1}{n+1}\cos(nt), \quad y(t) = \frac{n}{n+1}\sin t + \frac{1}{n+1}\sin(nt), \tag{1.1}$$

where n is a natural number with $n \geqslant 2$. For $s = (n-1)/(n+1)$ and $r = 1/(n+1)$, the curve (1.1) represents a rotated and translated epicycloid [21] with $(n-1)$ cusps. It is an algebraic curve of order $2n$. It can be easily seen that $x'(t_k) = 0$ and $y'(t_k) = 0$ for $t_k = (2k+1)\pi/(n-1)$, where $k \in \mathbb{Z}$. Also, $x''(t_k)$ and $y''(t_k)$ are not zero together. As a result, the curve (1.1) has cusp at the points t_k. If $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk in the complex plane $\mathbb{C}$, then the function $\phi_{n\mathcal{L}} : \mathbb{D} \to \mathbb{C}$ defined by

$$\phi_{n\mathcal{L}}(z) := 1 + \frac{nz}{n+1} + \frac{z^n}{n+1}, \quad (z \in \mathbb{D}) \tag{1.2}$$

maps the unit disk onto a domain bounded by the curve (1.1).

Let $\mathcal{A}$ denotes the class of all analytic functions f defined in $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be its subclass consisting of univalent functions. Ma and Minda [17] studied the unified class of starlike functions $\mathcal{S}^*(\phi)$ consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \phi(z)$ for all $z \in \mathbb{D}$, where ϕ is a univalent function having positive real part, $\phi(\mathbb{D})$ is symmetric about real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. It is easy to see that the function $\phi_{n\mathcal{L}}$ given by (1.2) satisfies all these conditions and hence we can define $\mathcal{S}^*_{n\mathcal{L}} := \mathcal{S}^*(\phi_{n\mathcal{L}})$ to be the class of functions $f : \mathbb{D} \to \mathbb{C}$ that satisfies

$$\frac{zf'(z)}{f(z)} \prec \phi_{n\mathcal{L}}(z) = 1 + \frac{nz}{n+1} + \frac{z^n}{n+1}, \quad (z \in \mathbb{D}).$$

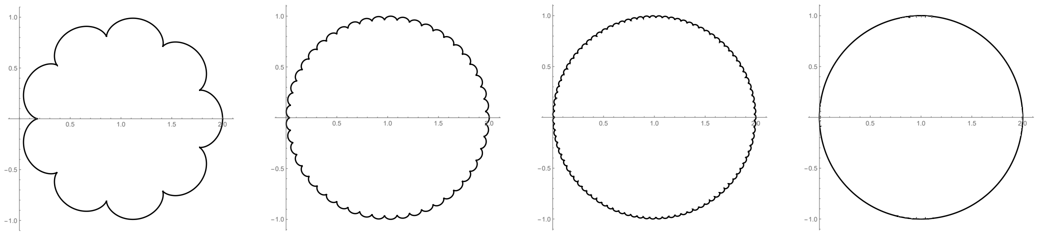
The function $f_{n\mathcal{L}} : \mathbb{D} \to \mathbb{C}$ defined as

$$f_{n\mathcal{L}}(z) = z \exp\left(\frac{n}{n+1}z + \frac{1}{n(n+1)}z^n\right) = z + \frac{n}{n+1}z^2 + \frac{n^2}{2(n+1)}z^3 + \cdots, \tag{1.3}$$

belongs to the class $\mathcal{S}^*_{n\mathcal{L}}$ and acts as an extremal function for most of the problems investigated for the class $\mathcal{S}^*_{n\mathcal{L}}$ in the present manuscript.

The sharp bounds for the initial four coefficients of a function $f \in \mathcal{S}^*_{n\mathcal{L}}$ are computed in Section 2. A preliminary lemma is proved in Section 3 to determine the largest radius of the disk centered on the real axis to lie inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$. Several inclusion relations between the class $\mathcal{S}^*_{n\mathcal{L}}$ and various subclasses of starlike functions which depends upon a parameter are established in Section 4. The cusp appearing in the boundary of the domain $\phi_{n\mathcal{L}}(\mathbb{D})$ at the angle $\pi/(n-1)$ plays a vital role in computing various radii constants concerning the class $\mathcal{S}^*_{n\mathcal{L}}$. Using the preliminary lemma proved in Section 3 and the notion of cusp, sharp $\mathcal{S}^*_{n\mathcal{L}}-$radii are evaluated for well-known classes of analytic functions in Section 5. The similar technique is employed to compute $\mathcal{S}^*(\phi)-$radius for the class $\mathcal{S}^*_{n\mathcal{L}}$ for different choices of the function ϕ in the last section of the paper. Throughout the paper (except Section 2), n is assumed to be an even natural number. A similar analysis can be carried out for the odd case as well.

It is worth to note that the class $\mathcal{S}^*_{n\mathcal{L}}$ reduces to the class $\mathcal{S}^*(1+z)$ as $n \to \infty$. In other words, the $(n-1)$ cusped domain transforms into the disk centered at $(1, 0)$ and radius 1 in the limiting case (see Figure 1).



(a) $n = 10$ (b) $n = 50$ (c) $n = 100$ (d) $n = 1000$

Figure 1. Limiting case

2. Coefficient bounds

In this section, sharp bounds for the first four initial coefficients of functions in the class $\mathcal{S}^*_{n\mathcal{L}}$ $(n \geqslant 2)$ are computed. For $0 \leqslant \alpha < 1$, let $\mathcal{P}(\alpha)$ be the class of analytic functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with $\operatorname{Re} p(z) > \alpha$ for all $z \in \mathbb{D}$ and let $\mathcal{P} := \mathcal{P}(0)$. The following estimates (see [11, 24, 26]) for the class $\mathcal{P}$ will be needed in our investigation.

Lemma 2.1. *If* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$, *then*

(i) $|c_2 - v c_1^2| \leqslant 2 \max\{1, |2v - 1|\}$,

(ii) $|c_3 - 2\beta c_1 c_3 + \delta c_1^3| \leqslant 2$ *if* $0 \leqslant \beta \leqslant 1$ *and* $\beta(2\beta - 1) \leqslant \delta \leqslant \beta$,

(iii) $|\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - (3/2)\beta c_1^2 c_2 - c_4| \leqslant 2$, *when* $0 < \alpha < 1$, $0 < a < 1$ *and* $8a(1-a)((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leqslant 4\alpha^2(1-\alpha)^2 a(1-a)$.

Theorem 2.2. *If* $n \geqslant 2$ *and* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}^*_{n\mathcal{L}}$, *then* $|a_2| \leqslant n/(n+1)$, $|a_3| \leqslant n/(2(n+1))$, $|a_4| \leqslant n/(3(n+1))$ *and* $|a_5| \leqslant n/(4(n+1))$. *All the bounds are best possible.*

Proof. The function $p(z) = zf'(z)/f(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is analytic in $\mathbb{D}$ and a simple calculation gives

$$(m-1)a_m = \sum_{k=1}^{m-1} b_k a_{m-k}, \text{ for } m \geqslant 2. \tag{2.1}$$

Since $\phi_{n\mathcal{L}}$ given by (1.2) is univalent in $\mathbb{D}$ and $p \prec \phi_{n\mathcal{L}}$, the function $p_1(z) = (1 + \phi_{n\mathcal{L}}^{-1}(p(z)))/(1 - \phi_{n\mathcal{L}}^{-1}(p(z))) = 1 + c_1 z + c_2 z^2 + \cdots$ belongs to the class $\mathcal{P}$ and it may be rewritten as $p(z) = \phi_{n\mathcal{L}}((p_1(z) - 1)/(p_1(z) + 1))$. This relation together with (2.1) yields

$$\begin{aligned}
a_2 &= b_1 = \frac{n}{2(n+1)} c_1 \\
a_3 &= \frac{n}{8(n+1)^2}(2(n+1)c_2 - c_1^2) \\
a_4 &= \frac{n}{48(n+1)^3}((n+2)c_1^3 - 2(n+1)(n+4)c_1 c_2 + 8(n+1)^2 c_3) \\
a_5 &= \frac{n}{384(n+1)^4}\big(48(n+1)^3 c_4 - 12(n+2)(n+1)^2 c_2^2 - 16(n+3)(n+1)^2 c_1 c_3 \\
&\qquad + 4(n^2 + 7n + 9)(n+1)c_1^2 c_2 - (n+2)(2n+3)c_1^4\big).
\end{aligned}$$

As $|c_1| \leqslant 2$, it follows that $|a_2| \leqslant n/(n+1)$. Using Lemma 2.1(i) with $v = 1/(2(n+1))$, we obtain

$$|a_3| \leqslant \frac{n}{4(n+1)} \left| c_2 - \left(\frac{1}{2(n+1)}\right) c_1^2 \right| \leqslant \frac{n}{2(n+1)}.$$

To compute the bound on a_4, observe that

$$|a_4| = \frac{n}{6(n+1)} \left| c_3 - \frac{n+4}{4(n+1)} c_1 c_2 + \frac{(n+2)}{8(n+1)^2} c_1^3 \right|.$$

Let us take $\beta = (n+4)/(8(n+1))$ and $\delta = (n+2)/(8(n+1)^2)$. For $n \geqslant 2$, it can be easily seen that $0 \leqslant \beta \leqslant 1$ and $\beta - \delta = (n^2+4n+2)/(8(n+1)^2) > 0$. Also, $\beta(2\beta-1) = -3n(n+4)/(32(n+1)^2) < 0 < \delta \leqslant \beta$. Thus, by Lemma 2.1(ii), $|a_4| \leqslant n/(3(n+1))$. Finally, note that

$$|a_5| = \frac{n}{8(n+1)}\left|\frac{(n+2)(2n+3)}{48(n+1)^3}c_1^4 + \frac{n+2}{4(n+1)}c_2^2 + \frac{n+3}{3(n+1)}c_1c_3 - \frac{n^2+7n+9}{12(n+1)^2}c_1^2c_2 - c_4\right|.$$

To complete the proof of the theorem, it suffices to show that the parameters

$$\gamma = \frac{(n+2)(2n+3)}{48(n+1)^3}, \quad a = \frac{n+2}{4(n+1)}, \quad \alpha = \frac{n+3}{6(n+1)}, \quad \beta = \frac{n^2+7n+9}{18(n+1)^2}$$

satisfy the hypothesis of Lemma 2.1(iii). For $n \geqslant 2$, it is clear that $0 < a, \alpha < 1$. Also, the quantity $8a(1-a)((\alpha\beta-2\gamma)^2+(\alpha(a+\alpha)-\beta)^2)+\alpha(1-\alpha)(\beta-2a\alpha)^2-4\alpha^2(1-\alpha)^2a(1-a)$ reduces to the expression $-(1301n^8+15816n^7+77806n^6+203428n^5+310942n^4+286536n^3+156564n^2+46656n+5832)/(93312(n+1)^8) \leqslant 0$ for all $n \in \mathbb{N}$. In view of Lemma 2.1(iii), it follows that $|a_5| \leqslant n/(4(n+1))$. For sharpness, the functions

$$f_i(z) = z\exp\left(\int_0^z \frac{\phi_{n\mathcal{L}}(t^{i-1})-1}{t}dt\right), \quad i = 2,3,4,5$$

are extremal for the coefficients a_i $(i = 2,3,4,5)$ respectively. □

For $n = 4$, Theorem 2.2 reduces to [5, Theorem 2.11, p. 179]. Also, in the limiting case $n \to \infty$, the bounds for the first four initial coefficients coincide with that of the result by Singh [29, Theorem 3, p. 79] for the class $\mathcal{S}^*(1+z)$.

3. Preliminary Lemma

From this section onwards, n is assumed to be an even natural number. For $2/(n+1) < a < 2$, the following lemma gives the radius of the largest disk centered at $(a,0)$ that can be inscribed inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$, where $\phi_{n\mathcal{L}}$ is given by (1.2). This lemma will be useful in investigating inclusion relations and various radii problems discussed in the next two sections of the paper.

Lemma 3.1. *For $2/(n+1) < a < 2$, let r_a be given by*

$$r_a = \begin{cases} a - \dfrac{2}{n+1}, & \dfrac{2}{n+1} < a \leqslant 1, \\ \sqrt{\beta_a}, & 1 \leqslant a < a_0, \\ 2-a, & a_0 \leqslant a < 2, \end{cases}$$

where

$$\beta_a := a^2 - 2a + \frac{2(n^2+1)}{(n+1)^2} + 2(1-a)\left(\frac{n-1}{n+1}\right)\cos\left(\frac{\pi}{n-1}\right)$$

and

$$a_0 := \frac{(n^2-1)\cos\left(\frac{\pi}{n-1}\right) - (n^2+4n+1)}{(n^2-1)\cos\left(\frac{\pi}{n-1}\right) - (n+1)^2}.$$

Then $\{w : |w-a| < r_a\} \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$.

