Communications in Advanced Mathematical Sciences
Vol. 5, No. 2, 48-77, 2022
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.1020515

Generalized Rayleigh-Quotient Formulas for the Real Parts, Imaginary Parts, and Moduli of Simple Eigenvalues of Compact Operators

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Abstract
In an earlier paper, the author derived generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of the eigenvalues of *diagonalizable matrices*. More precisely, max-, min-max-, min-, and max-min-formulas were obtained. In this paper, we extend these results to the eigenvalues of linear *nonsymmetric compact operators* with *simple eigenvalues* in a *Hilbert space*. As an application, a new formula for the spectral radius is derived. An example arising from a boundary value problem in Mathematical Physics illustrates the general results, and numerical computations underpin the theoretical findings. In addition, the Euler column is treated from the area of Elastomechanics, which is complemented by references to other examples from this area.

Keywords: Generalized Rayleigh-Quotients, Hilbert space, Real parts, Imaginary parts, and moduli of eigenvalues, Simple eigenvalues of compact operators
2010 AMS: Primary 65J05, Secondary 34L10, 47A12, 74B15

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Received: 8 October 2021, **Accepted:** 28 April 2022, **Available online:** 30 June 2022

1. Introduction

In [16], the author derived generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of the eigenvalues of diagonalizable nonsymmetric matrices, that is, in the case of a finite-dimensional space. In this paper, we extend these results to the eigenvalues of nonsymmetric compact operators with simple eigenvalues in an infinite-dimensional Hilbert space. Some arguments in the proofs are similar to those in the finite-dimensional case, but others are very different from them.

The paper is structured as follows. In Section 2, as a basis for what follows, functions of an operator in a Banach space are discussed which is taken from [18]. Section 3 contains the expansion of a linear nonsymmetric compact operator and of a pertinent projection operator in a Hilbert space. In Sections 4 - 6, generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli are given respectively followed in Section 7 by generalized Rayleigh-quotient formulas for real eigenvalues. In Section 8, the general results are employed to obtain a new formula for the spectral radius. Section 9 presents new generalized numerical ranges, and in Section 10 an example from the area of a boundary value problem is given along with the results of numerical computations. In Section 11, it is discussed what consequences changes in the arrangement of the eigenvalues will have. Section 12 contains the conclusion and an outlook to future work. Finally, the references follow. Besides the cited references, the following non-cited ones are given: [1] - [3], [6], [8], [9], [12] - [15], [17], [19], [21], [22], [29], and [30] since the author thinks that they could be of interest to the reader in the context of the treated subject. We mention that the Remarks are not enumerated.

2. Functions of an Operator in a Banach Space

This section is of fundamental importance for what follows; it is taken from the corresponding section in [18]. The results are obtained in a Banach space of which a Hilbert space is a particular case.

Let $\{0\} \neq E$ be a Banach space over the field $\mathbb{F} = \mathbb{C}$. Whereas in [10, Chapter I] it is supposed that $dim E < \infty$, here we assume that $dim E = \infty$. As was shown in [26] based on findings of [24], the following results taken from [10, Chapter I] are valid not only for $dim E < \infty$, but also for $dim E = \infty$ if the space is complete.

Let $p(\zeta)$ be the polynomial

$$p(\zeta) = \alpha_0 + \alpha_1 \zeta + \cdots \alpha_n \zeta^n, \; \zeta \in \mathbb{C} \tag{2.1}$$

with $\alpha_j \in \mathbb{C}, \; j = 0, 1, \cdots, n$. Then the polynomial $p(T) \in B(E)$ is defined by

$$p(T) = \alpha_0 + \alpha_1 T + \cdots \alpha_n T^n, \; \zeta \in \mathbb{C}, \tag{2.2}$$

see [10, Chapter I, §3.3]. Making use of the resolvent

$$R(\zeta) := (T - \zeta)^{-1}, \; \zeta \in \mathbb{C}, \tag{2.3}$$

one can now define the function $\phi(T)$ of T for a more general class of functions $\phi(\zeta)$.

Before we do this, we mention that linear compact operators need not have eigenvalues. For example, Volterra integral operators have no eigenvalues. On the other hand, consider a symmetric linear compact operator. Then, such an operator has at least one eigenvalue, and all eigenvalues are real and simple. For these operators, there may exit only a finite number of eigenvalues. Further, there is at most a countable set of eigenvalues with the only possible accumulation point zero, and there exists a set of pertinent pairwise orthonormal eigenvectors. Further, it is known that the non-zero elements of the spectrum consist solely of eigenvalues and that, if there is a countable set of eigenvalues, the associated sequence tends to zero. For all this, see [27, Chapter 6].

Further, according to [7, Theorem 44.1, p.191], one has $\sigma(T) \setminus \{0\} = \sigma_P(T) \setminus \{0\}$ where $\sigma(T)$ is the spectrum of T and $\sigma_P(T)$ the point spectrum consisting of the eigenvalues of T.

Taking this into account, for our general linear compact operator $T \in B(E)$, we suppose that the spectrum $\sigma(T)$ of T has a countable set of non-zero eigenvalues λ_j and that the sequence of eigenvalues tends to zero.

Additionally, we suppose that $0 \notin \sigma(T)$ so that $N(T) = \{0\}$ since without this condition, we cannot obtain relation (2.11) resp. (2.14) below.

Now, suppose that $\phi(\zeta)$ is holomorphic in a domain D of the complex plane containing all the eigenvalues $\lambda_j \neq 0$ of T, and let $C \subset D$ be a simple closed smooth curve with positive direction enclosing all the eigenvalues λ_j in its interior. Then, $\phi(T)$ is defined by the *Dunford-Taylor integral*

$$\phi(T) = -\frac{1}{2\pi i} \int_C \phi(\zeta) R(\zeta)\, d\zeta = -\frac{1}{2\pi i} \int_C \phi(\zeta)(T - \zeta)^{-1}\, d\zeta. \tag{2.4}$$

This is an analogue of the Cauchy integral formula in the Theory of Functions, see [11, Part I, §15, p. 61]. More generally, the curve C may consist of several simple closed rectifiable Jordan curves C_k having a positive direction with interiors D'_k such that the union of the D'_k contains all the eigenvalues of T. We note that (2.4) does not depend on C as long as C satisfies these conditions. For the C_k, we can use the circles $C_k = \{z \in \mathbb{C} \,|\, |z - \lambda_k| = r_k\}$ with sufficiently small radii r_k.

It can be verified that, for the polynomial

$$\phi(\zeta) = p(\zeta) = \alpha_0 + \alpha_1 \zeta + \cdots \alpha_n \zeta^n, \; \zeta \in \mathbb{C} \tag{2.5}$$

with $\alpha_j \in \mathbb{C}, \; j = 0, 1, \cdots, n$, the Dunford-Taylor integral (2.4) is equal to (2.2).

For the special case

$$\phi(\zeta) = p(\zeta) = \zeta, \tag{2.6}$$

we obtain

$$T = -\frac{1}{2\pi i} \int_C T R(\zeta)\, d\zeta = T \left(-\frac{1}{2\pi i} \int_C R(\zeta)\, d\zeta \right) = \left(-\frac{1}{2\pi i} \int_C R(\zeta)\, d\zeta \right) T. \tag{2.7}$$

Now, we set

$$P := -\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta. \tag{2.8}$$

According to [10, Chapter I, §5, Section 3], P is a continuous projection operator onto the *algebraic eigenspace* $X = P(E) = R(P)$, where $R(P)$ means the range of P. Thus, from (2.7) and (2.8), one obtains

$$T = TP = PT = PTP. \tag{2.9}$$

Now, let the radii r_k be chosen such that

$$C_j \cap C_k = \emptyset,\ j \neq k,\ j,k = 1,2,3,\cdots. \tag{2.10}$$

Then,

$$P = -\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta = \sum_{j=1}^{\infty}\left(-\frac{1}{2\pi i}\int_{C_j} R(\zeta)\,d\zeta\right) = \sum_{j=1}^{\infty} P_j \tag{2.11}$$

with

$$P_j = -\frac{1}{2\pi i}\int_{C_j} R(\zeta)\,d\zeta,\ j = 1,2,3,\cdots. \tag{2.12}$$

At this point, we needed the assumption $0 \notin \sigma(T)$ since otherwise any circle C_0 about $\lambda_0 = 0$ would eventually intersect with the circles C_k for sufficiently large k so that we would not have (2.10) for $j,k \in (0,1,2,3,\cdots)$. Let J be the sequence

$$J := (1,2,3,\cdots). \tag{2.13}$$

Then, (2.11) can be written as

$$\boxed{P = \sum_{j=1}^{\infty} P_j = \sum_{j\in J} P_j.} \tag{2.14}$$

Because of (2.10), one has

$$P_j P_k = P_k P_j = P_j \delta_{jk},\ j,k \in J. \tag{2.15}$$

Herewith,

$$P_j(E) =: X_j \tag{2.16}$$

is the *algebraic eigenspace of T associated with the eigenvalue* λ_j.

From (2.9), (2.11), and (2.15), we obtain

$$T = PT = TP = PTP = \sum_{j\in J} P_j T = \sum_{j\in J} TP_j = \sum_{j\in J} P_j T P_j, \tag{2.17}$$

and so

$$\begin{aligned} R(T) = T(E) &= (PT)(E) = (TP)(E) = (PTP)(E) \\ &= \sum_{j\in J}(P_j T)(E) = \sum_{j\in J}(TP_j)(E) = \sum_{j\in J}(P_j T P_j)(E). \end{aligned} \tag{2.18}$$

3. Expansion of a Linear Compact Operator and of a Pertinent Projection Operator in Hilbert Space

Together with Section 2, this section forms a basis for what follows. The statements are taken over from [18], but most of the proofs are omitted.

(i) The Conditions (C1) - (C4)

We assume the following conditions:

(C1) $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

(C2) $0 \neq T \in B(H)$ is compact having countably many simple non-zero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ with $\lim_{k\to\infty} \lambda_k = 0$ pertinent to the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$. Further, $0 \notin \sigma(T)$.

(C3) The eigenvectors of the adjoint T^* of T with the eigenvalues $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \cdots$ are $\psi_1, \psi_2, \psi_3, \cdots$

(C4) $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j = 1, 2, 3\cdots$

One has the following theorem.

