Fundamental Journal of Mathematics and Applications, 5 (2) (2022) 106-113
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1020437

Generalized Cylinder with Geodesic and Line of Curvature Parameterizations

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Article Info

Keywords: *Generalized cylinder, Geodesic, Line of curvature, Parameterization*
2010 AMS: *53A04, 53A05*
Received: *8 November 2021*
Accepted: *24 February 2022*
Available online: *28 March 2022*

Abstract

Constructing a surface with geodesic or line of curvature parameterization is an important problem in many practical applications. The present paper aims to design a generalized cylinder that is parametrized along the geodesics and lines of curvature curves in Euclidean 3- space. The main results show that the generalized cylinder with geodesic or line of curvature parameterization is a rectifying cylinder or a right cylinder respectively.

1. Introduction

A generalized cylinder is constructed by the constant motion of a straight line called the ruling through a given curve called the base curve. The generalized cylinders are a class of developable ruled surfaces that have no singularities points and can be produced from paper or sheet metal with no distortion. For this construction, the generalized cylinder has been investigated as a basic modeling surface in various fields of science including geometric modeling, computer graphic, architectural designing and manufacturing [1]-[4].

Geodesic and line of curvature are characteristic curves that lie on the surface. The geodesic curve gives the shortest path between two given points on curved spaces. A curve is a line of curvature if its direction always points in the principal directions, i.e., the direction in which the surface bends extremaly. Geodesics and lines of curvature have been used in shape analysis, therefore, the problems of computing and visualizing them on the surface have been investigated [5]-[7]. The rulings of the generalized cylinder are geodesics and lines of curvature.

Surface parameterization is the process of mapping a surface to a planar region [8]. Extracting and transferring the geometric information from shapes or between them depends on the parameterizations that are used as coordinate systems on the shapes. Several types of parameterizations are constructed on a surface and differ by their characterizing properties. During the parameterization some geometric quantities can be lost or distorted, therefore, designing and choosing the suitable parameterizations that minimize, maximize or preserve the desired geometrical properties is an interesting problem and hot topic in many areas of applications such as computer graphic [9]-[11], geometric modeling [12], and robot motion planning [13].

A parameterization on a surface is said to be geodesic or line of curvature if the two families of parametric curves are geodesics or lines of curvature. Parametrizations of smooth surfaces by curvature line exist on non-umbilical points as orthogonal curves on the surface. Geodesic and line of curvature parameterizations mean that the shape is charted or covered by two families of lines that are characterized by special directions. Parameterizing the surface along their geodesics or lines of curvature are widely investigated in many areas of sciences such as CAGD [14]-[16], surfaces motions [17, 18], architectural design

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[19, 20], and discrete differential geometry [21]-[23].

The main goal of this paper is to design a generalized cylinder whose parametric curves are geodesics or lines of curvature in Euclidean 3-space. A generalized cylinder has two families of parametric curves, rulings, and base curves. It is well known that the rulings are geodesics and lines of curvature on a generalized cylinder. Consequently, throughout this paper, our focus lies on the family of base curves. The generalized cylinders are a class of ruled surfaces, therefore, we start from a ruled surface parametrization, then with additional three conditions called the cylindrical conditions, the generalized cylinder is defined. After that, under some geometric constraints, we obtain the resulting cylinder that is parameterized by geodesic or line of curvature base curves. The main results show that the generalized cylinder with geodesic or line of curvature parameterization is a rectifying cylinder or a right cylinder respectively. In this article, we used the same approach that was used in [24] and with the developable surface.

The rest of this paper is organized as follows: In section 2, some basic notations, facts, and definitions of the space curve, regular surface, and special curves in Euclidean 3-space are reviewed. The main results are studied in section 3, where the generalized cylinder is defined in the first subsection, then the generalized cylinder with geodesic and line of curvature parameterizations are constructed subsequently in the other two subsections respectively. Examples to illustrate the main results are presented in section 4. Finally, the conclusion is given in section 5.

2. Preliminaries

This section introduces some basic concepts on the classical differential geometry of space curves and surfaces in three-dimensional Euclidean space. More details can be found in such standard references as [25]-[27].