Proof. Any point on the boundary of $\phi_{n\mathcal{L}}(\mathbb{D})$ is of the form $\phi_{n\mathcal{L}}(e^{it})$, $0 \leqslant t \leqslant 2\pi$. Since the curve $w = \phi_{n\mathcal{L}}(e^{it})$ is symmetric with respect to real axis, so it is sufficient to consider the interval $0 \leqslant t \leqslant \pi$. The square of the distance from the point $(a,0)$ to the points on the curve $\phi_{n\mathcal{L}}(e^{it})$ is given by:

$$\sigma(t) = \left(1+\frac{n}{n+1}\cos t+\frac{1}{n+1}\cos(nt)-a\right)^2+\left(\frac{n}{n+1}\sin t+\frac{1}{n+1}\sin(nt)\right)^2.$$

It can be easily seen that

$$\sigma'(t) = \frac{4n}{(n+1)^2}\cos\left(\frac{(n-1)t}{2}\right)\left[(n+1)(a-1)\sin\left(\frac{(n+1)t}{2}\right)-(n-1)\sin\left(\frac{(n-1)t}{2}\right)\right]$$

and

$$\sigma''(t) = \frac{2n((n-1)^2\cos((n-1)t)-(a-1)(n+1)\cos t-(a-1)n(n+1)\cos(nt))}{(n+1)^2}.$$

A calculation shows that $\sigma'(t)=0$ for

$$t=0,\frac{\pi}{n-1},\frac{3\pi}{n-1},\cdots,\frac{(n-3)\pi}{n-1},\pi.$$

By the geometry of the curve $\phi_{n\mathcal{L}}(e^{it})$, it is evident that the cusps are the most probable choices for the minimum of the function σ. As a result, the other roots of the equation $\sigma'(t)=0$ are not taken into consideration. For $n=2$, it can be easily seen that $\min\{\sigma(t): t\in[0,\pi]\}$ equals $\sigma(\pi)$ if $2/3<a\leqslant 4/3$ and $\sigma(0)$ if $4/3\leqslant a<2$. Let $n\geqslant 4$ be an even natural number. Observe that

$$\sigma''(0) = 2n\left(a-\frac{2(n^2+1)}{(n+1)^2}\right)>0, \text{ if } a>\frac{2(n^2+1)}{(n+1)^2}>1,$$
$$\sigma''(\pi) = \frac{2n(n-1)(a(n+1)-2)}{(n+1)^2}>0 \text{ if } a>\frac{2}{n+1},$$

and $\sigma''(\pi/(n-1))>0$ for $a<a_1$ where

$$a_1 := 1-\frac{(n-1)^2}{(n+1)\left(\cos\left(\frac{\pi}{n-1}\right)+n\cos\left(\frac{n\pi}{n-1}\right)\right)}.$$
$$=1+\frac{(n-1)}{(n+1)\cos\left(\frac{\pi}{n-1}\right)}.$$

Note that $a_1>2(n^2+1)/(n+1)^2>1$. Keeping in mind these observations, consider the following three cases:

Case 1: $2/(n+1)<a<1$. A straightforward calculation shows that

$$\sigma\left(\frac{k\pi}{n-1}\right)-\sigma(\pi)=2(1-a)\left(\frac{n-1}{n+1}\right)\left(1+\cos\left(\frac{k\pi}{n-1}\right)\right)>0$$

for $k=1,3,5,\ldots,n-3$. Therefore the minimum of σ is attained at $t=\pi$.

Case 2: $a=1$. In this case, the function σ assumes the same value at $k\pi/(n-1)$, for $k=1,3,5,\ldots,n-1$. Thus the minimum value is $\sigma(\pi)$.

Case 3: $1<a<2$. For this case, $\sigma(\pi)>\sigma(k\pi/(n-1))$ for $k=1,3,5,\ldots,n-3$ using the similar argument given in Case 1. Similarly, a simple calculation shows that

$$\sigma\left(\frac{k\pi}{n-1}\right)-\sigma\left(\frac{\pi}{n-1}\right)=2(a-1)\left(\frac{n-1}{n+1}\right)\left(\cos\left(\frac{\pi}{n-1}\right)-\cos\left(\frac{k\pi}{n-1}\right)\right)>0$$

for $k=3,5,\ldots,n-3$, since cosine is a decreasing function in $[0,\pi]$. Consequently, it follows that σ attains its minimum either at $t=\pi/(n-1)$ or $t=0$. For $1<a\leqslant 2(n^2+1)/(n+1)^2$, it is clear that $\min\{\sigma(t): t\in[0,\pi]\}=\sigma(\pi/(n-1))=\beta_a$. Let $a>2(n^2+1)/(n+1)^2$. Note that

$$\sigma\left(\frac{\pi}{n-1}\right)-\sigma(0)=2\left(1-\frac{n-1}{n+1}\cos\left(\frac{\pi}{n-1}\right)\right)(a-a_0).$$

Further, it is easy to deduce that $2(n^2+1)/(n+1)^2 < a_0 < a_1$. Thus, if $a \leqslant a_0$, then $\sigma(\pi/(n-1)) \leqslant \sigma(0)$ so that σ attains its minimum at $t = \pi/(n-1)$. If $a_0 \leqslant a < 2$, then $\sigma(\pi/(n-1)) \geqslant \sigma(0)$ and $\min\{\sigma(t) : t \in [0,\pi]\} = \sigma(0)$. □

4. Inclusion relations

This section deals with inclusion relation between the class $\mathcal{S}^*_{n\mathcal{L}}$ and various classes of starlike functions which depends on a parameter. For instance, $\mathcal{SS}^*(\beta)\,(0 < \beta \leqslant 1)$ is the class characterized by $|\arg(zf'(z)/f(z))| < \beta\pi/2$, $\mathcal{S}^*[A,B] := \mathcal{S}^*(1+Az)/(1+Bz)$ is the class of Janowski starlike functions for $-1 \leqslant B < A \leqslant 1$ and $\mathcal{S}^*(\alpha) := \mathcal{S}^*[1-2\alpha,-1]$ is the class of starlike functions of order $\alpha\,(0 \leqslant \alpha < 1)$. Sokół [31] introduced the class $\mathcal{S}^*(\sqrt{1+cz})$ which is associated with right loop of the Cassinian ovals given by $(u^2+v^2)^2 - 2(u^2-v^2) = c^2-1$, for $0 < c \leqslant 1$. For $c = 1$, this class reduces to the class $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+z})$. Also, for $0 \leqslant \alpha < 1$, the generalized class $\mathcal{S}^*_L(\alpha) = \mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z})$ was introduced by Khatter *et al.* [12] and this class also reduces to $\mathcal{S}^*_L$ for $\alpha = 0$. The following theorem gives various inclusion relations of the class $\mathcal{S}^*_{n\mathcal{L}}$ with the above mentioned classes, where n is an even natural number. For $a, r \in \mathbb{R}$ with $r > 0$, let $\mathbb{D}(a,r)$ denotes the open disk with centre $(a,0)$ and radius r. By Lemma 3.1, $\mathbb{D}(1,(n-1)/(n+1))$ lies inside $\phi_{n\mathcal{L}}(\mathbb{D})$ for each even natural number n.

Theorem 4.1. *For the class $\mathcal{S}^*_{n\mathcal{L}}$, the following relations holds:*

(a) $\mathcal{S}^*_{n\mathcal{L}} \subseteq \mathcal{S}^*(\alpha)$ *for* $0 \leqslant \alpha \leqslant 1 - \cos(\pi/(n+1))$.
(b) $\mathcal{S}^*_{n\mathcal{L}} \subseteq \mathcal{SS}^*(\beta)$ *for* $\beta \geqslant (n-1)/n$.
(c) $\mathcal{S}^*_{n\mathcal{L}} \subseteq \mathcal{S}^*[1,-(M-1)/M]$ *for* $M \geqslant 1$.
(d) $\mathcal{S}^*_L(\alpha) \subseteq \mathcal{S}^*_{n\mathcal{L}}$ *for* $\alpha \geqslant 2/(n+1)$.
(e) $\mathcal{S}^*\left(\sqrt{1+cz}\right) \subseteq \mathcal{S}^*_{n\mathcal{L}}$ *for* $0 < c \leqslant 1 - 4/(n+1)^2$.
(f) $\mathcal{S}^*[A,B] \subseteq \mathcal{S}^*_{n\mathcal{L}}$ $(-1 < B < A \leqslant 1)$ *if one of the following holds:*
  (i) $2(1-B^2) < (n+1)(1-AB) \leqslant (n+1)(1-B^2)$ *and* $(n+1)A \leqslant 2B + n - 1$,
  (ii) $1 - B^2 \leqslant 1 - AB \leqslant a_0(1-B^2)$ *and* $A \leqslant \sqrt{\beta_a}(1-B^2) + B$,
  (iii) $a_0(1-B^2) \leqslant 1 - AB < 2(1-B^2)$ *and* $A \leqslant 2B + 1$,
  where a_0 and β_a are same as defined in Lemma 3.1, with $a = (1-AB)/(1-B^2)$.

Proof. Let $f \in \mathcal{S}^*_{n\mathcal{L}}$. Then $zf'/f \prec \phi_{n\mathcal{L}}$, where $\phi_{n\mathcal{L}}$ is given by (1.2). For part (a), note that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \min_{|z|=1} \operatorname{Re}\left(\phi_{n\mathcal{L}}(z)\right).$$

For $z = e^{it}$, we have

$$\operatorname{Re}\left(\phi_{n\mathcal{L}}(e^{it})\right) = 1 + \frac{n\cos t}{n+1} + \frac{\cos(nt)}{n+1} := h(t),$$

where $t \in [0,2\pi]$. Since the curve $w = \phi_{n\mathcal{L}}(e^{it})$ is symmetric with respect to real axis, so in order to compute the minimum value of $h(t)$, we shall obtain all the critical points of the function h in the interval $[0,\pi]$. Note that

$$h'(t) = -\frac{2n}{n+1}\sin\left(\frac{(n+1)t}{2}\right)\cos\left(\frac{(n-1)t}{2}\right).$$

Therefore, the critical points are the roots of the equation $h'(t) = 0$. This gives $t_k = 2k\pi/(n+1)$ or $u_l = (2l+1)\pi/(n-1)$, for $k, l \in \mathbb{Z}$. Since $t_k, u_l \in [0,\pi]$, therefore $k = 0, 1, \ldots, n/2$ and $l = 0, 1, 2 \ldots, (n-2)/2$. A close inspection shows that h attains its minimum at t_k corresponding to $k = n/2$. Hence

$$\min_{|z|=1} \operatorname{Re}(\phi_{n\mathcal{L}}(z)) = h(t_{n/2}) = 1 + \frac{n}{n+1}\cos\left(\frac{n\pi}{n+1}\right) + \frac{1}{n+1}\cos\left(\frac{n^2\pi}{n+1}\right)$$
$$= 1 - \cos\left(\frac{\pi}{n+1}\right)$$

on simplification. Thus $f \in \mathcal{S}^*(\alpha)$, for $0 \leqslant \alpha \leqslant 1 - \cos(\pi/(n+1))$. For part (b), we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \max_{|z|=1} \arg(\phi_{n\mathcal{L}}(z)) = \max_{t\in(-\pi,\pi]} \arg(\phi_{n\mathcal{L}}(e^{it}))$$
$$= \max_{t\in(-\pi,\pi]} \tan^{-1}\left(\frac{n\sin t + \sin(nt)}{n+1+n\cos t + \cos(nt)}\right) = \tan^{-1}\left(\max_{t\in(-\pi,\pi]} g(t)\right),$$

where $g(t) = (n\sin t + \sin(nt))/(n+1+n\cos t+\cos(nt))$. As pointed out earlier, it is sufficient to compute the maximum value of $g(t)$ for $0 \leqslant t \leqslant \pi$. Indeed, we have

$$g'(t) = \frac{4n(n+1)\cos\left(\frac{nt}{2}\right)\cos\left(\frac{t}{2}\right)\cos\left(\frac{(n-1)t}{2}\right)}{(n+1+n\cos t+\cos(nt))^2}.$$

The possible critical points of g are $t_k = (2k+1)\pi/n$, $u_l = (2l+1)\pi$ and $v_m = (2m+1)\pi/(n-1)$, for $k, l, m \in \mathbb{Z}$. For t_k, u_l and v_m to lie in the interval $[0, \pi]$, we must have $k = 0, 1, 2\ldots, (n-2)/2$, $l = 0$ and $m = 0, 1, 2, \ldots, (n-2)/2$. A simple analysis of these critical points shows that the maximum of the function g is attained at the point t_k for $k = (n-2)/2$. Hence

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \tan^{-1}\left(g\left(\frac{(n-1)\pi}{n}\right)\right) = \tan^{-1}\left(\cot\left(\frac{\pi}{2n}\right)\right) = \frac{(n-1)\pi}{2n}.$$

So, $\mathcal{S}^*_{n\mathcal{L}} \subset \mathcal{SS}^*(\beta)$, where $\beta \geqslant (n-1)/n$. For proving the part (c), note that

$$\left|\frac{zf'(z)}{f(z)} - M\right| \leqslant |1-M| + \frac{n|z|}{n+1} + \frac{|z|^n}{n+1} < |1-M| + 1$$

Thus, for $M \geqslant 1$, $|zf'(z)/f(z) - M| < M$. The curves $\gamma_1 : \operatorname{Re} w = 1 - \cos(\pi/9)$, $\gamma_2 : \arg w = 7\pi/16$ and $\gamma_3 : |w-1| = 1$ for the case $n = 8$ in Figure 2(a)-(c) depict that the results are best possible.