Theorem 3.1. *(Biorthonormality relations for $\lambda_j \neq \lambda_k,\ j \neq k$)*

Let the conditions (C1) - (C4) be fulfilled. Then, with appropriate normalization, the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$ and $\psi_1, \psi_2, \psi_3, \cdots$ are biorthonormal, that is,

$$(\chi_j, \psi_k) = \delta_{jk},\ j,k \in J. \tag{3.1}$$

Proof. See [18, Theorem 3.1]. □

Furthermore, we obtain the following theorem.

Theorem 3.2. *(Expansion of Tu as well as of Pu in a series of eigenvectors) Let the conditions (C1) - (C4) be fulfilled. Then,*

$$Tu = \sum_{j\in J} \lambda_j (u, \psi_j)\chi_j,\ u \in H \tag{3.2}$$

as well as

$$Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H. \tag{3.3}$$

Proof. See [18, Theorem 3.2]. □

Remark. *From (3.2) we conclude that*

$$\overline{[\chi_1, \chi_2, \chi_3, \cdots]} = T(H) = R(T).$$

Further, from (3.3),

$$P : H \mapsto \overline{[\chi_1, \chi_2, \chi_3, \cdots]}.$$

□

Moreover, in [18, Theorem 3.3], we have proven the following theorem.

Theorem 3.3. *Let the conditions (C1) - (C4) be fulfilled. Then, we obtain*

$$u = Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H \tag{3.4}$$

and the projection operator

$$P_0 = I - P : H \mapsto N(T) = \{0\} \iff P_0 = 0. \tag{3.5}$$

For the next theorem, we define new subspaces of H. For every $j = 1,2,\ldots$, let

$$N_{\chi,j} := \{u \in H \,|\, u = \sum_{k=1}^{j} \alpha_k \chi_k \text{ with } \alpha_k \in \mathbb{C},\, k = 1,2,\ldots,j\} =: [\chi_1,\ldots,\chi_j], \tag{3.6}$$

$j = 1,2,\ldots$ and

$$N_{\chi,j,\mathbb{R}} := \{u \in H \,|\, u = \sum_{k=1}^{j} \beta_k \chi_k \text{ with } \beta_k \in \mathbb{R},\, k = 1,2,\ldots,j\} = [\chi_1,\ldots,\chi_j]_{\mathbb{R}},\ j = 1,2,\ldots \tag{3.7}$$

$j = 1,2,\ldots$ as well as

$$\begin{aligned} N_\chi &:= N_{\chi,\infty} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \alpha_k \chi_k \text{ exists in H with } \alpha_k \in \mathbb{C}, k = 1,2,\ldots\} \\ &= \overline{[\chi_1,\chi_2,\ldots]} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} N_{\chi,\mathbb{R}} &:= N_{\chi,\infty,\mathbb{R}} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \beta_k \chi_k \text{ exists in H with } \beta_k \in \mathbb{R}, k = 1,2,\ldots\} \\ &= \overline{[\chi_1,\chi_2,\ldots]}_{\mathbb{R}}. \end{aligned} \tag{3.9}$$

Likewise, we define

$$N_{\psi,j} := \{u \in H \,|\, u = \sum_{k=1}^{j} \alpha_k \psi_k \text{ with } \alpha_k \in \mathbb{C},\, k = 1,2,\ldots,j\} =: [\psi_1,\ldots,\psi_j], \tag{3.10}$$

$j = 1,2,\ldots$ and

$$N_{\psi,j,\mathbb{R}} := \{u \in H \,|\, u = \sum_{k=1}^{j} \beta_k \psi_k \text{ with } \beta_k \in \mathbb{R},\, k = 1,2,\ldots,j\} = [\psi_1,\ldots,\psi_j]_{\mathbb{R}}, \tag{3.11}$$

$j = 1,2,\ldots$ as well as

$$\begin{aligned} N_\psi &:= N_{\psi,\infty} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \alpha_k \psi_k \text{ exists in H with } \alpha_k \in \mathbb{C}, k = 1,2,\ldots\} \\ &= \overline{[\psi_1,\psi_2,\ldots]} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} N_{\psi,\mathbb{R}} &:= N_{\psi,\infty,\mathbb{R}} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \beta_k \psi_k \text{ exists in H with } \beta_k \in \mathbb{R}, k = 1,2,\ldots\} \\ &= \overline{[\psi_1,\psi_2,\ldots]}_{\mathbb{R}}. \end{aligned} \tag{3.13}$$

After these preparations, we are able to prove the following theorem.

Theorem 3.4. *Let the conditions (C1) - (C4) be fulfilled. Then,*

$$(Tu,v) = \sum_{j\in J} \lambda_j (u,\psi_j)(\chi_j,v),\ u,\, v \in H \tag{3.14}$$

and

$$(u,v) = (Pu,v) = \sum_{j\in J} (u,\psi_j)(\chi_j,v),\ u,\, v \in H \tag{3.15}$$

where

$$(u,\psi_j),\, (\chi_j,v) \in \mathbb{R},\ u \in N_{\chi,\mathbb{R}},\, v \in N_{\psi,\mathbb{R}},\ j \in J \tag{3.16}$$

leading to

$$Re\,(Tu,v) = \sum_{j\in J} Re\,\lambda_j (u,\psi_j)(\chi_j,v),\ u \in N_{\chi,\mathbb{R}},\, v \in N_{\psi,\mathbb{R}},\ j \in J. \tag{3.17}$$

Proof. Let $u \in N_{\chi,\mathbb{R}}$ and $v \in N_{\psi,\mathbb{R}}$. Then,

$$u = \sum_{j\in J} (u, \psi_j)\chi_j \tag{3.18}$$

and

$$v = \sum_{k\in J} (v, \chi_k)\psi_k \tag{3.19}$$

implying

$$(Tu, v) = \sum_{j,k\in J} \lambda_j (u, \psi_j)\overline{(v, \chi_k)}(\chi_j, \psi_k) \tag{3.20}$$

so that with (3.1) relation (3.14) follows.

Further, let $u \in N_{\chi,\mathbb{R}}$. Then,

$$u = \sum_{j\in J} \alpha_j \chi_j$$

with elements $\alpha_j \in \mathbb{R},\ j \in J$ so that

$$(u, \psi_j) = \sum_{k\in J} \alpha_k (\chi_k, \psi_j) = \alpha_j \in \mathbb{R}.$$

Correspondingly, for $v \in N_{\psi,\mathbb{R}}$, one has $(\chi_j, v) \in \mathbb{R}$ so that (3.16) is proven. Relation (3.17) is a direct consequence of (3.14) and (3.16). The expression in (3.15) follows in a similar way as that in (3.14) by using (3.4). □

Next, we want to define vector spaces similar to those in [16, (16), (17)], namely

$$M_{\chi,1,\mathbb{R}} := N_{\chi,\mathbb{R}} = \overline{[\chi_1, \chi_2, \ldots]}_{\mathbb{R}}, \tag{3.21}$$

$$\begin{aligned} M_{\chi,j,\mathbb{R}} &:= \{u \in N_{\chi,\mathbb{R}} \,|\, (u, \psi_k) = 0, k = 1, 2, \ldots, j-1\} \\ &= [\psi_1, \ldots, \psi_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}},\ j = 2, 3, \ldots \end{aligned} \tag{3.22}$$

where $M_{\chi,j,\mathbb{R}}$ is called an *orthogonal complement in* $N_{\chi,\mathbb{R}}$ and

$$M_{\psi,1,\mathbb{R}} := N_{\psi,\mathbb{R}} = \overline{[\psi_1, \psi_2, \ldots]}_{\mathbb{R}}, \tag{3.23}$$

$$\begin{aligned} M_{\psi,j,\mathbb{R}} &:= \{u \in N_{\psi,\mathbb{R}} \,|\, (u, \chi_k) = 0, k = 1, 2, \ldots, j-1\} \\ &= [\chi_1, \ldots, \chi_{j-1}]^{\perp}_{N_{\psi,\mathbb{R}}},\ j = 2, 3, \ldots \end{aligned} \tag{3.24}$$

where $M_{\psi,j,\mathbb{R}}$ *is called an orthogonal complement in* $N_{\psi,\mathbb{R}}$. The next lemma characterizes these spaces.

Lemma 3.5. *Let the conditions (C1) - (C4) be fulfilled as well as* $\{\chi_1, \chi_2, \ldots\}$ *and* $\{\psi_1, \psi_2, \ldots\}$ *be sets of biorthogonal eigenvectors of* T *and* T^* *respectively, i.e., such that*

$$(\chi_i, \psi_j) = \delta_{ij},\ i, j = 1, 2, \ldots. \tag{3.25}$$

Then,

$$M_{\chi,j,\mathbb{R}} = \overline{[\chi_j, \chi_{j+1}, \ldots]}_{\mathbb{R}}, j = 1, 2, \ldots \tag{3.26}$$

and

$$M_{\psi,j,\mathbb{R}} = \overline{[\psi_j, \psi_{j+1}, \ldots]}_{\mathbb{R}}, j = 1, 2, \ldots \tag{3.27}$$

Proof. The proof is done for (3.26) and $j = 3$. The general case can be made by induction. The proof of (3.27) is similar. So, we have to prove

$$\begin{aligned} M_{\chi,3,\mathbb{R}} &:= \{u \in N_{\chi,\mathbb{R}} \,|\, (u,\psi_k) = 0,\, k = 1,2\} = [\psi_1,\psi_2]^{\perp}_{N_{\chi,\mathbb{R}}} \\ &= \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}} \end{aligned} \tag{3.28}$$

(i) $\underline{\overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}} \subset M_{\chi,3,\mathbb{R}}:}$

Let $u \in \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}}$. Then, $u = \sum_{k=3}^{\infty} \beta_j \chi_j$ with elements $\beta_j \in \mathbb{R},\ j = 3,4,\ldots$. Let $s \in \{1,2\}$. This entails, due to Theorem 3.1, $(u,\psi_s) = \sum_{j=3}^{\infty} \beta_j(\chi_j,\psi_s) = 0$ so that $u \in M_{\chi,3,\mathbb{R}}$. Therefore, (i) is proven.