2.1. Curves in Euclidean 3-space

A smooth space curve in 3-dimensional Euclidean space is parameterized by a map $\gamma : I \subseteq \mathbb{R} \to E^3$, γ is called a regular curve if $\gamma' \neq 0$ for every point of an interval $I \subseteq \mathbb{R}$, and if $|\gamma'(s)| = 1$ where $|\gamma'(s)| = \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}$, then γ is said to be of unit speed (or parameterized by arc-length s). For a unit speed regular curve $\gamma(s)$ in E^3, the unit tangent vector $t(s)$ of γ at $\gamma(s)$ is given by $t(s) = \gamma'(s)$. If $\gamma''(s) \neq 0$, the unit principal normal vector $n(s)$ of the curve at $\gamma(s)$ is given by $n(s) = \frac{\gamma''(s)}{\|\gamma''\|}$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at $\gamma(s)$. For each point of $\gamma(s)$ where $\gamma''(s) \neq 0$, we associate the Serret-Frenet frame $\{t, n, b\}$ along the curve γ. As the parameter s traces out the curve, the Serret-Frenet frame moves along γ and satisfies the following Frenet-Serret formula :

$$\begin{aligned} t'(s) &= \kappa(s) n(s), \\ n'(s) &= -\kappa(s) t(s) + \tau b(s), \\ b'(s) &= -\tau(s) n(s), \end{aligned} \tag{2.1}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and torsion functions. When the point moves along the unit speed curve with non-vanishing curvature and torsion, the Serret-Frenet frame $\{t, n, b\}$ is drawn to the curve at each position of the moving point, this motion consists of translation with rotation and described by the following Darboux vector

$$\omega = \tau t + \kappa b$$

where the unit Darboux vector is given by

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b \tag{2.2}$$

Direction of Darboux vector is the direction of rotational axis and its magnitude gives the angular velocity of rotation. A necessary and sufficient condition that a curve be of constant slope (or general helix) is that the ratio of torsion to curvature is constant ($\frac{\tau}{\kappa} = c$). The general helix lies on a general cylinder and also known as a cylindrical helix. The circular helix (a helix on a circular cylinder) is a special helix with both of $\kappa(s) \neq 0$ and $\tau(s)$ are constants. The Darboux vector is constant for circular helix. For the cylindrical helix, the unit Darboux vector is constant as following

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b = \frac{c}{\sqrt{c^2 + 1}} t + \frac{1}{\sqrt{c^2 + 1}} b. \tag{2.3}$$

2.2. Surfaces in Euclidean 3-space

A smooth surface in 3-dimensional Euclidean space is parameterized by a map $X(u, v) : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. The variables (u, v) are called the (curvilinear) coordinates on the surface, the two families of u-curves ($v = const$), and v-curves ($u = const$), are called the parametric curves (or coordinate curves). Their directions are defined by the tangents vectors X_u and X_v respectively. The surface $X(u, v)$ is called a regular if the condition $X_u \times X_v \neq 0$ is satisfied for all points, that means the vectors X_u and X_v

do not vanish and have different directions. Consequently, the surface normal is defined at every point on the regular surface as a unit vector on the tangent plane and given by

$$N(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|}. \tag{2.4}$$

The first and second fundamental form of the parameterized regular surface are given by

$$I = Edu^2 + 2Fdudv + Gdv^2, \quad II = edu^2 + 2fdudv + gdv^2$$

where their coefficients can be calculated respectively as

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, e = \langle N, X_{uu} \rangle, f = \langle N, X_{uv} \rangle, and g = \langle N, X_{vv} \rangle.$$

The fundamental quantities I and II are important tools to describe the intrinsic and extrinsic geometry of surface. In particular, type of the parametric curves and their characteristics properties are described by the coefficients of the fundamental quantities I and II. For example, the coordinate curves are orthogonal if $F = 0$, conjugate if $f = 0$, and lines of curvature if satisfy both conditions.