For (d), let $f \in \mathcal{S}^*_L(\alpha)$. Then the quantity $zf'(z)/f(z)$ lies inside the domain $L_\alpha = \{w : |((w-\alpha)/(1-\alpha))^2 - 1| < 1\}$ and [12, Lemma 2.1, p. 236] gives

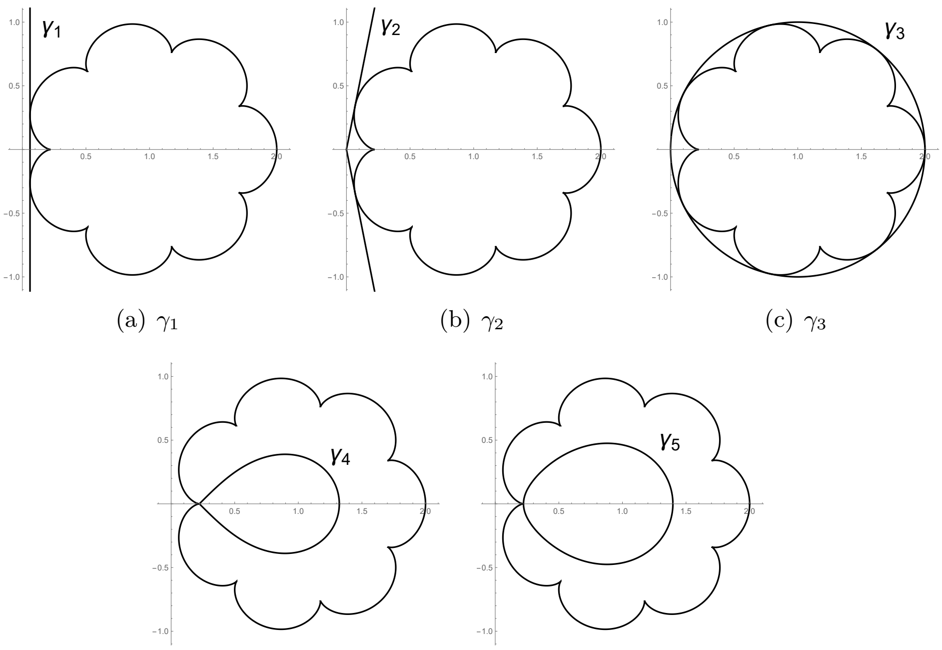
$$\alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha + (1-\alpha)\sqrt{2}.$$

The condition $\alpha + (1-\alpha)\sqrt{2} \leqslant 2$ is always satisfied. Therefore, for the function f to lie in the class $\mathcal{S}^*_{n\mathcal{L}}$, it is necessary that $\alpha \geqslant 2/(n+1)$. To prove that this condition is sufficient, note that if $\alpha_1 \geqslant \alpha_2$, then $L_{\alpha_1} \subseteq L_{\alpha_2}$. Consequently, it follows that if $\alpha \geqslant 2/(n+1)$, then $L_\alpha \subseteq L_{2/(n+1)}$. Now, it suffices to show that $L_{2/(n+1)} \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$. To verify this, we shall invoke [12, Lemma 2.3, p. 238] to prove that $L_{2/(n+1)}$ is contained in the disk $\mathbb{D}(1, (n-1)/(n+1))$. Consider the following two cases:

Case 1: $n = 2$. By taking $\alpha = 2/3$ and $a = 1$, the condition $\alpha < a < \sqrt{2}\alpha + (1-\alpha)/\sqrt{2}$ is satisfied. As a result, $L_{2/3} \subseteq \mathbb{D}(1, (\sqrt{2}-1)/3)$ which is obviously contained in the disk $\mathbb{D}(1, 1/3)$.

Case 2: $n \geqslant 4$. If we take $\alpha = 2/(n+1)$ and $a = 1$, then the condition $\sqrt{2}\alpha + (1-\alpha)/\sqrt{2} < a < \alpha + (1-\alpha)\sqrt{2}$ holds which implies that $L_{2/(n+1)} \subseteq \mathbb{D}(1, (n-1)/(n+1))$.

Combining these two cases, we conclude that $L_\alpha \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$ for $\alpha \geqslant 2/(n+1)$. The case $n = 8$ illustrated in Figure 2(d) by curve $\gamma_4 : \partial L_{2/9}$ shows that the bound is best possible.



(a) γ_1 (b) γ_2 (c) γ_3

(d) γ_4 (e) γ_5

Figure 2. Inclusion Relations for the class $\mathcal{S}^*_{8\mathcal{L}}$

For proving (e), if $f \in \mathcal{S}^*(\sqrt{1+cz})$, then $zf'(z)/f(z) \prec \sqrt{1+cz}$ and

$$\sqrt{1-c} < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \sqrt{1+c}.$$

Note that $\sqrt{1+c} < \sqrt{2} < 2$ and $\sqrt{1-c} \geqslant 2/(n+1)$ provided $c \leqslant 1-4/(n+1)^2$. To prove that this condition is sufficient as well, observe that if $Q_c = \{w \in \mathbb{C} : |w^2-1| < c\}$, then $Q_c \subseteq \mathbb{D}(a, R_a)$ where $0 < c \leqslant 1$, $\sqrt{1-c} < a < \sqrt{1+c}$ and R_a is given by

$$R_a = \begin{cases} \sqrt{1+c}-a, & \sqrt{1-c} < a \leqslant (\sqrt{1-c}+\sqrt{1+c})/2; \\ a-\sqrt{1-c}, & (\sqrt{1-c}+\sqrt{1+c})/2 \leqslant a < \sqrt{1+c}. \end{cases}$$

This can be proved on similar lines as that of [1, Lemma 2.2, p. 6559]. Since $Q_{c_1} \subseteq Q_{c_2}$ for $c_1 \leqslant c_2$, therefore it follows that if $c \leqslant 1-4/(n+1)^2$, then $Q_c \subseteq Q_{1-4/(n+1)^2}$. If we take $a = 1$ and $c = 1-4/(n+1)^2$, then the condition $(\sqrt{1-c}+\sqrt{1+c})/2 \leqslant a < \sqrt{1+c}$ is satisfied. Consequently, $Q_{1-4/(n+1)^2} \subseteq \mathbb{D}(1,(n-1)/(n+1))$ which is contained in $\phi_{n\mathcal{L}}(\mathbb{D})$. The sharpness is depicted for $n = 8$ by considering the curve $\gamma_5 : \partial Q_{77/81}$ in Figure 2(e).

For the last part, let $f \in \mathcal{S}^*[A,B]$. Then the quantity $zf'(z)/f(z)$ lies inside the disk $|w-a| < r_a$ with centre $a := (1-AB)/(1-B^2)$ and radius $r_a := (A-B)/(1-B^2)$. It is a simple exercise to show that this disk lies inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$ under the given three conditions in view of Lemma 3.1. □

Let us give an application of Theorem 4.1(f). By taking $A = 1-\alpha$ $(0 \leqslant \alpha < 1)$ and $B = 0$, the conditions $2(1-B^2) < (n+1)(1-AB) \leqslant (n+1)(1-B^2)$ and $(n+1)A \leqslant 2B+n-1$, are satisfied for $\alpha \geqslant 2/(n+1)$. Thus $\mathcal{S}^*[1-\alpha, 0] \subseteq \mathcal{S}^*_{n\mathcal{L}}$ for $2/(n+1) \leqslant \alpha < 1$.

5. $\mathcal{S}^*_{n\mathcal{L}}$−radius

In this section, the $\mathcal{S}^*_{n\mathcal{L}}$−radius for various well-known subclasses of analytic functions is investigated, where $n \in \mathbb{N}$ is an even natural number. By the notation $R_{\mathcal{M}_1}(\mathcal{M}_2)$, we mean the largest radius for which the functions $f \in \mathcal{M}_2$ satisfy the property of the functions of the set $\mathcal{M}_1$ in each subdisk $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ for $r \leqslant R_{\mathcal{M}_1}(\mathcal{M}_2)$.

MacGregor [18–20] studied (i) the class $\mathcal{W}$ of functions $f \in \mathcal{A}$ such that $f(z)/z \in \mathcal{P}$; (ii) the class $\mathcal{F}_1$ of functions $f \in \mathcal{A}$ such that $f/g \in \mathcal{P}$ for some $g \in \mathcal{A}$ with $g(z)/z \in \mathcal{P}$; and (iii) the class $\mathcal{F}_2$ of functions $f \in \mathcal{A}$ such that $g/f \in \mathcal{P}(1/2)$ for some $g \in \mathcal{A}$ with $g(z)/z \in \mathcal{P}$. For a function $p \in \mathcal{P}(\alpha)$ $(0 \leqslant \alpha < 1)$ and $z \in \mathbb{D}$ with $|z| = r$, the estimate

$$\left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2r(1-\alpha)}{(1-r)(1+(1-2\alpha)r)}, \tag{5.1}$$

is used to prove the following theorem, which evaluates the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the classes $\mathcal{W}$, $\mathcal{F}_1$ and $\mathcal{F}_2$.

Theorem 5.1. *If $n \in \mathbb{N}$ is an even natural number, then the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the classes $\mathcal{W}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ is given by*

(a) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{W}) = \dfrac{\sqrt{2(n^2+1)}-(n+1)}{n-1}$

(b) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{F}_1) = \dfrac{\sqrt{5n^2+6n+5}-2(n+1)}{n-1}$

(c) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{F}_2) = \dfrac{\sqrt{17n^2+10n+9}-3(n+1)}{4n}$.

Proof. (a) Let $f \in \mathcal{W}$. Then the function $p(z) = f(z)/z$ belongs to $\mathcal{P}$ and $zf'(z)/f(z) = 1 + zp'(z)/p(z)$ so that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant \frac{2r}{1-r^2}$$

by using (5.1). In view of Lemma 3.1, the function $f \in \mathcal{S}^*_{n\mathcal{L}}$ if $2r/(1-r^2) \leqslant 1 - 2/(n+1)$. This simplifies to $r \leqslant (\sqrt{2(n^2+1)} - (n+1))/(n-1)$. The result is sharp for the function $f_1(z) = z(1+z)/(1-z) \in \mathcal{W}$. For this function, $zf_1'(z)/f_1(z) = 2/(n+1) = \phi_{n\mathcal{L}}(-1)$, for $z = -(\sqrt{2(1+n^2)} - (n+1))/(n-1)$.

(b) For $f \in \mathcal{F}_1$, the functions $k_1(z) = f(z)/g(z)$ and $k_2(z) = g(z)/z$ belong to $\mathcal{P}$ and $f(z) = zk_1(z)k_2(z)$ so that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zk_1'(z)}{k_1(z)} + \frac{zk_2'(z)}{k_2(z)}.$$

Using (5.1), we obtain

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant \frac{4r}{1-r^2}.$$

By Lemma 3.1, $4r/(1-r^2) \leqslant 1 - 2/(n+1)$ which yields $r \leqslant (\sqrt{5n^2+6n+5} - 2(n+1))/(n-1)$. For sharpness, consider the functions $f_2(z) = z((1+z)/(1-z))^2$ and $g_2(z) = z(1+z)/(1-z)$. For $z = -(\sqrt{5n^2+6n+5} - 2(n+1))/(n-1)$, $zf_2'(z)/f_2(z) = 2/(n+1)$.