(ii) $\underline{M_{\chi,3,\mathbb{R}} \subset \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}}:}$

Let $u \in M_{\chi,3,\mathbb{R}}$. This implies $u \in N_{\chi,\mathbb{R}}$ and $(u,\psi_j) = 0,\ j = 1,2$. Now, $u = \sum_{k=1}^{\infty} \beta_k \chi_k$ with $\beta_k = (u,\psi_k) \in \mathbb{R}, k = 1,2,\ldots$ leading to $u = \sum_{k=3}^{\infty} \beta_k \chi_k$ since $(u,\psi_k) = 0,\ k = 1,2$ so that $u \in \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}}$. Therefore, (ii) is proven. □

Now, let $u \in N_{\chi,\mathbb{R}}$ with $u = \sum_{k=1}^{\infty} \alpha_k \chi_k$ and $\alpha_k \in \mathbb{R}$ as well as $v \in N_{\psi,\mathbb{R}}$ with $v = \sum_{k=1}^{\infty} \beta_k \psi_k$ and $\beta_k \in \mathbb{R}$. Then, due to Theorem 3.1,

$$(u,v) = \sum_{k=1}^{\infty} \alpha_k \beta_k. \tag{3.29}$$

In order to facilitate the manner of speaking, we say that the *scalar product* (u,v) *of* $u \in N_{\chi,\mathbb{R}}$ *and* $v \in N_{\psi,\mathbb{R}}$ *is strongly positive iff* $\alpha_k\beta_k \geq 0,\ k = 1,2,\ldots$ *and* $\sum_{k=1}^{\infty} \alpha_k\beta_k > 0$. For short, we write

$$(u,v) >> 0.$$

Remark. *One has* $\alpha_k = (u,\psi_k),\ u \in N_{\chi,\mathbb{R}}$ *and* $\beta_k = (\chi_k,v),\ v \in N_{\psi,\mathbb{R}}$ *for* $k = 1,2,\ldots$. *Therefore,* $(u,v) >> 0$ *means* $(u,\psi_k)(\chi_k,v) \geq 0,\ k = 1,2,\ldots$ *and* $(u,v) = \sum_{k=1}^{\infty}(u,\psi_k)(\chi_k,v) > 0$. □

Remark. *For* $(u,v) >> 0$, *one can admit linear combinations* $u = \sum_{k=1}^{\infty} \alpha_k \chi_k$ *and* $v = \sum_{k=1}^{\infty} \beta_k \psi_k$ *with* $\alpha_k, \beta_k \in \mathbb{C},\ k = 1,2,\ldots$ *such that* $\alpha_k \overline{\beta}_k = |\alpha_k \beta_k|,\ k = 1,2,\ldots$ *and* $\sum_{k=1}^{\infty} |\alpha_k \beta_k| > 0$. *For example, all elements* $\alpha_k, \beta_k \in \mathbb{C}$ *with* $\alpha_k = |\alpha_k| e^{i\varphi_k}$ *and* $\beta_k = |\beta_k| e^{i\varphi_k}$ *where* φ_k *is in* $0 \leq \varphi_k < 2\pi,\ k = 1,2,\ldots$ *are acceptable.* □

Remark. *At this point, we mention that, due to (2.14) and (2.17), it follows that we have the convergence*

$$P^{(n)} = \sum_{j=1}^{n} P_j \to P \ (n \to \infty)$$

and

$$T^{(n)} = \sum_{j=1}^{n} P_j T P_j \to T \ (n \to \infty)$$

in $B(H)$ *so that, e.g., the operators* T *and* P *defined in (3.2) and (3.3) are approximated by their partial sums not only pointwise, but even in the norm of* $B(H)$. □

4. Generalized Rayleigh-Quotient Formulas for the Real Parts of the Eigenvalues

In the sequel, we suppose that the non-zero eigenvalues are arranged according to

$$Re\lambda_1 \geq Re\lambda_2 \geq Re\lambda_3 \geq \ldots. \tag{4.1}$$

Such an arrangement is possible, for instance, if the real parts of all eigenvalues are positive. An arrangement that is always possible will be dealt with in Section 11.

One has the following generalized max-representation.

Theorem 4.1. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of* T *be arranged according to (4.1). Moreover, let the vector spaces* $M_{\chi,j,\mathbb{R}}$ *resp.* $M_{\psi,j,\mathbb{R}}$ *for* $j \in J$ *be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$Re\,\lambda_j = \max_{\substack{(u,v)>>0 \\ u \in M_{\chi,j,\mathbb{R}},\, v \in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)},\ j \in J. \tag{4.2}$$

The maximum is attained for $u = \chi_j,\ v = \psi_j$.

Proof. One uses (3.17) as starting point, i.e.,

$$Re\,(Tu,v) = \sum_{j\in J} Re\,\lambda_j (u,\psi_j)(\chi_j,v),\; u\in N_{\chi,\mathbb{R}},\, v\in N_{\psi,\mathbb{R}}$$

with

$$Re\,\lambda_j,\; (u,\psi_j),\; (\chi_j,v)\in\mathbb{R},\; u\in N_{\chi,\mathbb{R}},\; v\in N_{\psi,\mathbb{R}}.$$

Let $j\in J$ be arbitrarily chosen, but fixed as well as $u\in M_{\chi,j,\mathbb{R}}\subset N_{\chi,\mathbb{R}}$ and $v\in M_{\psi,j,\mathbb{R}}\subset N_{\psi,\mathbb{R}}$ also be arbitrarily chosen, but fixed with $(u,v)>>0$. Then,

$$\begin{aligned} Re\,(Tu,v) &= \sum_{k=j}^{\infty} Re\,\lambda_k (u,\psi_k)(\chi_k,v) \\ &\le \max_{k=j,j+1,\dots} Re\,\lambda_k \sum_{k=j}^{\infty}(u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j \sum_{k=1}^{\infty}(u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j (u,v),\; u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}},\; (u,v)>>0, \end{aligned}$$

that is,

$$\frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_j,\; u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}},\; (u,v)>>0$$

and thus

$$\max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_j.$$

Now, $Re\,\lambda_j$ is attained for $u=\chi_j\in M_{\chi,j,\mathbb{R}}$ and $v=\psi_j\in M_{\psi,j,\mathbb{R}}$. Thus, because of $(\chi_j,\psi_j)>>0$,

$$Re\,\lambda_j = \frac{Re\,(T\chi_j,\psi_j)}{(\chi_j,\psi_j)} \le \max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_j$$

so that (4.2) is proven. □

For the next theorem, we need the following *denotation of codimension*. A subspace $M\subset H$ has codimension j for $j\in J$ denoted by **codim M = j** if there exist linearly independent vectors $v_1,\dots,v_j\in H$ such that

$$M = [v_1,\dots,v_j]^{\perp} := [v_1,\dots,v_j]^{\perp}_H = \{u\in H\,|\,(u,v_k)=0,\; k=1,\dots,j\}.$$

Further, we set

$$\operatorname{codim} M = 0$$

if $M=H$. Next, we prove a generalized min-max-representation.

Theorem 4.2. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (4.1).*

Then, for every $j\in J$ and every subspace $M_\chi\subset N_{\chi,\mathbb{R}}$ and $M_\psi\subset N_{\psi,\mathbb{R}}$ with $\operatorname{codim} M_\chi = \operatorname{codim} M_\psi = j-1$, the following inequalities are valid:

$$Re\,\lambda_j \le \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_1, \tag{4.3}$$

and the following min-max-representation formulas hold:

$$Re\,\lambda_j = \min_{\substack{\operatorname{codim} M_\chi = j-1\\ \operatorname{codim} M_\psi = j-1}} \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)},\; j\in J. \tag{4.4}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\; M_\psi = M_{\psi,j,\mathbb{R}}. \tag{4.5}$$

Proof. (4.3): For all subspaces $M_\chi \subset N_{\chi,\mathbb{R}}$, one has

$$\max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \le \max_{\substack{(u,v)>>0\\ u\in N_{\chi,\mathbb{R}}, v\in N_{\psi,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} = Re\,\lambda_1. \tag{4.6}$$

In case $j=1$, it follows by definition of codim $M_\chi =$ codim $M_\psi = 0$ that $M_\chi = N_{\chi,\mathbb{R}}$ and $M_\psi = N_{\psi,\mathbb{R}}$ and thus the equal sign in (4.3); further, (4.4) reduces to (4.3) with the equal signs instead of the signs $\le$. Now, let $j\ge 2$. Then, there exist linearly independent vectors $u_1,\ldots,u_{j-1}$ and $v_1,\ldots,v_{j-1}$ with

$$M_\chi = [u_1,\ldots,u_{j-1}]^\perp_{N_{\chi,\mathbb{R}}}, \qquad M_\psi = [v_1,\ldots,v_{j-1}]^\perp_{N_{\psi,\mathbb{R}}}. \tag{4.7}$$

Define

$$z_\chi = \sum_{i=1}^{j} \alpha_i \chi_i$$

and determine the coefficients $\alpha_1,\ldots,\alpha_j$ by the $j-1$ linear equations

$$(z_\chi, u_k) = \sum_{i=1}^{j} \alpha_i (\chi_i, u_k) = 0,\ k=1,\ldots,j-1. \tag{4.8}$$

This system of $j-1$ linear equations and j unknowns has a nontrivial solution

$$z_\chi \ne 0,\ z_\chi \in M_\chi = [u_1,\ldots,u_{j-1}]^\perp_{N_{\chi,\mathbb{R}}}. \tag{4.9}$$

Now, define

$$z_\psi = \sum_{i=1}^{j} \alpha_i \psi_i \tag{4.10}$$

with the same coefficients α_i as in z_χ. Then,

$$z_\psi \ne 0. \tag{4.11}$$

Further,

$$(z_\chi, z_\psi) = \sum_{i=1}^{j} \alpha_i^2 > 0 \tag{4.12}$$

so that $(z_\chi, z_\psi) >> 0$. Moreover,

$$z_\psi \in [z_\psi]_{\mathbb{R}} \subset M_{\psi, z_\psi} \tag{4.13}$$

where M_{ψ,z_ψ} is any subspace of $N_{\psi,\mathbb{R}}$ with codimension $j-1$ containing the element z_ψ. From the above, it follows

$$Re\,(Tz_\chi, z_\psi) = \sum_{i,k=1}^{j} \alpha_i\, Re\,\lambda_i\, \alpha_k\, (\chi_i, \psi_k) = \sum_{i=1}^{j} Re\,\lambda_i\, \alpha_i^2. \tag{4.14}$$