Theorem 2.1. *[28] A necessary and sufficient condition for the coordinate curves of a parametrization to be lines of curvature in a neighborhood of a nonumbilical point is that $F = f = 0$.*

For a regular curve on a surface, there exists another frame $\{t(s), g(s), N(s)\}$ which is called Darboux frame. In this frame $t(s)$ is the unit tangent of the curve, $N(s)$ is the unit normal of the surface and g is a unit vector given by $g = N \times t$. The relations between Frenet frame and Darboux frame can be given by the following matrix representation

$$\begin{pmatrix} t \\ g \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{2.5}$$

A unit-speed curve on a surface is a geodesic if and only if the principal normal n to the curve and the surface normal N are parallel to each other at any point on the curve. Equivalently, a curve $\gamma(s)$ on the surface is a geodesic provided its acceleration vector $\gamma''(s)$ is always normal to the surface, i.e.

$$\gamma''(s) \times N = 0. \tag{2.6}$$

3. Generalized cylinder with geodesic and line of curvature parameterizations

This section is the main part of this paper, it consists of three subsections that are devoted to defining and covering the generalized cylinder with geodesics and lines of curvature parametrizations. A generalized cylinder has two families of parametric curves, rulings and base curves. It is well known that the rulings are geodesics and lines of curvature on a generalized cylinder. Consequently, this section is devoted to providing the necessary and sufficient conditions for the base curves to be geodesics or lines of curvature. We show that the generalized cylinder with geodesic parametrization is a rectifying cylinder, and the generalized cylinder with a line of curvature parametrization is a right cylinder. The following first subsection aims to parametrize the generalized cylinder, we start from the ruled parametrization, and with the cylindrical condition that is described by the constrains three equations that are must be satisfied, we obtain the cylindrical parametrization.

3.1. Generalized cylinder

A generalized cylinder is generated by a constant moving of a straight line on a given curve and defined by the following ruled parametrization

$$X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq \ell, v \in \mathbb{R}, \text{ where } D'(s) = 0. \tag{3.1}$$

A unit regular curve $\gamma(s)$ is called a base curve, and the line passing through $\gamma(s)$ that is parallel to $D(s)$ is called the ruling. $D(s)$ is a unit director vector field that gives the direction of the ruling, $D'(s) = 0$ is the cylindrical condition which means that the ruling moves in a constant direction. The unit normal vector field (shortly surface normal) of the generalized cylinder is defined by using (2.4) as

$$N(s,v) = \frac{X_s \times X_v}{|X_s \times X_v|} = \frac{(\gamma' \times D) + v(D' \times D)}{|(\gamma' \times D) + v(D' \times D)|} = \frac{\gamma' \times D}{|\gamma' \times D|}.$$

$D(s)$ is a unit vector field that lies in the space formed by the frame $\{t, n, b\}$ and can be written using (2.5) as following

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)g(s), \text{ where } g(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s).$$

Therefore $D(s)$ can be decomposed as the following [29]

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)), \tag{3.2}$$

where $\theta(s)$ and $\phi(s)$ are two scalar functions called the first and second angular functions [30]. The derivative of $D(s)$ is given by

$$D'(s) = -\sin\theta[\kappa\cos\phi + \frac{d\theta}{ds}]t + [\cos\theta(\kappa + \cos\phi\frac{d\theta}{ds}) - \sin\theta\sin\phi(s)(\frac{d\phi}{ds} + \tau)]n + [\sin\phi\cos\theta\frac{d\theta}{ds} + \sin\theta\cos\phi(\frac{d\phi}{ds} + \tau)]b.$$

Definition 3.1. *The ruled parametrization with base curve $\gamma(s)$ and a unit director vector D(s) (3.2) is defined by*

$$X(s,v) \quad = \gamma(s) + vD(s), 0 \le s \le L,\, v \in \mathbb{R}, \tag{3.3}$$

where

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)).$$

In the following theorem, we give the necessary and sufficient conditions to construct a generalized cylinder parametrization from a ruled parametrization (3.3), we call them the cylindrical conditions.

Theorem 3.2. *The ruled parametrization (3.1) is a generalized cylinder if and only if the following conditions are satisfied*

$$\begin{aligned} \kappa\cos\phi + \frac{d\theta}{ds} &= 0, \\ \cos\theta(\kappa + \cos\phi\frac{d\theta}{ds}) - \sin\theta\sin\phi(\frac{d\phi}{ds} + \tau) &= 0, \\ \sin\phi\cos\theta\frac{d\theta}{ds} + \sin\theta\cos\phi(\frac{d\phi}{ds} + \tau) &= 0. \end{aligned} \tag{3.4}$$

Definition 3.3. *The generalized cylinder with base curve $\gamma(s)$ and a unit director vector D(s) (3.2) is parameterized by*