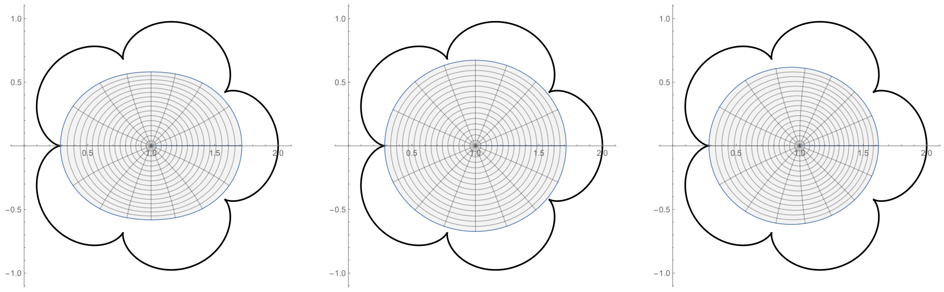
(c) Let $f \in \mathcal{F}_2$. Then the functions $k_3(z) = g(z)/f(z)$ and $k_4(z) = g(z)/z$ belong to $\mathcal{P}(1/2)$ and $\mathcal{P}$ respectively and $f(z) = zk_4(z)/k_3(z)$. A simple computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zk_4'(z)}{k_4(z)} - \frac{zk_3'(z)}{k_3(z)}.$$

so that (5.1) gives

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant \frac{3r+r^2}{1-r^2}.$$

This disk lies inside $\phi_{n\mathcal{L}}(\mathbb{D})$ provided $(3r+r^2)/(1-r^2) \leqslant 1 - 2/(n+1)$, by Lemma 3.1, which holds for $r \leqslant (\sqrt{17n^2+10n+9} - 3(n+1))/(4n)$. The bound is sharp for the function $f_3(z) = z(1+z)^2/(1-z)$ with $g_3(z) = z(1+z)/(1-z)$. For $z = -(\sqrt{17n^2+10n+9} - 3(n+1))/(4n)$, the quantity $zf_3'(z)/f_3(z)$ equals $2/(n+1)$.



(a) $\mathcal{W}$ (b) $\mathcal{F}_1$ (c) $\mathcal{F}_2$

Figure 3. $\mathcal{S}^*_{n\mathcal{L}}-$radius for classes $\mathcal{W}$, $\mathcal{F}_1$ and $\mathcal{F}_2$

The sharpness in all the three parts is illustrated in Figure 3 by depicting the image domains of the functions $zf_i'(z)/f_i(z)$ $(i = 1, 2, 3)$ under the specified subdisks $|z| < (\sqrt{74} - 7)/5$, $|z| < (\sqrt{221} - 14)/5$ and $|z| < (\sqrt{681} - 21)/24$ respectively for $n = 6$. □

Apart from the classes discussed in Section 4, several other subclasses of starlike functions are studied by various authors for an appropriate choice of the function ϕ to define the class $\mathcal{S}^*(\phi)$ (as investigated by Ma and Minda [17]). The classes needed in our investigation are $\mathcal{S}^*_{RL} := \mathcal{S}^*(\sqrt{2} - (\sqrt{2} - 1)((1 - z)/(1 + 2(\sqrt{2} - 1)z))^{1/2})$, $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$, $\mathcal{S}^*_C = \mathcal{S}^*(1 + 4z/3 + 2z^2/3)$, $\mathcal{S}^*_{\mathbb{C}} := \mathcal{S}^*(z + \sqrt{1 + z^2})$, $\mathcal{S}^*_R := \mathcal{S}^*((k^2 + z^2)/(k^2 - kz))$ $(k = \sqrt{2} + 1)$, $\mathcal{S}^*_{sin} := \mathcal{S}^*(1 + \sin z)$, $\mathcal{S}^*_{lim} = \mathcal{S}^*(1 + \sqrt{2}z + z^2/2)$, $\mathcal{S}^*_{SG} := \mathcal{S}^*(2/(1 + e^{-z}))$, $\mathcal{S}^*_{ne} := \mathcal{S}^*(1 + z - z^3/3)$, $\mathcal{S}^*_{\wp} := \mathcal{S}^*(1 + ze^z)$, $\mathcal{S}^*_{cosh} := \mathcal{S}^*(\cosh z)$, $\mathcal{S}^*_{\rho} := \mathcal{S}^*(1 + \sinh^{-1}(z))$ and $\mathcal{S}^*_{car} := \mathcal{S}^*(1 + z + z^2/2)$. These classes are studied in [2–7,9,13–15,22,23,27,28,33,34]. The $\mathcal{S}^*_{n\mathcal{L}}-$radius for the classes $\mathcal{S}^*_{SG}$ and $\mathcal{S}^*_{cosh}$ is 1 as the functions $2/(1 + e^{-z})$ and $\cosh z$ map $\mathbb{D}$ inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$. By making use of Lemma 3.1, the next theorem determines $\mathcal{S}^*_{n\mathcal{L}}-$radius for three subclasses of starlike functions.

Theorem 5.2. *For an even natural number n, the sharp $\mathcal{S}^*_{n\mathcal{L}}-$radii are given by*

(a) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_L(\alpha)) = \dfrac{(n-1)\left((n+3) - 2\alpha(n+1)\right)}{(n+1)^2(1-\alpha)^2}$, $0 \leqslant \alpha < \dfrac{2}{n+1}$.

(b) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}\left(\mathcal{S}^*(\sqrt{1+cz})\right) = \dfrac{n^2 + 2n - 3}{c(n+1)^2}$, $1 - \dfrac{4}{(n+1)^2} < c \leqslant 1$.

(c) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{RL}) = \dfrac{7n^2 - 4\sqrt{2}n - 2n - 5 + 4\sqrt{2}}{7n^2 - 24\sqrt{2}n + 30n + 47 - 32\sqrt{2}}$.

Proof. (a) For $2/(n+1) \leqslant \alpha < 1$, the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_L(\alpha)$ is 1 by Theorem 4.1(d). Let $0 \leqslant \alpha < 2/(n+1)$ and $f \in \mathcal{S}^*_L(\alpha)$. Then $zf'(z)/f(z) \prec \alpha + (1-\alpha)\sqrt{1+z}$ and for $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant (1-\alpha)\left(1 - \sqrt{1-r}\right).$$

This disk lies inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$ provided $(1-\alpha)(1-\sqrt{1-r}) \leqslant 1 - 2/(n+1)$ by Lemma 3.1, which yields $r \leqslant (n-1)\left(n+3-2\alpha(n+1)\right)/((n+1)^2(1-\alpha)^2) := r_1$. The result is sharp for the function g_1 with $zg_1'(z)/g_1(z) = \alpha + (1-\alpha)\sqrt{1+z}$ as $-r_1 g_1'(-r_1)/g_1(-r_1) = 2/(n+1) = \phi_{n\mathcal{L}}(-1)$. For $\alpha = 0$ and $n = 6$, the sharpness is illustrated in Figure 4(a).

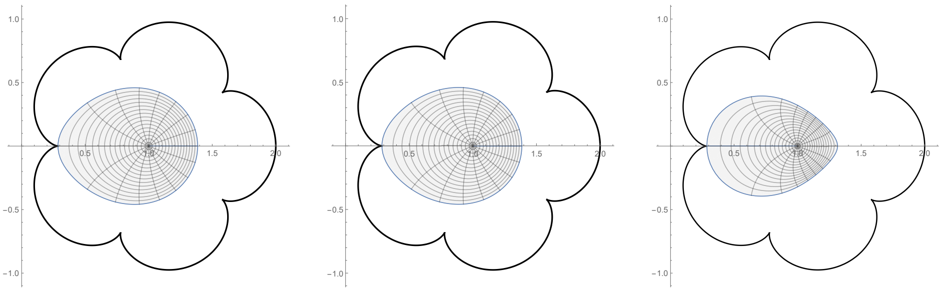
(b) For $0 < c \leqslant 1 - 4/(n+1)^2$, the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*(\sqrt{1+cz})$ is 1 by Theorem 4.1(e). Assume that $1 - 4/(n+1)^2 < c \leqslant 1$ and $f \in \mathcal{S}^*(\sqrt{1+cz})$. Then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1 - \sqrt{1-cr}, \quad |z| = r.$$

By using Lemma 3.1, we get $1 - \sqrt{1-cr} \leqslant 1 - 2/(n+1)$ which simplifies to $r \leqslant (n^2 + 2n - 3)/(c(n+1)^2) := r_2$. The result is sharp for the function g_2 defined as

$$g_2(z) = \frac{4z \exp(2\sqrt{1+cz} - 2)}{(\sqrt{1+cz} + 1)^2}$$

and $zg_2'(z)/g_2(z) = 2/(n+1)$ for $z = -r_2$ (see Figure 4(b) for $n = 6$ and $c = 45/49$).



(a) $\mathcal{S}^*_L$ (b) $\mathcal{S}^*(\sqrt{1+(45/49)z})$ (c) $\mathcal{S}^*_{RL}$

Figure 4. $\mathcal{S}^*_{n\mathcal{L}}-$radius of the classes $\mathcal{S}^*_L$, $\mathcal{S}^*(\sqrt{1+cz})$ and $\mathcal{S}^*_{RL}$

(c) Let $f \in \mathcal{S}^*_{RL}$. Then it is easy to see that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1 - \left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right),$$

where $|z| = r$. In view of Lemma 3.1, the above disk lies inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$ provided

$$1 - \left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right) \leqslant 1 - \frac{2}{n+1}.$$

This holds true for $r \leqslant (7n^2 - 4\sqrt{2}n - 2n - 5 + 4\sqrt{2})/(7n^2 - 24\sqrt{2}n + 30n + 47 - 32\sqrt{2}) := r_3$. The result is sharp for the function g_3 defined by

$$\frac{zg_3'(z)}{g_3(z)} = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$$

which assumes the value $2/(n+1)$ at $z = -r_3$ as illustrated in Figure 4(c) for $n = 6$. □

As pointed out in Section 1, the cusp at the angle $\pi/(n-1)$ will play a pivotal role in finding the radii constants for some of the classes. This technique will be employed in the following theorem.

Theorem 5.3. *For an even natural number n and $\gamma = e^{i\pi/(n-1)}$, the sharp $\mathcal{S}^*_{n\mathcal{L}}-$radius for various Ma-Minda type subclasses of starlike functions is given by*

(a) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_C) = \left|\sqrt{\dfrac{2(2+3\gamma^n + (2+3\gamma)n)}{n+1}} - 1\right|.$

(b) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_e) = \left|\log\left(1 + \dfrac{n\gamma + \gamma^n}{n+1}\right)\right|.$

(c) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{\mathfrak{C}}) = \left|\dfrac{\gamma^{2n}+2n\gamma^{n+1}+n^2\gamma^2+2(n+1)\gamma^n+2n(n+1)\gamma}{2(n+1)(1+\gamma^n+n(1+\gamma))}\right|.$

(d) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{R}) = \begin{cases} 1, & n=2 \\ r_4, & n \geqslant 4 \end{cases}$, *where*

$$r_4 := \frac{1}{2(n+1)}\left|\sqrt{(3+2\sqrt{2})(\gamma^{2n}+2n\gamma^{n+1}+n^2\gamma^2+6(n+1)\gamma^n+6n(n+1)\gamma+(n+1)^2} -(1+\sqrt{2})(\gamma^n+1+n(1+\gamma)))\right|.$$

(e) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{lim}) = \sqrt{2}\left|\sqrt{\dfrac{1+n(1+\gamma)+\gamma^n}{n+1}}-1\right|.$

(f) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{car}) = \left|\sqrt{\dfrac{1+n(1+2\gamma)+2\gamma^n}{n+1}}-1\right|.$

(g) $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{\wp}) = \begin{cases} r_7, & n=2, \\ \tilde{r}_7, & n \geqslant 4, \end{cases}$
where r_7 and $\tilde{r}_7$ are the solutions of the equations $re^r = 1$ and $re^r = (n\gamma+\gamma^n)/(n+1)$, respectively.

Proof. (a) Let $f \in \mathcal{S}^*_C$. Then $zf'(z)/f(z) \prec \phi_C(z)$ where $\phi_C(z) = 1+4z/3+2z^2/3$. We need to find the value of r such that the function ϕ_C maps the subdisk $\mathbb{D}_r$ into the domain $\phi_{n\mathcal{L}}(\mathbb{D})$. Since the epicycloid curve $\partial\phi_{n\mathcal{L}}(\mathbb{D})$ has a cusp at $\pi/(n-1)$, therefore for the cardioid domain $\phi_C(\mathbb{D}_r)$ to lie inside $\phi_{n\mathcal{L}}(\mathbb{D})$, it is necessary that $r \leqslant r_1$, where r_1 is the absolute value of the solution of the equation $\phi_C(z) = \phi_{n\mathcal{L}}(\gamma)$ or

$$\frac{4z}{3}+\frac{2z^2}{3} = \frac{n\gamma}{n+1}+\frac{\gamma^n}{n+1}$$

for z, which is given by $r_1 = |\sqrt{(2(2+3\gamma^n+(2+3\gamma)n))/(n+1)}-1|$.