Now, $\alpha_i \in \mathbb{R}$, $i=1,\ldots,j$. Therefore,

$$Re\,(Tz_\chi, z_\psi) \ge (\min_{i=1,\ldots,j} Re\,\lambda_i) \sum_{i=1}^{j} \alpha_i^2 = Re\,\lambda_j\, (z_\chi, z_\psi) \tag{4.15}$$

leading to

$$\frac{Re\,(Tz_\chi, z_\psi)}{(z_\chi, z_\psi)} \ge Re\,\lambda_j. \tag{4.16}$$

Moreover, due to (4.1),

$$Re\,(Tu,v) \le (\max_{j=1,2,\ldots} Re\,\lambda_j)\sum_{j=1}^{\infty}(u,\psi_j)\,(\chi_j,v) = Re\,\lambda_1\,(u,v),$$

$(u,v) >> 0,\, u\in N_{\chi,\mathbb{R}},\ v\in N_{\psi,\mathbb{R}}$ so that

$$Re\,\lambda_1 \ge \frac{(Tu,v)}{(u,v)},\ (u,v) >> 0,\, u\in N_{\chi,\mathbb{R}},\ v\in N_{\psi,\mathbb{R}}. \tag{4.17}$$

This implies

$$Re\,\lambda_j \le \frac{Re\,(Tz_\chi,z_\psi)}{(z_\chi,z_\psi)} \le \max_{\substack{(u,v)>>0\\ u\in M_{\chi,z_\chi},v\in M_{\psi,z_\psi}}} \frac{Re\,(Tu,v)}{(u,v)} \le \max_{\substack{(u,v)>>0\\ u\in M_\chi,v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_1. \tag{4.18}$$

Therefore, (4.3) is proven.

Proof of (4.4): From (4.3), we conclude

$$\min_{\substack{\operatorname{codim} M_\chi=j-1\\ \operatorname{codim} M_\psi=j-1}}\ \max_{\substack{(u,v)>>0\\ u\in M_\chi,v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \ge Re\,\lambda_j. \tag{4.19}$$

On the other hand, from Theorem 4.1,

$$Re\,\lambda_j = \max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}},v\in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \ge \min_{\substack{\operatorname{codim} M_\chi=j-1\\ \operatorname{codim} M_\psi=j-1}}\ \max_{\substack{(u,v)>>0\\ u\in M_\chi,v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \tag{4.20}$$

since

$$M_{\chi,j,\mathbb{R}} = \overline{[\chi_j,\chi_{j+1},\ldots]}_{\mathbb{R}} = [\psi_1,\ldots,\psi_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}} \tag{4.21}$$

and

$$M_{\psi,j,\mathbb{R}} = \overline{[\psi_j,\psi_{j+1},\ldots]}_{\mathbb{R}} = [\chi_1,\ldots,\chi_{j-1}]^{\perp}_{N_{\psi,\mathbb{R}}} \tag{4.22}$$

so that codim $M_{\chi,j,\mathbb{R}} = j-1$ and codim $M_{\psi,j,\mathbb{R}} = j-1$.

Relations (4.19) and (4.20) imply (4.4).

The last assertion follows from (4.21) and (4.22). □

The next theorem contains a generalized min-representation.

Theorem 4.3. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (4.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j\in J$ be defined by (3.7) resp. (3.11). Then,*

$$Re\,\lambda_j = \min_{\substack{(u,v)>>0\\ u\in N_{\chi,j,\mathbb{R}},v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)},\ j\in J. \tag{4.23}$$

The minimum is attained for $u=\chi_j$, $v=\psi_j$.

Proof. Due to (3.17),

$$Re\,(Tu,v) = \sum_{j\in J} Re\,\lambda_j(u,\psi_j)(\chi_j,v),\ u\in N_{\chi,\mathbb{R}},\, v\in N_{\psi,\mathbb{R}}$$

with

$$Re\,\lambda_j,\ (u,\psi_j),\ (\chi_j,v)\in\mathbb{R},\ u\in N_{\chi,\mathbb{R}},\ v\in N_{\psi,\mathbb{R}}.$$

Let $j \in J$ be arbitrarily chosen, but fixed as well as $u \in N_{\chi,j,\mathbb{R}} \subset N_{\chi,\mathbb{R}}$ and $v \in N_{\psi,j,\mathbb{R}} \subset N_{\psi,\mathbb{R}}$ also be arbitrarily chosen, but fixed with $(u,v) >> 0$. Then, with (4.1),

$$\begin{aligned} Re\,(Tu,v) &= \sum_{k=1}^{j} Re\,\lambda_k(u,\psi_k)(\chi_k,v) \\ &\geq \min_{k=1,\ldots,j} Re\,\lambda_k \sum_{k=1}^{j}(u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j \sum_{k=1}^{j}(u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j(u,v), \end{aligned}$$

that is,

$$\frac{Re\,(Tu,v)}{(u,v)} \geq Re\,\lambda_j,\; u \in N_{\chi,j,\mathbb{R}}, v \in N_{\psi,j,\mathbb{R}},\; (u,v) >> 0$$

and therefore,

$$\min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \geq Re\,\lambda_j.$$

Now, $Re\,\lambda_j$ is attained for $u = \chi_j \in N_{\chi,j,\mathbb{R}}$ and $v = \psi_j \in N_{\psi,j,\mathbb{R}}$, that is,

$$Re\,\lambda_j = \frac{Re\,(T\chi_j,\psi_j)}{(\chi_j,\psi_j)} \geq \min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \geq Re\,\lambda_j$$

so that (4.23) is proven. □

Next, we derive the following generalized max-min-representation of $Re\lambda_j$.

Theorem 4.4. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (4.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11).*

Then, for every $j \in J$ and every subspace $N_\chi \subset N_{\chi,\mathbb{R}}$ and $N_\psi \subset N_{\psi,\mathbb{R}}$ with $\dim N_\chi = \dim N_\psi = j$, the following inequalities are valid:

$$\min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq Re\,\lambda_j, \tag{4.24}$$

and the following max-min-representation formulas hold:

$$Re\,\lambda_j = \max_{\substack{dim N_\chi = j \\ dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)},\; j \in J. \tag{4.25}$$

The maximum is attained for

$$N_\chi = N_{\chi,j,\mathbb{R}},\; N_\psi = N_{\psi,j,\mathbb{R}}. \tag{4.26}$$

Proof. Let $j \in J$, and let $N_\chi \subset N_{\chi,\mathbb{R}}$ as well as $N_\psi \subset N_{\psi,\mathbb{R}}$ be subspaces with $dim N_\chi = dim N_\psi = j$. Then,

$$\min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq \min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \leq \frac{Re\,(T\chi_j,\psi_j)}{(\chi_j,\psi_j)} = Re\,\lambda_j \tag{4.27}$$

so that (4.24) follows. From (4.27), we conclude

$$\max_{\substack{dim N_\chi = j \\ dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq Re\,\lambda_j. \tag{4.28}$$

Further, (4.23) implies

$$Re\,\lambda_j = \min_{\substack{(u,v)>>0\\ u\in N_{\chi,j,\mathbb{R}},v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \le \max_{\substack{dim N_\chi=j\\ dim N_\psi=j}} \min_{\substack{(u,v)>>0\\ u\in N_\chi,v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\lambda_j. \tag{4.29}$$

From (4.28) and (4.29), we deduce (4.25) and that the maximum is attained for $N_\chi = N_{\chi,j,\mathbb{R}}$, $N_\psi = N_{\psi,j,\mathbb{R}}$. □

Changes in the Finite-Dimensional Case

In this case, the Hilbert space H over the field $\mathbb{F}=\mathbb{C}$ can be identified with $\mathbb{C}^n$ and the compact operator with an $n\times n$-matrix. Further, one has $J=(1,\ldots,n)$ instead of $J=(1,2,\ldots)$, and (4.1) is replaced by

$$Re\,\lambda_1 \ge \ldots \ge Re\,\lambda_n.$$

Moreover, the condition $\lim_{j\to\infty}\lambda_j = 0$ is omitted.

As a consequence, Theorems 4.1 - 4.4 deliver [16, Theorems 4 - 7] where the proofs of the theorems in this paper are essentially different from those in [16]. Beyond this, the proof of Theorem 4.2 is more detailed than the proof of [16, Theorem 5.]

5. Generalized Rayleigh-Quotient Formulas for the Imaginary Parts of the Eigenvalues

In this section, we want to state formulas for the representation of the imaginary parts of the eigenvalues of the compact operator $0\ne T\in B(H)$ by Rayleigh quotients that generalize existing ones. We remind the reader that, in this paper beginning with Section 3, all eigenvalues are assumed to be simple. We obtain max-, min-max-, min-, and max-min-representations corresponding to those in Section 4.

Similarly to (4.1) we suppose that the eigenvalues of the compact operator T are arranged such that

$$Im\,\lambda_1 \ge Im\,\lambda_2 \ge Im\,\lambda_3 \ge \ldots. \tag{5.1}$$

First, we want to state a relation corresponding to that of (3.17).

Lemma 5.1. *Let the conditions (C1) - (C4) be fulfilled. Then, with the denotations of Theorem 3.1,*

$$Im\,(Tu,v) = \sum_{j\in J} Im\,\lambda_j (u,\psi_j)(\chi_j,v),\; u\in N_{\chi,\mathbb{R}},\; v\in N_{\psi,\mathbb{R}}. \tag{5.2}$$

Proof. Equation (5.2) follows directly from Theorem 3.4, Formulas (3.14) and (3.16). □

One has a series of theorems for the imaginary parts of the eigenvalues corresponding to those of Theorems 4.1 - 4.4 in Section 4. These Theorems 5.2 - 5.5 are stated without proofs since the only difference is that (5.1) and (5.2) are used instead of (4.1) and (3.17).

Theorem 5.2. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}}$ resp. $M_{\psi,j,\mathbb{R}}$ for $j\in J$ be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$Im\,\lambda_j = \max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}},v\in M_{\psi,j,\mathbb{R}}}} \frac{Im\,(Tu,v)}{(u,v)},\; j\in J. \tag{5.3}$$

The maximum is attained for $u=\chi_j$, $v=\psi_j$.