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))], 0 \le s \le L,\, v \in \mathbb{R}, \tag{3.5}$$

where

$$\kappa\cos\phi + \frac{d\theta}{ds} = 0, \cos\theta(\kappa + \cos\phi\frac{d\theta}{ds}) - \sin\theta\sin\phi(\frac{d\phi}{ds} + \tau) = 0, \textit{ and } \sin\phi\cos\theta\frac{d\theta}{ds} + \sin\theta\cos\phi(\frac{d\phi}{ds} + \tau) = 0.$$

The first and second derivatives of the generalized cylinder parameterized by (3.5) are given in the following equations

$$X_s = t(s), X_{ss} = \kappa(s)n(s), X_{sv} = 0, X_v = D(s), X_{vv} = 0. \tag{3.6}$$

The inner and cross products of the tangents vectors X_s and X_v are given by

$$\begin{aligned} &\langle X_s, X_v\rangle = \cos\theta(s), \\ &X_s \times X_v = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \end{aligned}$$

By using (2.4), the unit normal of the generalized cylinder (3.5) is defined everywhere and given by the following

$$N(s,v) = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \tag{3.7}$$

The main result of this paper is the following theorem which is proved in the next subsections.

Theorem 3.4. *Let $X(s,v) = \gamma(s) + vD(s), 0 \le s \le L, v \in \mathbb{R}$ be a generalized cylinder, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature , $D(s)$ is a unit director vector defined by (3.2) satisfying $D'(s) = 0$. Then the generalized cylinder with geodesic or line of curvature parameterization is a rectifying cylinder or a right cylinder respectively.*

3.2. Generalized cylinder with geodesic parameterization

Theorem 3.5. *All base curves of the generalized cylinder parameterized by (3.5) are geodesics if and only if the following conditions are satisfied.*

$$\cos\phi(s)=0, \qquad \frac{d\theta}{ds}=0, \qquad \cos\theta(s)\kappa(s)-\sin\theta(s)\tau=0. \tag{3.8}$$

Proof. According to (2.6), the base curves on a generalized cylinder (3.5) are geodesics if and only if their acceleration vector X_{ss} is normal to the surface, or equivalently $N(s,v)\times X_{ss}=0$. From (3.6) and (3.7), it follows that $N(s,v)\times X_{ss}=-\cos\phi\, t(s)$, the geodesic condition $N(s,v)\times X_{ss}=0$ is satisfied if and only if $\cos\phi(s)=0$ which is the first condition of (3.8). By substitution it in the cylindrical conditions (3.4), we get the other conditions of (3.8). □

Definition 3.6. *A generalized cylinder with geodesic base curves is defined by*

$$X(s,v)=\gamma(s)+v[\cos\theta(s)t(s)+\sin\theta(s)b(s)],\, 0\le s\le L,\, v\in\mathbb{R}, \tag{3.9}$$

$$\tau(s)\sin\theta(s)-\kappa(s)\cos\theta(s)=0, \textit{ and } \quad \theta'(s)=0.$$

Proposition 3.7. *[24] Suppose that $D(s)=\cos\theta(s)t(s)+\sin\theta(s)b(s)$ is a unit rectifying vector defined along a unit speed curve $\gamma(s)$ with non vanishing curvature and torsion, then $D(s)$ is a unit Darboux vector field if and only if $\kappa\cos\theta-\tau\sin\theta=0$.*

Proof. Let $D(s)=\cos\theta(s)t(s)+\sin\theta(s)b(s)$ be a unit Darboux vector. From (2.2),

$$\cos\theta=\frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \quad \sin\theta(s)=\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \quad \textit{and} \quad \cot\theta=\frac{\tau}{\kappa}.$$

This implies that $\kappa\cos\theta-\tau\sin\theta=0$, and vice versa. □

Definition 3.8. *A generalized cylinder with geodesic base curves is defined by*

$$X(s,v)=\gamma(s)+vD(s), 0\le s\le L, v\in\mathbb{R}$$

where

$$D(s)=\frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s)+\frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s), \quad D'(s)=0.$$

As discussed in (2.3), the condition for unit Darboux vector to be constant is equivalent to the base curve is a helix. As well known, the base curve and director vector are responsible to build the generalized cylinder, so the following theorem gives the conditions that can be applied on the base curve and director vector at the same time to generate a generalized cylinder with geodesic base curves.