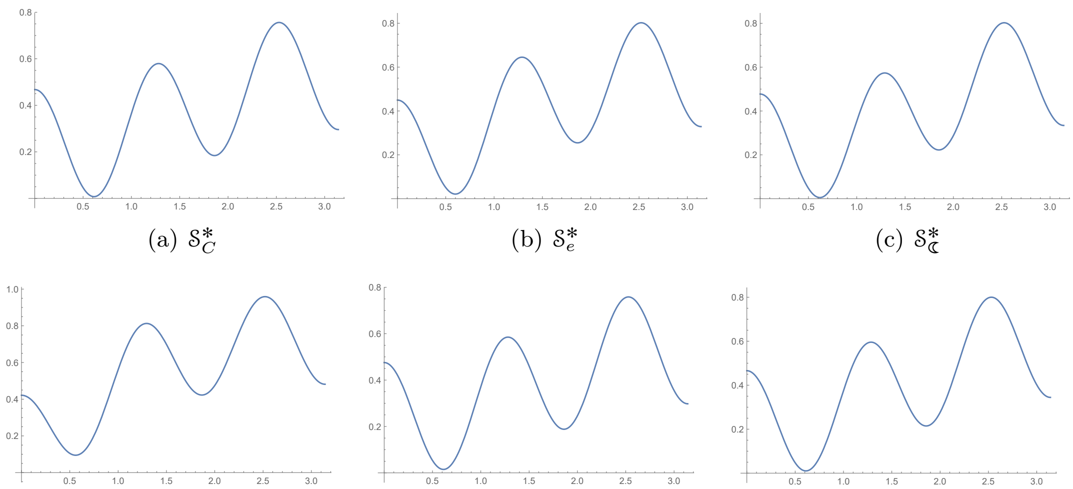
In order to prove that the condition $r \leqslant r_1$ is sufficient as well, it suffices to show that the image of $\mathbb{D}_{r_1}$ under the function ϕ_C lies inside $\phi_{n\mathcal{L}}(\mathbb{D})$. To prove this, we consider the difference of the square of the distances of points on the boundary curves $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_C(r_1e^{it})$ with the point $(1,0)$, which is denoted by $d(t,n)$ and defined as

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it})-1|^2 - |\phi_C(r_1e^{it})-1|^2 \\ &= \frac{n^2+1+2n\cos((n-1)t)}{(n+1)^2} - \frac{4r_1^2(4+r_1^2+4r_1\cos t)}{9}, \quad t \in [0,\pi] \text{ and } n \geqslant 2. \end{aligned}$$

Using the technique of calculus, it can be shown that $d(t,n) \geqslant 0$ for all $t \in [0,\pi]$ and $n \geqslant 2$ (its graph for $n=6$ in plotted in Figure 5(a)). This implies that $\phi_C(\mathbb{D}_{r_1}) \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$. The result is sharp for the function $\tilde{f}_1(z) = z\exp(4z/3+z^2/3)$, the case $n=6$ being depicted in Figure 6(a).

(b) If $f \in \mathcal{S}^*_e$, then $zf'(z)/f(z) \prec \phi_e(z) := e^z$. Proceeding as in part (a), the $\mathcal{S}^*_{n\mathcal{L}}$-radius for the class $\mathcal{S}^*_e$ is at most $r_2 = |\log(1+(n\gamma+\gamma^n)/(n+1))|$ which is the absolute value of the solution of the equation $\phi_e(z) = \phi_{n\mathcal{L}}(\gamma)$ for z. It remains to show that the $\mathcal{S}^*_{n\mathcal{L}}$-radius for the class $\mathcal{S}^*_e$ is at least r_2, that is, $\phi_e(\mathbb{D}_{r_2}) \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$. The difference between the square of the distances of the points on the boundary curves $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_e(r_2e^{it})$ from the point $(1,0)$ is given by

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it})-1|^2 - |\phi_e(r_2e^{it})-1|^2 \\ &= \frac{2n(\cos((n-1)t)-1)+(n+1)^2e^{r_2\cos t}(2\cos(r_2\sin t)-e^{r_2\cos t})}{(n+1)^2}. \end{aligned}$$



(a) $\mathcal{S}^*_C$ (b) $\mathcal{S}^*_e$ (c) $\mathcal{S}^*_{\mathfrak{C}}$

(d) $\mathcal{S}^*_R$ (e) $\mathcal{S}^*_{lim}$ (f) $\mathcal{S}^*_{car}$

Figure 5. Graph of distance function $d(t,6)$

A computation shows that $d(t,n)$ is non-negative for $t \in [0,\pi]$ and $n \geqslant 2$ (see the graph of $d(t,6)$ in Figure 5(b)). The result is sharp for the function

$$\tilde{f}_2(z) = \exp\left(\int_0^z \frac{e^t-1}{t}dt\right)$$

and the case $n = 6$ is illustrated in Figure 6(b).

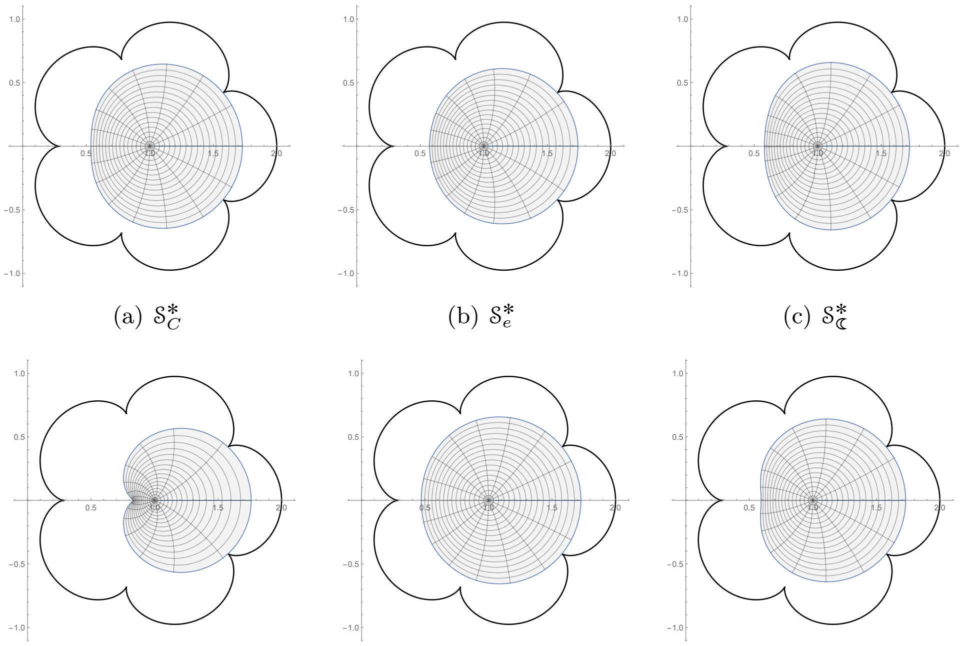
(c) If $f \in \mathcal{S}^*_{\mathfrak{C}}$, then $zf'(z)/f(z) \prec \phi_{\mathfrak{C}}(z) := z + \sqrt{1+z^2}$. A necessary condition on r for the inclusion relation $\phi_{\mathfrak{C}}(\mathbb{D}_r) \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$ to hold is $r \leqslant r_3$, where $r_3 := |(\gamma^{2n} + 2n\gamma^{n+1} + n^2\gamma^2 + 2(n+1)\gamma^n + 2n(n+1)\gamma)/(2(n+1)(1+\gamma^n+n(1+\gamma))|$ is the absolute value of the solution of the equation $\phi_{\mathfrak{C}}(z) = \phi_{n\mathcal{L}}(\gamma)$ for z. The sufficiency of the condition $r \leqslant r_3$ can be proved if we show that $\phi_{\mathfrak{C}}(\mathbb{D}_{r_3}) \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$. To see this, define a function $d(t,n)$ as the difference between the square of the distances from the point $(1,0)$ to the points on the boundary curves $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_{\mathfrak{C}}(r_3e^{it})$. Then

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it}) - 1|^2 - |\phi_{\mathfrak{C}}(r_3e^{it}) - 1|^2 \\ &= \frac{n^2+1+2n\cos((n-1)t)}{(n+1)^2} - (r_3^2 + 1 + \mu(t) - 2r_3\cos t \\ &\quad - \sqrt{2}(r_3\cos t + 1)\sqrt{\nu(t)+\mu(t)} + \sqrt{2}r_3\sqrt{\mu(t)-\nu(t)}\sin t), \end{aligned}$$

where $\mu(t) = \sqrt{1 + r_3^4 + 2r_3^2\cos 2t}$ and $\nu(t) = 1 + r_3^2\cos 2t$. A calculation shows that $d(t,n) \geqslant 0$ for each $t \in [0,\pi]$ and $n \geqslant 2$ (see Figure 5(c) for $n = 6$). This shows that r_3 is the required $\mathcal{S}^*_{n\mathcal{L}}$−radius with sharpness holding for the function $\tilde{f}_3$ which satisfies $z\tilde{f}_3'(z)/\tilde{f}_3(z) = z + \sqrt{1+z^2}$. The sharpness is illustrated in Figure 6(c) for $n = 6$.

(d) Let $\phi_R(z) := (k^2+z^2)/(k^2-kz)$, $k = \sqrt{2}+1$. For $n = 2$, the $\mathcal{S}^*_{2\mathcal{L}}$−radius for the class $\mathcal{S}^*_R$ is 1 as $\phi_R(\mathbb{D}) \subseteq \phi_{2\mathcal{L}}(\mathbb{D})$. Suppose that $n \geqslant 4$. Geometrical considerations show that $r \leqslant r_4$ for $\phi_R(\mathbb{D}_r)$ to lie inside $\phi_{n\mathcal{L}}(\mathbb{D})$, where r_4 is the absolute value of the solution of the equation $\phi_R(z) = \phi_{n\mathcal{L}}(\gamma)$ for z. In fact, $\phi_R(\mathbb{D}_{r_4})$ lie inside $\phi_{n\mathcal{L}}(\mathbb{D})$. To see this, we define the function $d(t,n)$ as

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it}) - 1|^2 - |\phi_R(r_4e^{it}) - 1|^2 \\ &= \frac{n^2+1+2n\cos((n-1)t)}{(n+1)^2} - \frac{r_4^2(3+2\sqrt{2}+(3-2\sqrt{2})r_4^4 - 2r_4^2\cos(2t))}{(3+2\sqrt{2}+r_4^2-2(1+\sqrt{2})r_4\cos t)^2}, \end{aligned}$$



(a) $\mathcal{S}^*_C$ (b) $\mathcal{S}^*_e$ (c) $\mathcal{S}^*_{\leftmoon}$

(d) $\mathcal{S}^*_R$ (e) $\mathcal{S}^*_{lim}$ (f) $\mathcal{S}^*_{car}$

Figure 6. $\mathcal{S}^*_{n\mathcal{L}}$−radius for various classes $\mathcal{S}^*(\phi)$

which is the difference between the square of the distances from the point $(1,0)$ to the points on the boundary curves $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_R(r_4 e^{it})$. For $t \in [0,\pi]$ and $n \geqslant 4$, the function $d(t,n)$ is non-negative (which is depicted in Figure 5(d) for $n=6$). Thus, $\phi_R(\mathbb{D}_r) \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$ for $r \leqslant r_4$. The result is sharp for the function $\tilde{f}_4(z) = k^2 z e^{-z/k}/(k-z)^2$ with the case $n=6$ being plotted in Figure 6(d).