Theorem 5.3. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1).*
Then, for every $j \in J$ and every subspace $M_\chi \subset N_{\chi,\mathbb{R}}$ and $M_\psi \subset N_{\psi,\mathbb{R}}$ with $\operatorname{codim} M_\chi = \operatorname{codim} M_\psi = j-1$, the following inequalities are valid:

$$Im\,\lambda_j \le \max_{\substack{(u,v)>>0 \\ u\in M_\chi, v\in M_\psi}} \frac{Im\,(Tu,v)}{(u,v)} \le Im\,\lambda_1, \tag{5.4}$$

and the following min-max-representation formulas hold:

$$Im\,\lambda_j = \min_{\substack{\operatorname{codim} M_\chi = j-1 \\ \operatorname{codim} M_\psi = j-1}} \max_{\substack{(u,v)>>0 \\ u\in M_\chi, v\in M_\psi}} \frac{Im\,(Tu,v)}{(u,v)},\ j \in J. \tag{5.5}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\ M_\psi = M_{\psi,j,\mathbb{R}}. \tag{5.6}$$

The next theorem contains a generalized min-representation of $Im\,\lambda_j$.

Theorem 5.4. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11). Then,*

$$Im\,\lambda_j = \min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Im\,(Tu,v)}{(u,v)},\ j \in J. \tag{5.7}$$

The minimum is attained for $u = \chi_j$, $v = \psi_j$.

Next, we derive the following generalized max-min-representation of $Im\,\lambda_j$.

Theorem 5.5. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11).*
Then, for every $j \in J$ and every subspace $N_\chi \subset N_{\chi,\mathbb{R}}$ and $N_\psi \subset N_{\psi,\mathbb{R}}$ with $\dim N_\chi = \dim N_\psi = j$, the following inequalities are valid:

$$\min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Im\,(Tu,v)}{(u,v)} \le Im\,\lambda_j, \tag{5.8}$$

and the following max-min-representation formulas hold:

$$Im\,\lambda_j = \max_{\substack{dim N_\chi = j \\ dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Im\,(Tu,v)}{(u,v)},\ j \in J. \tag{5.9}$$

The maximum is attained for

$$N_\chi = N_{\chi,j,\mathbb{R}},\ N_\psi = N_{\psi,j,\mathbb{R}}. \tag{5.10}$$

Changes in the Finite-Dimensional Case

In this case, the Hilbert space H over the field $\mathbb{F} = \mathbb{C}$ can be identified with $\mathbb{C}^n$ and the compact operator with an $n \times n$-matrix. Further, one has $J = (1,\ldots,n)$ instead of $J = (1,2,\ldots)$, and (5.1) is replaced by

$$Im\,\lambda_1 \ge \ldots \ge Im\,\lambda_n.$$

Further, the condition $\lim_{j\to\infty} \lambda_j = 0$ is omitted.
As a consequence, Theorems 5.2 - 5.5 deliver [16, Theorems 9 - 12].

6. Generalized Rayleigh-Quotient Formulas for the Moduli of the Eigenvalues

Whereas in Sections 4 and 5 max-, min-max-, min-, and max-min-representations with generalized Rayleigh quotients could be obtained, it seems that, for the moduli of the eigenvalues, only a max-representation is possible.

We now deduce this max-representation. For this, we suppose that the eigenvalues $\lambda_1, \lambda_2, \ldots$ are arranged such that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots . \tag{6.1}$$

Herewith, one has the following theorem.

Theorem 6.1. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (6.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}}$ resp. $M_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$|\lambda_j| = \max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)},\ j \in J. \tag{6.2}$$

The maximum is attained for $u = \chi_j$, $v = \psi_j$.

Proof. One uses (3.14) and (3.16) as starting point, i.e.,

$$(Tu,v) = \sum_{j\in J} \lambda_j (u,\psi_j)(\chi_j,v),\ u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}}$$

with

$$(u,\psi_j),\ (\chi_j,v) \in \mathbb{R},\ u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}}.$$

Let $j \in J$ be arbitrarily chosen, but fixed as well as $u \in M_{\chi,j,\mathbb{R}} \subset N_{\chi,\mathbb{R}}$ and $v \in M_{\psi,j,\mathbb{R}} \subset N_{\psi,\mathbb{R}}$ also be arbitrarily chosen, but fixed with $(u,v) >> 0$. Then,

$$\begin{aligned}
|(Tu,v)| &= |\sum_{k=j}^{\infty} \lambda_k (u,\psi_k)(\chi_k,v)| \\
&\leq \max_{k=j,j+1,\ldots} |\lambda_k| \sum_{k=j}^{\infty} (u,\psi_k)(\chi_k,v) \\
&= |\lambda_j| \sum_{k=1}^{\infty} (u,\psi_k)(\chi_k,v) \\
&= |\lambda_j|\,(u,v),\ u \in M_{\chi,j,\mathbb{R}}, v \in M_{\psi,j,\mathbb{R}},\ (u,v) >> 0,
\end{aligned}$$

that is,

$$\frac{|(Tu,v)|}{(u,v)} \leq |\lambda_j|,\ u \in M_{\chi,j,\mathbb{R}}, v \in M_{\psi,j,\mathbb{R}},\ (u,v) >> 0$$

and thus

$$\max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)} \leq |\lambda_j|.$$

Now, $|\lambda_j|$ is attained for $u = \chi_j \in M_{\chi,j,\mathbb{R}}$ and $v = \psi_j \in M_{\psi,j,\mathbb{R}}$. Thus, because of $(\chi_j,\psi_j) >> 0$,

$$|\lambda_j| = \frac{|(T\chi_j,\psi_j)|}{(\chi_j,\psi_j)} \leq \max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)} \leq |\lambda_j|$$

so that (6.2) is proven. □

7. Generalized Rayleigh-Quotient Formulas for Real Eigenvalues

When all eigenvalues of a compact operator T are real and simple, then

$$\sigma(T) \subset \mathbb{R}$$

and

$$Re\,\lambda_j = \lambda_j,\ j = 1, 2, \ldots.$$

We mention that, in particular, $\lambda_j(T^*T) \in \mathbb{R}_0^+ := \{x \in \mathbb{R}\,|\,x \geq 0\} \subset \mathbb{R}$. For $\sigma(T) \subset \mathbb{R}$, from Section 4 one gets the following corollaries where correspondingly to (4.1), we suppose that the eigenvalues are arranged such that

$$\lambda_1 \geq \lambda_2 \geq \ldots. \tag{7.1}$$

The corollaries are obtained as follows.

Corollary 7.1. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}}$ resp. $M_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$\lambda_j = \max_{\substack{(u,v)>>0 \\ u \in M_{\chi,j,\mathbb{R}}, v \in M_{\psi,j,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)},\ j \in J. \tag{7.2}$$

The maximum is attained for $u = \chi_j$, $v = \psi_j$.

Corollary 7.2. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1).*

Then, for every $j \in J$ and every subspace $M_\chi \subset N_{\chi,\mathbb{R}}$ and $M_\psi \subset N_{\psi,\mathbb{R}}$ with $\operatorname{codim} M_\chi = \operatorname{codim} M_\psi = j-1$, the following inequalities are valid:

$$\lambda_j \leq \max_{\substack{(u,v)>>0 \\ u \in M_\chi, v \in M_\psi}} \frac{(Tu,v)}{(u,v)} \leq \lambda_1, \tag{7.3}$$

and the following min-max-representation formulas hold:

$$\lambda_j = \min_{\substack{\operatorname{codim} M_\chi = j-1 \\ \operatorname{codim} M_\psi = j-1}} \max_{\substack{(u,v)>>0 \\ u \in M_\chi, v \in M_\psi}} \frac{(Tu,v)}{(u,v)},\ j \in J. \tag{7.4}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\ M_\psi = M_{\psi,j,\mathbb{R}}. \tag{7.5}$$

The next corollary contains a generalized min-representation of λ_j.

Corollary 7.3. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11). Then,*

$$\lambda_j = \min_{\substack{(u,v)>>0 \\ u \in N_{\chi,j,\mathbb{R}}, v \in N_{\psi,j,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)},\ j \in J. \tag{7.6}$$

The minimum is attained for $u = \chi_j$, $v = \psi_j$.

Corollary 7.4. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11).*

Then, for every $j \in J$ and every subspace $N_\chi \subset N_{\chi,\mathbb{R}}$ and $N_\psi \subset N_{\psi,\mathbb{R}}$ with $\dim N_\chi = \dim N_\psi = j$, the following inequalities are valid:

$$\min_{\substack{(u,v)>>0\\ u\in N_\chi, v\in N_\psi}} \frac{(Tu,v)}{(u,v)} \le \lambda_j, \tag{7.7}$$

and the following max-min-representation formulas hold:

$$\lambda_j = \max_{\substack{dim N_\chi = j\\ dim N_\psi = j}} \min_{\substack{(u,v)>>0\\ u\in N_\chi, v\in N_\psi}} \frac{(Tu,v)}{(u,v)}, \ j \in J. \tag{7.8}$$

The maximum is attained for $N_\chi = N_{\chi,j,\mathbb{R}}$, $N_\psi = N_{\psi,j,\mathbb{R}}$.

Changes in the Finite-Dimensional Case

In this case, the Hilbert space H over the field $\mathbb{F} = \mathbb{C}$ can be identified with $\mathbb{C}^n$ and the compact operator with an $n \times n$-matrix. Further, one has $J = (1,\ldots,n)$ instead of $J = (1,2,\ldots,$ and (7.1) is replaced by

$$\lambda_1 \ge \ldots \ge \lambda_n.$$

Further, the condition $\lim_{j\to\infty} \lambda_j = 0$ is omitted.

As a consequence, Corollaries 7.1 - 7.4 deliver [16, Corollaries 14 - 17].

8. Application to New Formula for Spectral Radius

In this section, an application of the obtained results is presented. More precisely, a new formula for the spectral radius $\rho(T)$ is derived. First, known formulas for this quantity are recapitulated.

Known formulas for the spectral radius $\rho(T)$

Let the conditions (C1) - (C4) be fulfilled. One formula is given by

$$\rho(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}, \tag{8.1}$$

see [10, Chapter I, p. 27], where the expression on the right-hand member of (8.1) is independent of the norm $\|\cdot\|$.

If $\mathbb{F} = \mathbb{C}$, another representation is

$$\rho(T) = \max_{j=1,2,\ldots} |\lambda_j|, \tag{8.2}$$

cf. [10, Chapter I, (5.10), p. 38].

New formula for the spectral radius $\rho(T)$

Let the conditions (C1) - (C4) be fulfilled, and let the eigenvalues of T be arranged according to (6.1).