Theorem 3.9. *Let $X(s,v)=\gamma(s)+vD(s), 0\le s\le L, v\in\mathbb{R}$ be a generalized cylinder, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature and torsion, $D(s)$ is a unit director vector defined by (3.2) satisfying D'(s) =0. Then every ruling is a geodesic and the base curves are geodesics if and only if $\gamma(s)$ is a helix and $D(s)$ is a unit Darboux vector.*

Definition 3.10. *A generalized cylinder with geodesic parameterization is defined by*

$$X(s,v)=\gamma(s)+vD(s), 0\le s\le L, v\in\mathbb{R}, \tag{3.10}$$

where

$$D(s)=\frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s)+\frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s), \quad \textit{and } \gamma(s) \textit{ is a helix.}$$

The developable ruled surface whose director vector is a unit Darboux vector has been studied by many researchers and it has been called the rectifying developable surface, (see, e.g., [31]). The generalized cylinder defined by (3.10) is a special case where the unit Darboux vector is a constant and we call it the rectifying cylinder. The base curve is a geodesic on its rectifying developable is a classical result has been stated in the classical differential geometry books [26], but according to theorem (3.9) all base curves are geodesics on their rectifying cylinder.

Corollary 3.11. *A generalized cylinder with geodesic parameterization (3.10) is a rectifying cylinder.*

Theorem 3.12. *Among all generalized cylinders parameterized by (3.5), only the rectifying cylinder (3.10) can be equipped with geodesic parameterization.*

In the above definition (3.10) we remark that for the rectifying cylinder (3.10) whose parametric curves are geodesics, the base geodesic curves have the same curvature and torsion, and differ only by the rigid motion modeled by a constant unit Darboux vector with fixed direction and fixed angular velocity. Therefore, it is interesting to end this subsection with the following result

Corollary 3.13. *The geodesic parametric curves of the the rectifying cylinder (3.10) are lines and helices.*

3.3. Generalized cylinder with line of curvature parameterization

Theorem 3.14. *All base curves of the generalized cylinder parameterized by (3.5) are lines of curvature if and only if the following conditions are satisfied*

$$\cos\theta(s) = 0, \qquad \cos\phi(s) = 0, \qquad \tau(s) = 0. \tag{3.11}$$

Proof. By Theorem (2.1), the base curves on a generalized cylinder (3.5) are lines of curvature if and only if $F = f = 0$. From (3.6) and (3.7), $f = \langle N, X_{vs}\rangle = 0$ is satisfied without further condition, and $F = \langle X_s, X_v\rangle = \cos\theta$, therefore, $F = 0$ if and only if $\cos\theta = 0$ which is the first condition of (3.11). By substitution it in the cylindrical conditions (3.4), we get the other conditions of (3.11). □

Definition 3.15. *A generalized cylinder with line of curvature base curves is defined by*

$$X(s,v) = \gamma(s) + vb(s), 0 \le s \le L, v \in \mathbb{R}, \quad \textit{where} \quad \tau(s) = 0.$$

The plane curve ($\tau(s) = 0$) has no binormal unit vector $b(s)$, therefore, the binormal of plane curve coincides with the normal vector to the plane of the curve. Without loss in generality we may assume that the unit vector $\langle 0,0,1\rangle$ is the normal to the plane of planar curve $\gamma(s)$.

Theorem 3.16. *Let $X(s,v) = \gamma(s) + vD(s), 0 \le s \le L, v \in \mathbb{R}$ be a generalized cylinder, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature, $D(s)$ is a unit director vector defined by (3.2) satisfying D'(s) =0. Then every ruling is a line of curvature and the base curves are lines of curvature if and only if $\gamma(s)$ is a plane curve and $D(s)$ is a unit normal vector to the plane of $\gamma(s)$.*

Definition 3.17. *A generalized cylinder with line of curvature parameterization is defined by*

$$X(s,v) = \gamma(s) + vD(s), 0 \le s \le L, v \in \mathbb{R}, \tag{3.12}$$

where

$$D(s) = \langle 0,0,1\rangle \quad \textit{and} \quad \gamma(s) \textit{ is a plane curve.}$$

The generalized cylinder whose base curve is a plane curve and the director vector is a unit normal vector to the plane of the base curve is called a right generalized cylinder [32] or shortly right cylinder.