(e) Note that $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{lim}) \leqslant r_5 := \sqrt{2}|\sqrt{(1+n(1+\gamma)+\gamma^n)/(n+1)}-1|$ which is the absolute value of the solution of the equation $\phi_{lim}(z) = \phi_{n\mathcal{L}}(\gamma)$ for z, where $\phi_{lim}(z) = 1+\sqrt{2}z+z^2/2$. Now, let $d(t,n)$ be the difference of the square of the distance of the boundary points $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_{lim}(r_5 e^{it})$ from the point $(1,0)$. Then

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it})-1|^2 - |\phi_{lim}(r_5 e^{it})-1|^2 \\ &= \frac{n^2+1+2n\cos((n-1)t)}{(n+1)^2} - \frac{r_5^2(8+r_5^2+4\sqrt{2}r_5\cos t)}{4}. \end{aligned}$$

A calculation gives $d(t,n) \geqslant 0$ for $t \in [0,\pi]$ and $n \geqslant 2$ (the graph for $n=6$ is illustrated in Figure 5(e)). Thus, image of the subdisk $\mathbb{D}_{r_5}$ under the function ϕ_{lim} lies inside $\phi_{n\mathcal{L}}(\mathbb{D})$. This shows that $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{lim}) \geqslant r_5$. The sharpness for $n=6$ is depicted in Figure 6(e) for the function $\tilde{f}_5(z) = z\exp(\sqrt{2}z + z^2/4)$.

(f) To compute the $\mathcal{S}^*_{n\mathcal{L}}$−radius for the class $\mathcal{S}^*_{car}$, let $\phi_{car}(z) := 1+z+z^2/2$. As proceeded in the earlier parts, we will solve the equation $\phi_{car}(z) = \phi_{n\mathcal{L}}(\gamma)$ for z and take its absolute value which turns out to be $r_6 := |\sqrt{(1+n(1+2\gamma)+2\gamma^n)/(n+1)}-1|$. It remains to show that ϕ_{car} maps $\mathbb{D}_{r_6}$ inside $\phi_{n\mathcal{L}}(\mathbb{D})$. For this, we consider the difference between the squares of distances from the point on the boundary curves $\phi_{n\mathcal{L}}(e^{it})$ and

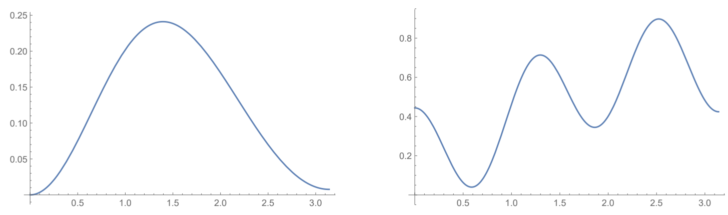
$\phi_{car}(r_6e^{it})$ with the point $(1,0)$ as

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it}) - 1|^2 - |\phi_{car}(r_6e^{it}) - 1|^2 \\ &= \frac{n^2+1+2n\cos((n-1)t)}{(n+1)^2} - \frac{r_6^2}{4}(4 + r_6^2 + 4r_6\cos t). \end{aligned}$$

Since $d(t,n)$ is non-negative for $t \in [0,\pi]$ and $n \geqslant 2$, it follows that r_6 is the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_{car}$, which is attained by the function $\tilde{f}_6(z) = z\exp(z + z^2/4)$. The graph of $d(t,n)$ and sharpness are illustrated in Figure 5(f) and Figure 6(f) respectively for $n = 6$.

(g) If $d(t,n,r)$ denotes the difference between the squares of the distances of the boundary points of the curves $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_{\wp}(re^{it})$ from the point $(1,0)$, where $\phi_{\wp}(z) = 1 + ze^z$, then

$$d(t,n,r) = |\phi_{n\mathcal{L}}(e^{it}) - 1|^2 - |\phi_{\wp}(re^{it}) - 1|^2 = \frac{n^2+1+2n\cos((n-1)t)}{(n+1)^2} - e^{2r\cos t}r^2.$$



(a) $d(t,2,r_7)$ (b) $d(t,6,\tilde{r}_7)$

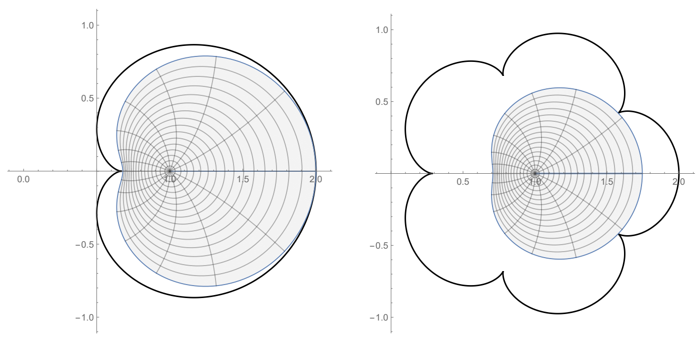
Figure 7. Distance function corresponding to the class $\mathcal{S}^*_{\wp}$

For $n = 2$, a necessary condition for the subordination $\phi_{\wp} \prec \phi_{n\mathcal{L}}$ to hold in $\mathbb{D}_r$ is

$$1 + re^r = \phi_{\wp}(r) \leqslant \phi_{2\mathcal{L}}(r) \leqslant \phi_{2\mathcal{L}}(1) = 2.$$

This simplifies to $r \leqslant r_7$, where r_7 is the root of the equation $re^r = 1$ in $(0,1)$. Also, note that $d(t,2,r_7) \geqslant 0$ for all $t \in [0,\pi]$ (see Figure 7(a)). Hence $\mathcal{R}_{\mathcal{S}^*_{2\mathcal{L}}}(\mathcal{S}^*_{\wp}) = r_7$.

If $n \geqslant 4$, then the condition $r \leqslant \tilde{r}_7$ is necessary for the inclusion relation $\phi_{\wp}(\mathbb{D}_r) \subseteq \phi_{n\mathcal{L}}(\mathbb{D})$, where $\tilde{r}_7$ is the absolute value of the solution of the equation $\phi_{\wp}(z) = \phi_{n\mathcal{L}}(\gamma)$ for z. Since the function $d(t,n,\tilde{r}_7)$ is non-negative for $t \in [0,\pi]$ (see Figure 7(b)), it follows that $\tilde{r}_7$ is the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_{\wp}$.



(a) $n = 2$ (b) $n = 6$

Figure 8. Sharpness of $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_{\wp}$

For both the cases, the result is sharp for the function $\tilde{f}_7(z) = e^{e^z}$ and is illustrated in Figure 8. □

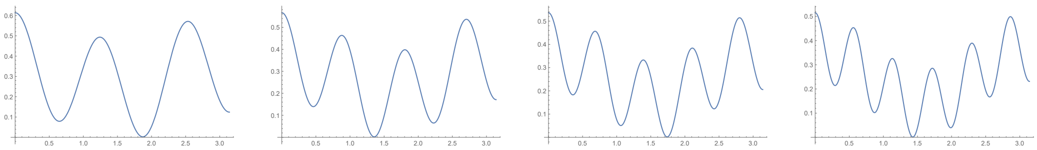
The last theorem of this section tackles the radius problem by considering different cusps depending upon the geometry of the associated classes.

Theorem 5.4. *Let $n = 2k$, $k \in \mathbb{N}$ and $\gamma = e^{i\pi/(n-1)}$. Then*

(a) *The $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_{sin}$ is $|\sin^{-1}((\gamma^{nk} + n\gamma^k)/(n+1))|$ if k is odd; and $|\sin^{-1}((\gamma^{n(k-1)} + n\gamma^{k-1})/(n+1))|$ if k is even.*

(b) *The $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_{ne}$ is the absolute value of the solution of the equation for z in the interval $(0,1)$: $z - z^3/3 = (\gamma^{nk} + n\gamma^k)/(n+1)$ if k is odd; and $z - z^3/3 = (\gamma^{n(k-1)} + n\gamma^{k-1})/(n+1)$ if k is even.*

Proof. (a) Let $f \in \mathcal{S}^*_{sin}$. Then $zf'(z)/f(z) \prec \phi_{sin}(z) := 1 + \sin z$. If k is odd, then the cusp to be considered is at the angle $k\pi/(n-1)$. For the image of the function $1 + \sin z$ under $\mathbb{D}_r$ to lie inside the domain $\phi_{n\mathcal{L}}(\mathbb{D})$, it is necessary that $r \leqslant |z_1|$, where z_1 is the solution of the equation $\phi_{sin}(z) = \phi_{n\mathcal{L}}(\gamma^k)$, for z. Similarly, if k is even, then the cusp at the angle $(k-1)\pi/(n-1)$ will be considered and the necessary condition in this case is $r \leqslant |z_2|$, where z_2 is the solution of the equation $\phi_{sin}(z) = \phi_{n\mathcal{L}}(\gamma^{k-1})$, for z.



(a) $n = 6$ (b) $n = 8$ (c) $n = 10$ (d) $n = 12$

Figure 9. Distance function $d(t, n)$ for $n = 6, 8, 10, 12$

In order to prove that these conditions are sufficient as well, we will consider the difference of the square of the distances of the points on the boundary curves $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_{sin}(Re^{it})$ from the point $(1, 0)$ as

$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it}) - 1|^2 - |\phi_{sin}(Re^{it}) - 1|^2 \\ &= \frac{n^2 + 1 + 2n\cos((n-1)t)}{(n+1)^2} - \sin^2(R\cos t) - \sinh^2(R\sin t). \end{aligned}$$

where $R = |z_1|$ if k is odd and $R = |z_2|$ if k is even. A calculation shows that $d(t, n)$ is non-negative for $t \in [0, \pi]$. The graphs of $d(t, n)$ are plotted in Figure 9 for $n = 6, 8, 10, 12$. Hence, the $\mathcal{S}^*_{n\mathcal{L}}-$radius for the class $\mathcal{S}^*_{sin}$ is R and this result is sharp for the function $\tilde{g}_1$ given by

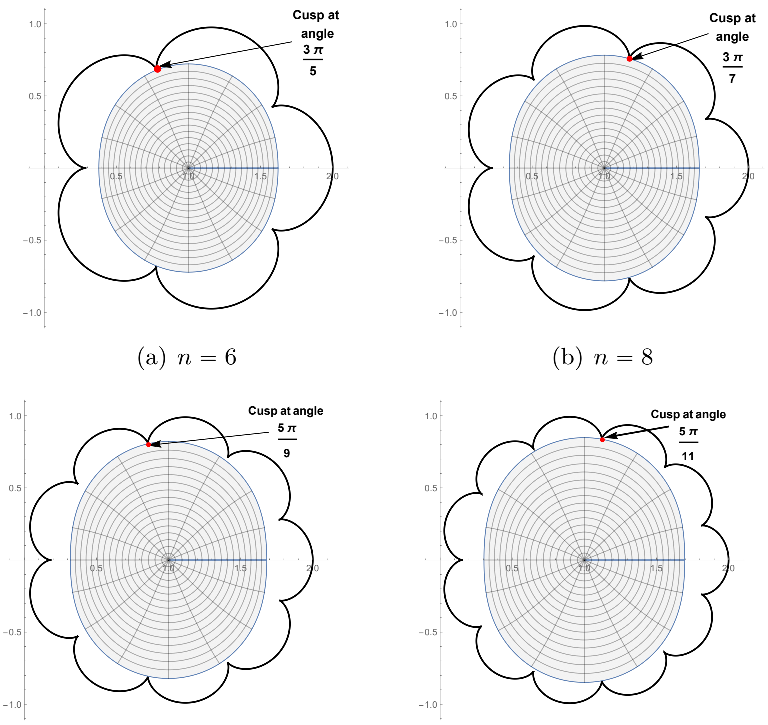
$$\tilde{g}_1(z) = z \exp\left(\int_0^z \frac{\sin t}{t} dt\right)$$

and is depicted in Figure 10 for some choices of n.

(b) Let $f \in \mathcal{S}^*_{ne}$ and $\phi_{ne}(z) := 1 + z - z^3/3$. Proceeding in the similar fashion as in part (a), the cusps at the angle $k\pi/(n-1)$ if k is odd; and $(k-1)\pi/(n-1)$ if k is even, need to be taken into consideration. Consequently, $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{ne}) \leqslant \tilde{R}$, where $\tilde{R} = |z_3|$ if k is odd and $\tilde{R} = |z_4|$ if k is even, where z_3 and z_4 are solutions of the equations $\phi_{ne}(z) = \phi_{n\mathcal{L}}(\gamma^k)$ and $\phi_{ne}(z) = \phi_{n\mathcal{L}}(\gamma^{k-1})$ respectively, for z.