Then, from Theorem 6.1, as Application, we deduce the new formula

$$\rho(T) = \max_{\substack{(u,v)>>0\\ u\in N_{\chi,\mathbb{R}}, v\in N_{\psi,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)}. \tag{8.3}$$

Proof. This follows from (6.2) as well as $M_{\chi,1,\mathbb{R}} = N_{\chi,\mathbb{R}}$, $M_{\psi,1,\mathbb{R}} = N_{\psi,\mathbb{R}}$ according to (3.21) and (3.23) as well as (3.9) and (3.13) since

$$\rho(T) = \max_{j=1,2,\ldots} |\lambda_j| = |\lambda_1| = \max_{\substack{(u,v)>>0\\ u\in M_{\chi,1,\mathbb{R}}, v\in M_{\psi,1,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)} = \max_{\substack{(u,v)>>0\\ u\in N_{\chi,\mathbb{R}}, v\in N_{\psi,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)}. \tag{8.4}$$

□

9. New Generalized Numerical Ranges

In this section, a series of known numerical ranges is recapitulated, and new numerical ranges of a compact operator are defined. The new generalized numerical ranges are defined for compact operators with simple eigenvalues similarly as for diagonalizable matrices in [16].

Known numerical range of $T \in B(H)$ with respect to the Hilbert space H
Following [25, Section 5.4,(5)], the *numerical range* of $T \in B(H)$ is defined by

$$W_H(T) = \{z \in \mathbb{C} \,|\, z = \frac{(Tu,u)}{(u,u)},\ 0 \neq u \in H\} \tag{9.1}$$

which is a convex subset of $\mathbb{C}$. Applying this definition to T^*T instead to T, we obtain

$$W_H(T^*T) = \{x \in \mathbb{R}_0^+ \,|\, x = \frac{(T^*Tu,u)}{(u,u)} = \frac{(Tu,Tu)}{(u,u)} \geq 0,\ 0 \neq u \in H\} \tag{9.2}$$

which is a convex subset of $\mathbb{R}_0^+$. One has

$$W_H(T^*T) = [\inf_{j=1,2,\ldots} \lambda_j(T^*T),\ \sup_{j=1,2,\ldots} \lambda_j(T^*T)] = [\frac{1}{\|T^{-1}\|_2^2}, \|T\|_2^2] \tag{9.3}$$

where $1/\|T^{-1}\|_2^2$ has to be interpreted as zero if T^{-1} does not exist.

Generalized numerical range of $T \in B(H)$ with respect to the subspaces N_χ and N_ψ
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *generalized range* of T with respect to the subspaces N_χ and N_ψ by

$$W_{N_\chi,N_\psi,gen.}(T) = \{z \in \mathbb{C} \,|\, z = \frac{(Tu,v)}{(u,v)},\ (u,v) >> 0,\ u \in N_\chi, v \in N_\psi\} \tag{9.4}$$

Real part of the numerical range of $T \in B(H)$ with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *real part of the generalized numerical range* of T with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$ by

$$Re\,[W_{N_{\chi,\mathbb{R}},N_{\psi,\mathbb{R}},gen.}](T) = \{x \in \mathbb{R} \,|\, x = \frac{Re\,(Tu,v)}{(u,v)},\ (u,v) >> 0,\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}\}. \tag{9.5}$$

Imaginary part of the numerical range of $T \in B(H)$ with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *imaginary part of the generalized numerical range* of T with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$ by

$$Im\,[W_{N_{\chi,\mathbb{R}},N_{\psi,\mathbb{R}},gen.}](T) = \{y \in \mathbb{R} \,|\, y = \frac{Im\,(Tu,v)}{(u,v)},\ (u,v) >> 0,\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}\}. \tag{9.6}$$

Modulus of the generalized numerical range of $T \in B(H)$ with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *modulus of the generalized numerical range* of T with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$ by

$$|\,W_{N_{\chi,\mathbb{R}},N_{\psi,\mathbb{R}},gen.}(T)\,| = \{x \in \mathbb{R}_0^+ \,|\, x = \frac{|(Tu,v)|}{(u,v)},\ (u,v) >> 0,\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}\}. \tag{9.7}$$

10. Examples from the Area of Boundary Eigenvalue Problems

In this section, we check some of the formulas of Section 7 on an example of a nonsymmetric compact operator with nonnegative simple eigenvalues from the area of Mathematical Physics. More precisely, we check the validity of the following relation

$$\frac{(Tu,v)}{(u,v)} \in [\inf_{j=1,2,\ldots} \lambda_j(T),\ \sup_{j=1,2,\ldots} \lambda_j(T)]$$

for a series of vectors $u \in N_{\chi,\mathbb{R}}$, $v \in N_{\psi,\mathbb{R}}$ with $(u,v) >> 0$, which is a consequence of Theorems 7.2 and 7.4.

10.1 A Non-Selfadjoint BEVP with Ordinary Differential Operator of 2nd Order

(i) *The Differential Operators* L *and* L_+ *and Pertinent BEVPs*

As an example, we choose the non-selfadjoint Boundary Eigenvalue Problem (for short: BEVP) with ordinary differential operator of 2nd order in [18]. The differential operator L is given by

$$(Lu)(x) = -u''(x) + p_0\, u'(x) + q_0\, u(x),\; 0 \le x \le l \tag{10.1}$$

with the real constants p_0, q_0 where we restrict q_0 to $q_0 > 0$ and with the boundary conditions

$$u(0) = u(l) = 0. \tag{10.2}$$

The formally adjoint differential operator L_+ is given by

$$(L_+ v)(x) = -v''(x) - p_0\, v'(x) + q_0\, v(x),\; 0 \le x \le l \tag{10.3}$$

with the boundary conditions

$$v(0) = v(l) = 0. \tag{10.4}$$

The pertinent BEVPs read

$$\pi_{2,\mu} :\; Lu = \mu u,\; u \in H_D = D(L) \tag{10.5}$$

where

$$H_D = \{u \in C^2[0,l] \,|\, u(0) = u(l) = 0\} \tag{10.6}$$

and

$$\pi_{2,\overline{\mu},+} :\; L_+ v = \overline{\mu} v,\; v \in H_{D,+} = D(L_+). \tag{10.7}$$

where

$$H_{D,+} = H_D. \tag{10.8}$$

(ii) The Eigenvalues and Eigenfunctions

The eigenvalues of L and L_+ are given by

$$\mu = \overline{\mu} = \mu_j = \overline{\mu}_j = \frac{j^2\pi^2}{l^2} + D,\; j \in J \tag{10.9}$$

with the quantity

$$D = D(p_0, q_0) = (\frac{p_0}{2})^2 + q_0 \tag{10.10}$$

so that

$$\lambda_j = \frac{1}{\mu_j} = \frac{1}{\dfrac{j^2\pi^2}{l^2} + D},\; j \in J. \tag{10.11}$$

The biorthonormal eigenfunctions are found to be

$$\chi_j(x) = \sqrt{\frac{2}{l}}\, exp(\frac{p_0}{2} x) \sin j\pi \frac{x}{l},\; 0 \le x \le l,\; j \in J \tag{10.12}$$

and

$$\psi_j(x) = \sqrt{\frac{2}{l}}\, exp(-\frac{p_0}{2} x) \sin j\pi \frac{x}{l},\; 0 \le x \le l,\; j \in J \tag{10.13}$$

so that we have

$$(\chi_j, \psi_k) = \int_0^l \chi_j(x)\, \psi_k(x)\, dx = \frac{2}{l} \int_0^l \sin j\,\pi \frac{x}{l} \sin k\,\pi \frac{x}{l}\, dx = \delta_{jk},\ 0 \le x \le l,\ j,k \in J. \tag{10.14}$$

(iii) The Green's Function of $L_{p_0,q_0}\,u = 0,\ u(0) = u(l) = 0$
A set of fundamental solutions of $L_{p_0 \cdot q_0} = 0$, i.e., when $\mu = 0$, is given by

$$u_1(x) = e^{\frac{p_0}{2}x} \sinh \sqrt{D}x \tag{10.15}$$

$$u_2(x) = e^{\frac{p_0}{2}x} \cosh \sqrt{D}x \tag{10.16}$$

with

$$D = D(p_0, q_0) = (\frac{p_0}{2})^2 + q_0$$

in (10.10). Based on these fundamental solutions, the Green's functions pertinent to the BVPs $L_{p_0 \cdot q_0}\,u = 0,\ u(0) = u(l) = 0$ resp. $L_{+,p_0 \cdot q_0}\,v = 0,\ v(0) = v(l) = 0$ are given by

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{\sinh\sqrt{D}x\ \sinh\sqrt{D}(l-s)}{\sqrt{D}\sinh\sqrt{D}\,l} \exp\left(\dfrac{p_0}{2}(x-s)\right),\ 0 \le x \le s \le l, \\ \\ G_2(x,s) = \dfrac{\sinh\sqrt{D}(l-x)\ \sinh\sqrt{D}s}{\sqrt{D}\sinh\sqrt{D}\,l} \exp\left(\dfrac{p_0}{2}(x-s)\right)\ 0 \le s \le x \le l, \end{cases} \tag{10.17}$$

resp.