Corollary 3.18. *A generalized cylinder with line of curvature parameterization (3.12) is a right cylinder.*

Theorem 3.19. *Among all generalized cylinders parameterized by (3.5), only the right cylinder (3.12) can be equipped with line of curvature parameterization.*

Corollary 3.20. *The line of curvature parametric curves of the the right cylinder (3.12) are lines and plane curves.*

4. Examples

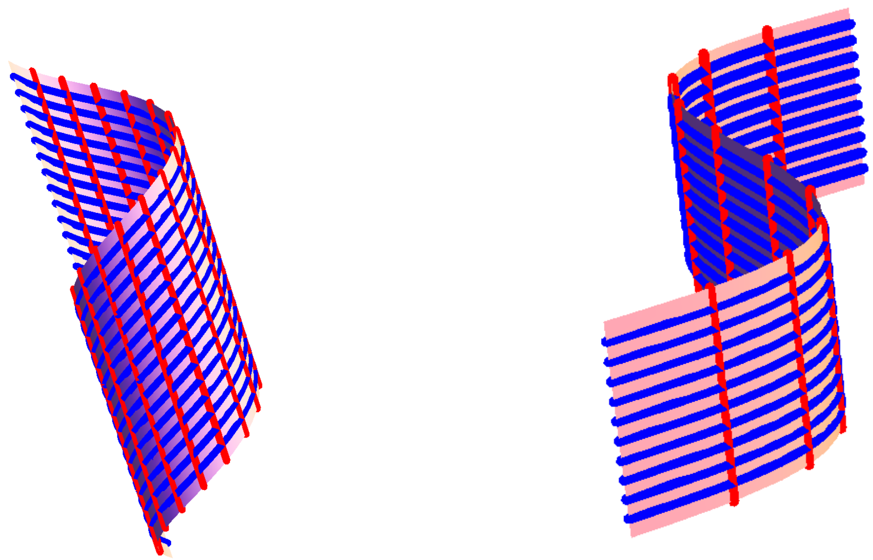
In this section, we give two examples of a generalized cylinder with geodesic and line of curvature parametrization and draw their pictures by using Mathematica. It is worth noting that the results are satisfied even the base curve is not a unit speed as shown in the second example.

Example 4.1. *Let $\gamma(s) = (\frac{\sqrt{3}}{2}\sin(s), \frac{s}{2}, \frac{\sqrt{3}}{2}\cos(s))$ be a unit speed helix curve, therefore the unit tangent and binormal vectors are given respectively by $t = \left(\frac{\sqrt{3}}{2}\cos(s), \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin(s)\right)$ and $b = (-\frac{1}{2}\cos(s), \frac{\sqrt{3}}{2}, \frac{1}{2}\sin(s))$. Their curvature and torsion are $\kappa = \frac{\sqrt{3}}{2}$ and $\tau = \frac{1}{2}$. According to definition (3.10), the generalized cylinder with geodesic parametrization is defined by*

$$X(s,v) = \gamma(s) + v[\frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s) + \frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s)],\ 0 \le s \le L, v \in \mathbb{R}.$$

By substitution $\frac{\tau}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{2}$ and $\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{\sqrt{3}}{2}$, and for $0 \le s \le 2\pi$, $0 \le v \le \pi$, the constructed cylinder is a rectifying cylinder with geodesic parametrization as shown in Figure 1(a).

Example 4.2. *Let $\gamma(s) = (s, \sin(s), 0)$ be a plane curve. According to definition (3.12), the generalized cylinder with line of curvature parametrization can be defined by $X(s,v) = \gamma(s) + v\,(0,0,1)$, $0 \le s \le 2\pi, 0 \le v \le \pi/2$. The constructed cylinder is a right cylinder with line of curvature parametrization as shown in Figure 1(b).*



(a) Rectifying cylinder with geodesic parametrization

(b) Right cylinder with line of curvature parametrization

Figure 4.1: Generalized cylinder with geodesic or line of curvature parametrizations

5. Conclusion

In this paper, using a ruled parametrization (3.1), and with three conditions called the cylindrical conditions (3.4) we constructed a generalized cylinder parametrization (3.5). After that, through many geometric constraints we obtained the resulting cylinder that is parameterized by geodesics or line of curvatures. The main results asserted that the generalized cylinder with geodesic or line of curvature parametrization is a rectifying cylinder (3.10) or a right cylinder (3.12) respectively.

Acknowledgements

The author would like to express his sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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