Now, in order to show that $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{S}^*_{ne}) \geqslant \tilde{R}$, let us define the difference of the squares of the distances on the boundary points $\phi_{n\mathcal{L}}(e^{it})$ and $\phi_{ne}(re^{it})$ from the point $(1, 0)$ as

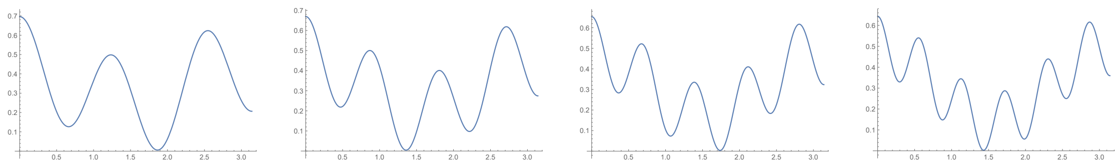
$$\begin{aligned} d(t,n) &= |\phi_{n\mathcal{L}}(e^{it}) - 1|^2 - |\phi_{ne}(\tilde{R}e^{it}) - 1|^2 \\ &= \frac{n^2 + 1 + 2n\cos((n-1)t)}{(n+1)^2} - \frac{\tilde{R}^2}{9}(\tilde{R}^4 - 6\tilde{R}^2\cos 2t + 9). \end{aligned}$$



(a) $n = 6$ (b) $n = 8$

(c) $n = 10$ (d) $n = 12$

Figure 10. Image domain of $1 + \sin z$ lying in $\phi_{n\mathcal{L}}(\mathbb{D})$ for various choices of n



(a) $n = 6$ (b) $n = 8$ (c) $n = 10$ (d) $n = 12$

Figure 11. Distance function $d(t, n)$ for $n = 6, 8, 10, 12$

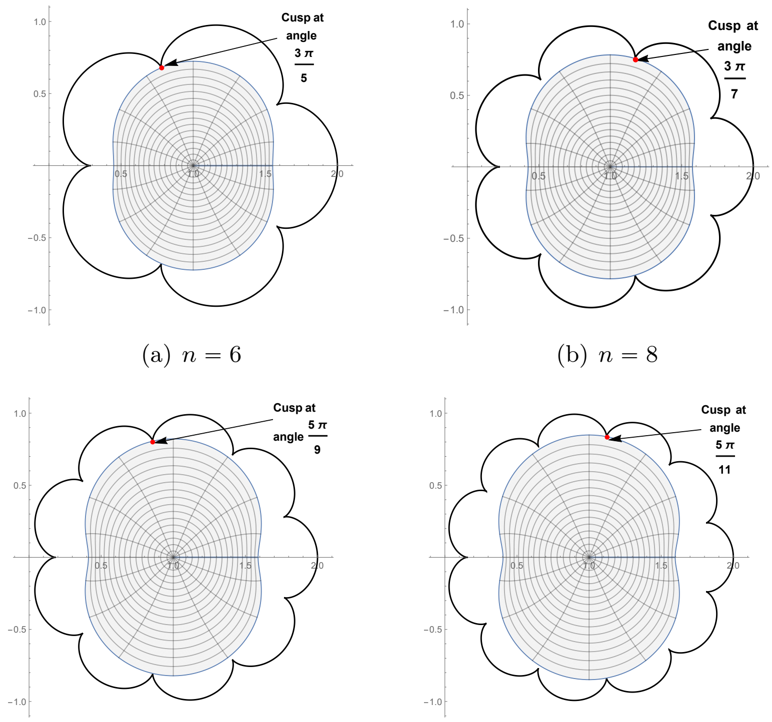
It can be deduced that $d(t, n) \geqslant 0$ for $t \in [0, \pi]$ (See Figure 11).

The sharpness is attained by the function $\tilde{g}_2(z) = z \exp(z - z^3/9)$ and is illustrated for some choices of n in Figure 12. □

The numerical values of various radii computed in Theorems 5.2, 5.3 and 5.4 are enlisted in Table 1 for $n = 2, 4, 6, 8$.

Apart from the classes discussed in this section, two subclasses of analytic functions, namely $\mathcal{BS}(\alpha)$, $0 < \alpha \leqslant 1$ [10] and $\mathcal{M}(\beta)$, $\beta > 1$ [32] are also widely studied, consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec 1 + z/(1 - \alpha z^2)$ and $\operatorname{Re}(zf'(z)/f(z)) < \beta$ respectively, for all $z \in \mathbb{D}$. Using the similar technique employed in Theorem 5.1, it can be shown that $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{M}(\beta)) = (n-1)/((2\beta - 1)n + (2\beta - 3))$ and $\mathcal{R}_{\mathcal{S}^*_{n\mathcal{L}}}(\mathcal{BS}(\alpha)) = (\sqrt{(1 + 4\alpha)(n^2 + 1) + 2(1 - 4\alpha)n} - (n+1))/(2\alpha(n-1))$. Since its proof is similar, therefore the details are omitted. We close this section with the following important remark.

Remark 5.5. The class $\mathcal{S}^*_{n\mathcal{L}}$ reduces to the class $\mathcal{S}^*(1 + z)$ in the limiting case. Thus, $\mathcal{S}^*(1+z)$−radius for various classes has been summarized in Table 2 which can be obtained



(a) $n = 6$ (b) $n = 8$

(c) $n = 10$ (d) $n = 12$

Figure 12. Nephroid domain lying in various $\phi_{n\mathcal{L}}(\mathbb{D})$ domain

Class	$n = 2$	$n = 4$	$n = 6$	$n = 8$
$\mathcal{S}^*_L$	0.55556	0.84000	0.91837	0.95062
$\mathcal{S}^*_{RL}$	0.61031	0.83720	0.89838	0.92632
$\mathcal{S}^*_C$	0.29289	0.39718	0.44719	0.47635
$\mathcal{S}^*_e$	0.40547	0.50775	0.55498	0.58419
$\mathcal{S}^*_{\mathfrak{C}}$	0.41667	0.50528	0.57129	0.61045
$\mathcal{S}^*_R$	1.00000	0.83372	0.86637	0.89273
$\mathcal{S}^*_{lim}$	0.25951	0.39041	0.44440	0.47533
$\mathcal{S}^*_{car}$	0.42265	0.50906	0.56901	0.60441
$\mathcal{S}^*_{\wp}$	0.56714	0.43608	0.46731	0.48812
$\mathcal{S}^*_{sin}$	0.33983	0.58075	0.67047	0.71887
$\mathcal{S}^*_{ne}$	0.34729	0.56131	0.63763	0.67907

Table 1. $\mathcal{S}^*_{n\mathcal{L}}-$radii for $n = 2, 4, 6, 8$

by taking the limit $n \to \infty$ in the results proved in this section, except for the classes $\mathcal{S}^*_L(\alpha)$, $0 \leqslant \alpha < 1$ and $\mathcal{S}^*(\sqrt{1+cz})$, $0 < c \leqslant 1$, whose radius turns out to be 1 by parts (d) and (e) of Theorem 4.1.

Class	$\mathcal{S}^*(1+z)-$radius
$\mathcal{W}$	$\sqrt{2}-1$
$\mathcal{F}_1$	$\sqrt{5}-2$
$\mathcal{F}_2$	$(\sqrt{17}-3)/4$
$\mathcal{S}^*_L(\alpha)$, $\mathcal{S}^*(\sqrt{1+cz})$, $\mathcal{S}^*_{RL}$	1
$\mathcal{S}^*_C$	$\sqrt{5/2}-1$
$\mathcal{S}^*_e$	$\log 2$
$\mathcal{S}^*_{\mathbb{C}}$	$3/4$
$\mathcal{S}^*_R$	$-1-\sqrt{2}+(6+4\sqrt{2})^{1/2}$
$\mathcal{S}^*_{lim}$	$2-\sqrt{2}$
$\mathcal{S}^*_{car}$	0.732051
$\mathcal{S}^*_{\wp}$	0.567143
$\mathcal{S}^*_{sin}$	0.881374
$\mathcal{S}^*_{ne}$	0.817732
$\mathcal{BS}(\alpha)$	$(\sqrt{1+4\alpha}-1)/(2\alpha)$
$\mathcal{M}(\beta)$	$1/(2\beta-1)$

Table 2. Radii in the limiting case

6. Radius constants for $\mathcal{S}^*_{n\mathcal{L}}$

In this section, the $\mathcal{S}^*(\phi)-$radii are determined for the class $\mathcal{S}^*_{n\mathcal{L}}$ for various choices of ϕ. The notations introduced in Sections 4 and 5 will have the same meaning for the following theorem as well.

Theorem 6.1. *The sharp radii constants for the class $\mathcal{S}^*_{n\mathcal{L}}$ are as follows:*

(a) *The $\mathcal{S}^*_L(\alpha)-$radius $(0 \leqslant \alpha < 1)$ is the smallest positive real root of the equation $r^n + rn - (\sqrt{2}-1)(1-\alpha)(n+1) = 0$ in $(0,1)$.*
(b) *The $\mathcal{S}^*_{RL}-$radius is the smallest positive real root of the equation $r^n + rn - (n+1)(\sqrt{\gamma}-\gamma)^{1/2} = 0$ in $(0,1)$, where $\gamma = 2\sqrt{2}-2$.*
(c) *The $\mathcal{S}^*_R-$radius is the smallest positive real root of the equation $r^n - rn - (n+1)(2\sqrt{2}+3) = 0$ in $(0,1)$.*
(d) *The $\mathcal{S}^*_{sin}-$radius is the smallest positive real root of the equation $r^n + rn - (n+1)\sin 1 = 0$ in $(0,1)$.*
(e) *The $\mathcal{S}^*_{SG}-$radius is the smallest positive real root of the equation $r^n + rn - (n+1)(e-1)/(e+1) = 0$ in $(0,1)$.*
(f) *The $\mathcal{S}^*_{ne}-$radius is the smallest positive real root of the equation $r^n + rn - 2(n+1)/3 = 0$ in $(0,1)$.*
(g) *The $\mathcal{S}^*_{\wp}-$radius is the smallest positive real root of the equation $r^n - rn + (n+1)/e = 0$ in $(0,1)$, for $n \geqslant 4$ and $\mathcal{R}_{\mathcal{S}^*_{\wp}}(\mathcal{S}^*_{2\mathcal{L}}) = 1$.*
(h) *The $\mathcal{S}^*_{\rho}-$radius is the smallest positive real root of the equation $r^n + rn - (n+1)\sinh^{-1}(1) = 0$ in $(0,1)$.*

Proof. Let $f \in \mathcal{S}^*_{n\mathcal{L}}$. Then $zf'(z)/f(z) \prec \phi_{n\mathcal{L}}(z)$ and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant \frac{nr}{n+1} + \frac{r^n}{n+1}, \quad |z| = r. \tag{6.1}$$

Suppose that $f_{n\mathcal{L}} \in \mathcal{S}^*_{n\mathcal{L}}$ is given by (1.3).

(a) By using [12, Lemma 2.3, p. 6], it can be easily seen that the disk (6.1) lies inside the lemniscate of Bernoulli $|((w-\alpha)/(1-\alpha))^2 - 1| = 1$ if

$$\frac{nr}{n+1} + \frac{r^n}{n+1} \leqslant (\sqrt{2}-1)(1-\alpha).$$

This gives $r \leqslant s_1$, where s_1 is the smallest positive real root of the equation $r^n + rn - (\sqrt{2}-1)(1-\alpha)(n+1) = 0$ in $(0,1)$. For sharpness, the value of $zf'_{n\mathcal{L}}(z)/f_{n\mathcal{L}}(z)$ equals $\alpha + (1-\alpha)\sqrt{2}$ for $z = s_1$.

(b) The disk (6.1) lies inside the left-half of shifted lemniscate of Bernoulli $|(w-\sqrt{2})^2 - 1| = 1$ if

$$\frac{nr}{n+1} + \frac{r^n}{n+1} \leqslant \sqrt{\sqrt{2\sqrt{2}-2} - 2\sqrt{2} + 2},$$

by [22, Lemma 3.2, p. 10]. This simplifies to $r \leqslant s_2$, where s_2 is the smallest positive real root of the equation $r^n + rn - (n+1)(\sqrt{\gamma} - \gamma)^{1/2} = 0$ in $(0,1)$, where $\gamma = 2\sqrt{2} - 2$. The result is sharp for the function $f_{n\mathcal{L}}$.