$$G_+(x,s) = \begin{cases} G_{+,1}(x,s) = \dfrac{\sinh\sqrt{D}(l-x)\ \sinh\sqrt{D}s}{\sqrt{D}\sinh\sqrt{D}\,l} \exp\left(\dfrac{p_0}{2}(s-x)\right),\ 0 \le x \le s \le l, \\ \\ G_{+,2}(x,s) = \dfrac{\sinh\sqrt{D}x\ \sinh\sqrt{D}(l-s)}{\sqrt{D}\sinh\sqrt{D}\,l} \exp\left(\dfrac{p_0}{2}(s-x)\right)\ 0 \le s \le x \le l, \end{cases} \tag{10.18}$$

so that, because of $D = D(p_0, q_0)$,

$$G(x,s) = G(x,s;p_0,q_0) \tag{10.19}$$

and

$$G_+(x,s) = G^T(x,s) = G(s,x) = G(s,x;-p_0,q_0) \tag{10.20}$$

in accordance with the fact that, for the pertinent operators, one has $G_+ = G^T$, see [18].
(iv) The Compact Operators T and $T_+ = T^* = T^T$
The inverse operators $T := G := L_+^{-1}$ and $T_+ := G_+ := L_+^{-1}$ are given by

$$(Tu)(x) = (Gu)(x) = (L^{-1}u)(x) = \int_0^l G(x,s;p_0,q_0)\, u(s)\, ds,\ u \in C([0,l],\mathbb{R}) \subset C[0,l] \tag{10.21}$$

where $C([0,l],\mathbb{R})$ is the set of real-valued continuous functions on $[0,l]$ endowed with the norm $\|\cdot\|_2$, and

$$(T_+u)(x) = (G_+u)(x) = (L_+^{-1}u)(x) = \int_0^l G^T(x,s;-p_0,q_0)\, u(s)\, ds,\ u \in C([0,l],\mathbb{R}) \tag{10.22}$$

with the eigenvalues

$$\lambda_j(T) = \lambda_j(G) = \lambda_j(T^T) = \lambda_j(G^T) = \frac{1}{\mu_j(L)} = \frac{1}{\dfrac{j^2\pi^2}{l^2} + D},\ j \in J, \tag{10.23}$$

and the same eigenfunctions χ_j in (10.12) resp. ψ_j in (10.13). From (10.23), we have

$$\lim_{j\to\infty} \lambda_j(T) = 0. \tag{10.24}$$

Further,

$$\inf_{j=1,2,\ldots} \lambda_j(T) = 0, \qquad \sup_{j=1,2,\ldots} \lambda_j(T) = \lambda_1(T) = \frac{1}{\dfrac{\pi^2}{l^2} + D} = \frac{1}{\dfrac{\pi^2}{l^2} + (\dfrac{p_0}{2})^2 + q_0}. \tag{10.25}$$

Now, due to [18, Theorem 3.3, (3.14)] and since $\chi_j(x) \in \mathbb{R}$, $0 \le x \le l$, one has

$$C([0,l],\mathbb{R}) \subset N_{\chi,\mathbb{R}} \subset L_2(0,l).$$

Therefore, from (7.1) and (7.3), we obtain

$$0 \le \frac{(Tu,v)}{(u,v)} \le \frac{1}{\dfrac{\pi^2}{l^2} + (\dfrac{p_0}{2})^2 + q_0},\ (u,v) >> 0,\ u,v \in C([0,l],\mathbb{R}). \tag{10.26}$$

(v) <u>Special case $p_0 = q_0 = 0$</u>
We mention that, in the particular case $p_0 = q_0 = 0$, we obtain

$$\mu_j = \frac{j^2\pi^2}{l^2},\ j \in J,$$

$$\lambda_j = \frac{l^2}{j^2\pi^2},\ j \in J,$$

$$\chi_j(x) = \psi_j(x) = \varphi_j(x) = \sqrt{\frac{2}{l}}\,\sin j\pi\,\frac{x}{l},\ 0 \le x \le l,\ j \in J,$$

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{x\,(l-s)}{l},\ 0 \le x \le s \le l, \\[2ex] G_2(x,s) = \dfrac{s\,(l-x)}{l},\ 0 \le s \le x \le l. \end{cases}$$

In this special case, we have

$$\frac{(Tu,v)}{(u,v)} \in [0; l^2/\pi^2],\ (u,v) >> 0,\ u,v \in C([0,l];\mathbb{R})). \tag{10.27}$$

10.2 Computations with Computer Algebra

In the particular case $p_0 = q_0 = 0$, using the symbolic-function feature of Matlab, one obtains the following Table 10.1.

i_u	i_v	u	v	$(Tu,v))$	(u,v)	$(Tu,v))/(u,v)$
1	1	1	1	$\frac{l^3}{12}$	l	$\frac{1}{12}\,l^2$
2	1	x	1	$\frac{l^4}{24}$	$\frac{l^2}{2}$	$\frac{1}{12}\,l^2$
1	2	1	x	$\frac{l^4}{24}$	$\frac{l^2}{2}$	$\frac{1}{12}\,l^2$
3	1	x^2	1	$\frac{l^5}{40}$	$\frac{l^3}{3}$	$\frac{3}{40}\,l^2$
2	2	x	x	$\frac{l^5}{45}$	$\frac{l^3}{3}$	$\frac{1}{15}\,l^2$
1	3	1	x^2	$\frac{l^5}{40}$	$\frac{l^3}{3}$	$\frac{3}{40}\,l^2$
4	1	x^3	1	$\frac{l^6}{60}$	$\frac{l^4}{4}$	$\frac{1}{15}\,l^2$
3	2	x^2	x	$\frac{l^6}{72}$	$\frac{l^4}{4}$	$\frac{1}{18}\,l^2$
2	3	x	x^2	$\frac{l^6}{72}$	$\frac{l^4}{4}$	$\frac{1}{18}\,l^2$
1	4	1	x^3	$\frac{l^6}{60}$	$\frac{l^4}{4}$	$\frac{1}{15}\,l^2$
5	1	x^4	1	$\frac{l^7}{84}$	$\frac{l^5}{5}$	$\frac{5}{84}\,l^2$
4	2	x^3	x	$\frac{l^7}{105}$	$\frac{l^5}{5}$	$\frac{1}{21}\,l^2$
3	3	x^2	x^2	$\frac{l^7}{112}$	$\frac{l^5}{5}$	$\frac{5}{112}\,l^2$
2	4	x	x^3	$\frac{l^7}{105}$	$\frac{l^5}{5}$	$\frac{1}{21}\,l^2$
1	5	1	x^4	$\frac{l^7}{84}$	$\frac{l^5}{5}$	$\frac{5}{84}\,l^2$
6	1	$1+x$	1	$\frac{(l+2)\,l^3}{24}$	$\frac{l\,(l+2)}{2}$	$\frac{1}{12}\,l^2$
1	6	1	$1+x$	$\frac{(l+2)\,l^3}{24}$	$\frac{l\,(l+2)}{2}$	$\frac{1}{12}\,l^2$
6	6	$1+x$	$1+x$	$\frac{(4l^2+15l+15)\,l^3}{180}$	$l+l^2\,(\frac{l}{3}+1)$	$f(l)=g(l)\,l^2$

Table 10.1: Computer-Algebra Results

with

$$f(l)=\frac{(4l^2+15l+15)\,l^3}{180\,(l+l^2\,(\frac{l}{3}+1))}=\frac{4l^5+15l^4+15l^3}{60l^3+180l^2+180l}=l^2\,\frac{4l^2+15l+15}{60l^2+180l+180}=g(l)\,l^2$$

where

$$g(l)=\frac{4l^2+15l+15}{60l^2+180l+180}.$$

The function $y=g(x)$ for $0\leq x\leq 10$ is illustrated in *Fig. 10.1*.

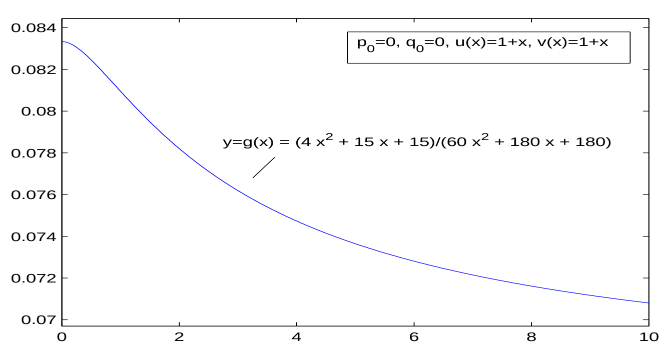

Fig. 10.1: Curve y=g(x) for $0\leq x\leq 10$

One has

$$\lim_{l\to 0}g(l)=\frac{15}{180}=\frac{1}{12}\in[0;\frac{1}{\pi^2}]$$

and

$$\lim_{l\to\infty} g(l) = \lim_{l\to\infty} \frac{4+\frac{15}{l}+\frac{15}{l^2}}{60+\frac{180}{l}+\frac{180}{l^2}} = \frac{4}{60} = \frac{1}{15} \in [0;\frac{1}{\pi^2}]$$

as well as

$$g'(x) = -\frac{x(x+2)}{20(x^2+3x+3)} < 0,\ x>0$$

so that $y=g(x), x>0$ is strictly monotonically decreasing. In *Fig. 10.2*, the curve $y=g(x)$ for $1 \le x \le 3$ is shown.

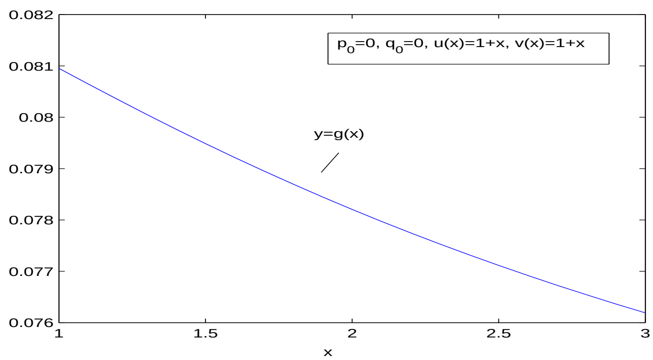

Fig. 10.2: Curve y=g(x) for $1 \le x \le 3$

At this point, we introduce the denotation of *reduced length*. Apparently,

$$\frac{(Tu,v)}{(u,v)} \in [0;\frac{1}{\pi^2}l^2]$$

for all values in *Table 10.1* which confirms (10.27) for $p_0 = q_0 = 0$, and for the largest eigenvalue of T, one has

$$\lambda_1(T) = \max_{\substack{(u,v)>>0\\ u\in N_{\chi,\mathbb{R}},v\in N_{\psi,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)} = \max_{\substack{(u,v)>>0\\ u,v\in C([0,l;\mathbb{R}])}} \frac{(Tu,v)}{(u,v)} = \frac{(T\chi_1,\psi_1)}{(\chi_1,\psi_1)} = \frac{l^2}{\pi^2}.$$

Correspondingly to this formula, for $u,v\in C([0,l],\mathbb{R})$ with $(u,v)>>0$, we define the *reduced length* $l_{red,D=0}$ by

$$Q_{Ray} := \frac{(Tu,v)}{(u,v)} = \frac{l^2_{red,D=0}}{\pi^2}$$

implying

$$l^2_{red,D=0} = Q_{Ray}\,\pi^2.$$

For $u=\chi_1$, $v=\psi_1$, we get back

$$l^2_{red,D=0} = \frac{l^2}{\pi^2}\pi^2 = l^2$$

or

$$l_{red,D=0} = l,$$

as it must be. For $i_u = 5$, $i_v = 1$, i.e., for $u(x) = x^4,\ v(x) = 1$, *Table 10.1* delivers $Q_{Ray} = 5/84\,l^2$ and therefore

$$l_{red,D=0} = \pi\sqrt{\frac{5}{84}}\,l \doteq 0.766470\,l < l$$

and for $i_u = 6$, $i_v = 6$, i.e., for $u(x) = 1+x,\ v(x) = 1+x$, *Table 10.1* gives $Q_{Ray} \in [1/15\,l^2, 1/12\,l^2]$ so that

$$l_{red,D=0} \in \pi[\frac{1}{\sqrt{15}}\,l; \frac{1}{\sqrt{12}}\,l] \doteq [0.2581988\,l; 0.288675\,l] \subset [0,l]$$

The interpretation of $l_{red,D=0}$ is as follows. If the length l is replaced by $l_{red,D=0}$ for the index pair (i_u, i_v) resp. the pair of functions $u, v \in C([0,l], R)$ with $(u,v) >> 0$, then $\lambda_1(T) = \max\limits_{\substack{(u,v)>>0 \\ u,v\in C([0,l];R)}} \frac{(Tu,v)}{(u,v)}$ is attained for the pair of functions pertinent to the pair of indices (i_u, i_v) in *Table 10.1*.