(c) A necessary condition for the subordination $\phi_{n\mathcal{L}} \prec \phi_R$ to hold in $\mathbb{D}_r$ is

$$2(\sqrt{2}-1) = \phi_R(-1) \leqslant \phi_{n\mathcal{L}}(-r) = 1 - \frac{nr}{n+1} + \frac{r^n}{n+1}.$$

This gives $r \leqslant s_3$, where s_3 is the smallest positive real root of the equation $r^n - rn - (n+1)(2\sqrt{2}-3) = 0$ in $(0,1)$. To show that the condition $r \leqslant s_3$ is sufficient as well for the subordination $\phi_{n\mathcal{L}} \prec \phi_R$ to hold in $\mathbb{D}_r$, we will prove that the image domain $\phi_{n\mathcal{L}}(\mathbb{D}_{s_3})$ is contained in $\phi_R(\mathbb{D})$. To see this, we define a function $d(t,n)$ as the difference of the squares of the boundary points on the curves $\phi_R(e^{it})$ and $\phi_{n\mathcal{L}}(s_3 e^{it})$ from the point $(1,0)$ which takes the form

$$\begin{aligned} d(t,n) &= |\phi_R(e^{it}) - 1|^2 - |\phi_{n\mathcal{L}}(s_3 e^{it}) - 1|^2 \\ &= \frac{2+\sqrt{2} + (1+\sqrt{2})\cos t}{10 + 7\sqrt{2} - (7+5\sqrt{2})\cos t} - \frac{n^2 s_3^2 + s_3^{2n} + 2n s_3^{n+1}\cos((n-1)t)}{(n+1)^2}. \end{aligned}$$

Since $d(t,n)$ is a decreasing function of $t \in [0,\pi]$ (see Figure 14(a)), it follows that $d(t,n)$ is non-negative if and only if $d(\pi,n) \geqslant 0$. But

$$d(\pi,n) = (3-2\sqrt{2})^2 - \frac{(s_3^n - s_3 n)^2}{(n+1)^2} = 0.$$

Hence $\phi_{n\mathcal{L}}(\mathbb{D}_r) \subseteq \phi_R(\mathbb{D})$ for $r \leqslant s_3$. This bound is best possible as $s_3 f'_{n\mathcal{L}}(s_3)/f_{n\mathcal{L}}(s_3) = 2(\sqrt{2}-1) = \phi_R(-1)$.

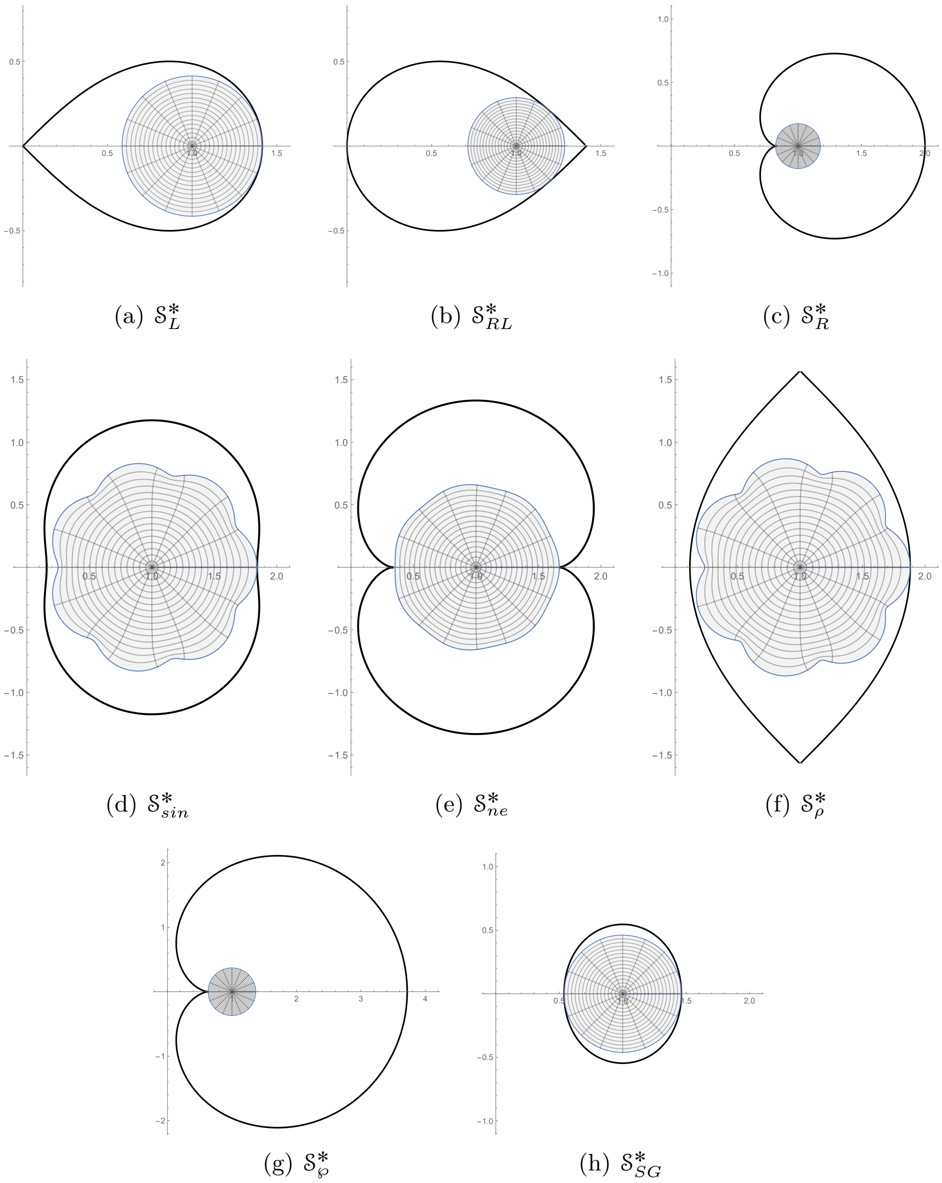
(d) In view of [4, Lemma 3.3, p. 7], the disk (6.1) lies in the image domain $\phi_{sin}(\mathbb{D})$ if

$$\frac{nr}{n+1} + \frac{r^n}{n+1} \leqslant \sin 1.$$

This is true for $r \leqslant s_4$, where s_4 is the smallest positive real root of the equation $r^n + rn - (n+1)\sin 1 = 0$ in $(0,1)$. The result is best possible as seen by the function $f_{n\mathcal{L}}$ which satisfies $s_4 f'_{n\mathcal{L}}(s_4)/f_{n\mathcal{L}}(s_4) = 1 + \sin 1 = \phi_{sin}(1)$.

(e) By [6, Lemma 2.2, p. 5], the disk (6.1) lies in the modified sigmoid $|\log(w/(2-w))| = 1$ if

$$\frac{nr}{n+1} + \frac{r^n}{n+1} \leqslant \frac{e-1}{e+1}.$$



(a) $\mathcal{S}^*_L$ (b) $\mathcal{S}^*_{RL}$ (c) $\mathcal{S}^*_R$

(d) $\mathcal{S}^*_{sin}$ (e) $\mathcal{S}^*_{ne}$ (f) $\mathcal{S}^*_\rho$

(g) $\mathcal{S}^*_\wp$ (h) $\mathcal{S}^*_{SG}$

Figure 13. Sharpness of various radii for class $\mathcal{S}^*_{n\mathcal{L}}$

This simplifies to $r \leqslant s_5$, where s_5 is the smallest positive real root of the equation $r^n + rn - (n+1)(e-1)/(e+1) = 0$ in $(0,1)$. The bound cannot be improved further as for $z = s_5$, $zf'_{n\mathcal{L}}(z)/f_{n\mathcal{L}}(z)$ assumes the value $2e/(e+1)$.

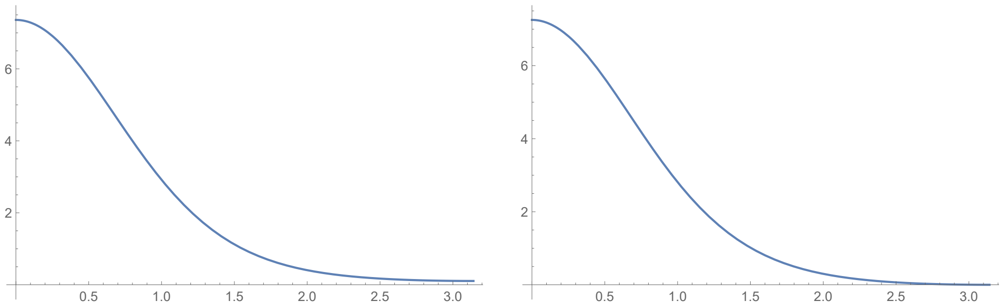
(f) By [33, Lemma 2.2, p. 8], the following condition implies that the disk (6.1) lie inside the nephroid domain $\phi_{ne}(\mathbb{D})$:

$$\frac{nr}{n+1} + \frac{r^n}{n+1} \leqslant \frac{2}{3}.$$

This gives $r \leqslant s_6$, where s_6 is the smallest positive real root of the equation $r^n + rn - 2(n+1)/3 = 0$ in $(0,1)$. For $z = s_6$, $zf'_{n\mathcal{L}}(z)/f_{n\mathcal{L}}(z) = 5/3 = \phi_{ne}(1)$.

(g) Since $\phi_{2\mathcal{L}}(\mathbb{D}) \subseteq \phi_\wp(\mathbb{D})$, the $\mathcal{S}^*_\wp-$radius for $\mathcal{S}^*_{2\mathcal{L}}$ is 1. Suppose that $n \geqslant 4$. A necessary condition for the subordination $\phi_{n\mathcal{L}} \prec \phi_\wp$ to hold on $\mathbb{D}_r$ is

$$1 - \frac{1}{e} = \phi_\wp(-1) \leqslant \phi_{n\mathcal{L}}(-r) = 1 - \frac{nr}{n+1} + \frac{r^n}{n+1}.$$



(a) $\mathcal{S}^*_R$ (b) $\mathcal{S}^*_\wp$

Figure 14. Graph of distance function $d(t, 8)$

This simplifies to $r \leqslant s_7$, where s_7 is the smallest positive real root of the equation $r^n - rn + (n+1)/e = 0$ in $(0,1)$. Proceeding as in part (c), let $d(t,n)$ denote the difference of the squares of the distances from the point $(1,0)$ to the points on the boundary curves $\phi_\wp(e^{it})$ and $\phi_{n\mathcal{L}}(s_7 e^{it})$. Then

$$\begin{aligned} d(t,n) &= |\phi_\wp(e^{it}) - 1|^2 - |\phi_{n\mathcal{L}}(s_7 e^{it}) - 1|^2 \\ &= e^{2\cos t} - \frac{n^2 s_7^2 + s_7^{2n} + 2n s_7^{n+1}\cos((n-1)t)}{(1+n)^2} \end{aligned}$$

A computation yields that the function $d(t,n)$ is a decreasing function (see Figure 14(b)). As a result, the function $d(t,n)$ is non-negative if and only if $d(\pi,n) \geqslant 0$. Since

$$d(\pi,n) = \frac{1}{e^2} - \frac{(s_7^n - n s_7)^2}{(n+1)^2} = 0,$$

the $\mathcal{S}^*_\wp$−radius is s_7. The result is sharp for the function $f_{n\mathcal{L}}$ as $zf'_{n\mathcal{L}}(z)/f_{n\mathcal{L}}(z) = 1 - 1/e = \phi_\wp(-1)$, for $z = -s_7$.

(h) By using [14, Lemma 2.1, p. 4], the disk (6.1) lie inside the image domain of the function $1 + \sinh^{-1}(z)$ if

$$\frac{r^n}{n+1} + \frac{nr}{n+1} \leqslant \sinh^{-1}(1),$$

which simplifies to $r \leqslant s_8$, where s_8 is the smallest positive real root of the equation $r^n + rn - (n+1)\sinh^{-1}(1) = 0$ in $(0,1)$. The function $f_{n\mathcal{L}}$ shows that the bound is best possible as $zf'_{n\mathcal{L}}(z)/f_{n\mathcal{L}}(z) = 1 + \sinh^{-1}(1)$, for $z = s_8$.

The sharpness of the results proved in this theorem is illustrated by Figure 13 for $n = 8$ and the numerical value of the computed radii are tabulated for some choices of n in Table 3.

□

Class	$n=2$	$n=4$	$n=6$	$n=8$
$\mathcal{S}^*_L$	0.497545	0.501903	0.48118	0.465714
$\mathcal{S}^*_{RL}$	0.363001	0.353501	0.333349	0.32165
$\mathcal{S}^*_R$	0.303379	0.213942	0.200158	0.193019
$\mathcal{S}^*_{sin}$	0.877342	0.892917	0.895669	0.895131
$\mathcal{S}^*_{SG}$	0.544782	0.554083	0.535219	0.519222
$\mathcal{S}^*_{ne}$	0.732051	0.752971	0.748475	0.738894
$\mathcal{S}^*_{\wp}$	1	0.472288	0.43025	0.413972
$\mathcal{S}^*_{\rho}$	0.908958	0.921471	0.924325	0.924715

Table 3. Radii constants for $n = 4, 6$ and 8

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