10.3 Numerical Computations

If $p_0 \neq 0$ or $q_0 \neq 0$, then the results obtained by the Computer Algebra using the symbolic-function feature of Matlab get complicated. So, in this subsection, we use numerical integration methods to compute the Rayleigh quotients $(Tu,v)/(u,v)$. For the computation of

$$(Tu)(x) = \int_0^l G(x,s)\,u(s)\,ds = \int_0^x G_2(x,s)\,u(s)\,ds + \int_x^l G_1(x,s)\,u(s)\,ds,$$

we employ the Matlab routine *dblquad*, and for $(Tu,v) = \int_0^l (Tu)(x)\,v(x)\,dx$ as well as $(u,v) = \int_0^l u(x)\,v(x)\,dx$ the Matlab routine *quadl* that is based on the Simpson rule.

As to the *reduced length* $l_{red,D}$ for the general case when $D = (\frac{p_0}{2})^2 + q_0$ is not necessary equal to zero, we depart from the formula

$$\lambda_1(T) = \max_{\substack{(u,v)>>0 \\ u\in N_{\chi,\mathbb{R}},\, v\in N_{\psi,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)} = \frac{1}{\frac{\pi^2}{l^2} + D} = \frac{l^2}{\pi^2 + D\,l^2}$$

since the maximum is attained for $u = \chi_1 \in C([0,l];\mathbb{R}) \subset N_{\chi,\mathbb{R}}$ and $v = \psi_1 \in C([0,l];\mathbb{R}) \subset N_{\chi,\mathbb{R}}$. In analogy to this formula, we define

$$Q_{Ray} = \frac{(Tu,v)}{(u,v)} := \frac{l^2_{red,D}}{\pi^2 + D\,l^2_{red,D}}$$

leading to

$$l^2_{red,D} = Q_{Ray}\,(\pi^2 + D\,l^2_{red,D}).$$

This implies

$$l^2_{red,D}\,(1 - D\,Q_{Ray}) = \pi^2\,Q_{Ray}$$

or

$$l^2_{red,D} = \pi^2\,\frac{Q_{Ray}}{1 - D\,Q_{Ray}}$$

leading to

$$l_{red,D} = \pi\,\frac{\sqrt{Q_{Ray}}}{\sqrt{1 - D\,Q_{Ray}}} = \pi\,\frac{\sqrt{\frac{(Tu,v)}{(u,v)}}}{\sqrt{1 - D\,\frac{(Tu,v)}{(u,v)}}}.$$

Special Case: $\boxed{u = \chi_1,\ v = \psi_1}$

In this case, we obtain

$$Q_{Ray} = \frac{(T\chi_1, \psi_1)}{(\chi_1, \psi_1)} = \lambda_1 = \frac{l^2}{\pi^2 + D\,l^2} = \frac{l^2_{red,D}}{\pi^2 + D\,l^2_{red,D}}$$

implying

$$l_{red,D} = l,$$

as it must be. In order to test the numerical computations, we begin with the special case $\boxed{p_0 = q_0 = 0}$. The pertinent computations for $y = (Tu,v)/(u,v)\,\frac{1}{l^2}$ with $u(x) = 1+x,\, v(x) = 1+x,\ 1 \le x \le 3$ deliver the same numerical values as for $y = g(l),\ 1 \le l \le 3$ in *Table 10.2*. This is illustrated in *Fig. 10.3*.

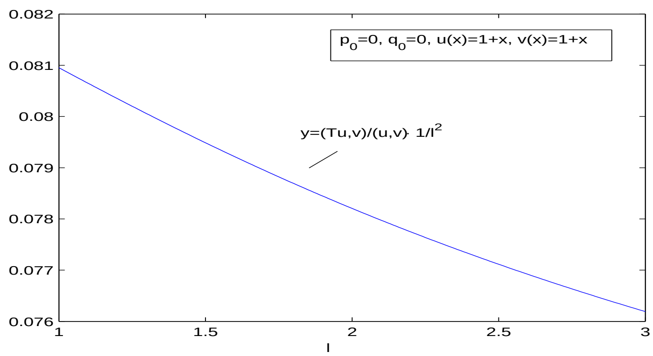

Fig. 10.3: Curve $y = ((Tu,v)/(u,v))/l^2$ for $0 \le l \le 3$

From this, one can expect that the numerical computations for the other pairs u, v of functions are reliable.

For $\boxed{p_0 = 0,\, q_0 = 1}$ and $\boxed{u(x) = 1,\ v(x) = 1}$, we have computed a series of variants for $l = 1.0(0.1)3.0$ given in *Table 10.2*.

k	l	$\frac{(Tu,v)}{(u,v))}$	$\frac{(Tu,v)}{(u,v))}\ \frac{1}{l^2}$	$\frac{1}{\pi^2+Dl^2}$	$l_{red,D}$	$\frac{l_{red,D}}{l}$
1	1.000000	0.083334	0.08333422	0.052995	0.947231	0.947231
2	1.100000	0.100834	0.08333422	0.052995	1.052045	0.956405
3	1.200000	0.120001	0.08333422	0.052995	1.160117	0.966764
4	1.300000	0.140834	0.08333387	0.052995	1.271937	0.978413
5	1.400000	0.163334	0.08333358	0.052995	1.388071	0.991480
6	1.500000	0.187501	0.08333358	0.052995	1.509175	1.006117
7	1.600000	0.213334	0.08333350	0.052995	1.636004	1.022503
8	1.700000	0.240834	0.08333350	0.052995	1.769458	1.040857
9	1.800000	0.270000	0.08333346	0.052995	1.910604	1.061447
10	1.900000	0.300834	0.08333346	0.052995	2.060739	1.084599
11	2.000000	0.333334	0.08333346	0.052995	2.221444	1.110722
12	2.100000	0.367501	0.08333346	0.052995	2.394687	1.140327
13	2.200000	0.403334	0.08333346	0.052995	2.582953	1.174070
14	2.300000	0.440834	0.08333346	0.052995	2.789440	1.212800
15	2.400000	0.480001	0.08333346	0.052995	3.018349	1.257645
16	2.500000	0.520834	0.08333341	0.052995	3.275340	1.310136
17	2.600000	0.563334	0.08333341	0.052995	3.568273	1.372413
18	2.700000	0.607501	0.08333341	0.052995	3.908442	1.447571
19	2.800000	0.653334	0.08333341	0.052995	4.312825	1.540295
20	2.900000	0.700834	0.08333341	0.052995	4.808408	1.658072
21	3.000000	0.750001	0.08333341	0.052995	5.441409	1.813803

Table 10.2: Computational Results for $p_0 = 0,\, q_0 = 1,\ u(x) = 1,\, v(x) = 1$

For $\boxed{p_0 = 0,\, q_0 = 1}$ and $\boxed{u(x) = 1+x,\ v(x) = 1+x}$, we have computed a series of variants for $l = 1.0(0.1)3.0$ given in

Table 10.3. This is illustrated in *Fig. 10.4*.

k	l	$\frac{(Tu,v)}{(u,v))}$	$\frac{(Tu,v)}{(u,v))}\ \frac{1}{l^2}$	$\frac{1}{\pi^2+Dl^2}$	$l_{red,D}$	$\frac{l_{red,D}}{l}$
1	1.000000	0.080953	0.08095276	0.052995	0.932388	0.932388
2	1.100000	0.097584	0.08064818	0.052995	1.033086	0.939169
3	1.200000	0.115702	0.08034841	0.052995	1.136372	0.946976
4	1.300000	0.135292	0.08005448	0.052995	1.242657	0.955890
5	1.400000	0.156344	0.07976723	0.052995	1.352407	0.966005
6	1.500000	0.178846	0.07948727	0.052995	1.466148	0.977432
7	1.600000	0.202790	0.07921501	0.052995	1.584482	0.990301
8	1.700000	0.228167	0.07895064	0.052995	1.708107	1.004769
9	1.800000	0.254969	0.07869424	0.052995	1.837836	1.021020
10	1.900000	0.283189	0.07844575	0.052995	1.974631	1.039279
11	2.000000	0.312821	0.07820516	0.052995	2.119642	1.059821
12	2.100000	0.343858	0.07797231	0.052995	2.274262	1.082982
13	2.200000	0.376296	0.07774703	0.052995	2.440198	1.109181
14	2.300000	0.410129	0.07752910	0.052995	2.619579	1.138948
15	2.400000	0.445354	0.07731832	0.052995	2.815102	1.172959
16	2.500000	0.481965	0.07711445	0.052995	3.030249	1.212100
17	2.600000	0.519961	0.07691725	0.052995	3.269615	1.257544
18	2.700000	0.559336	0.07672649	0.052995	3.539424	1.310898
19	2.800000	0.600089	0.07654194	0.052995	3.848361	1.374415
20	2.900000	0.642216	0.07636335	0.052995	4.209008	1.451382
21	3.000000	0.685714	0.07619049	0.052995	4.640441	1.546814

Table 10.3: Computational Results for $p_0 = 0$, $q_0 = 1$, $u(x) = 1+x$, $v(x) = 1+x$

m

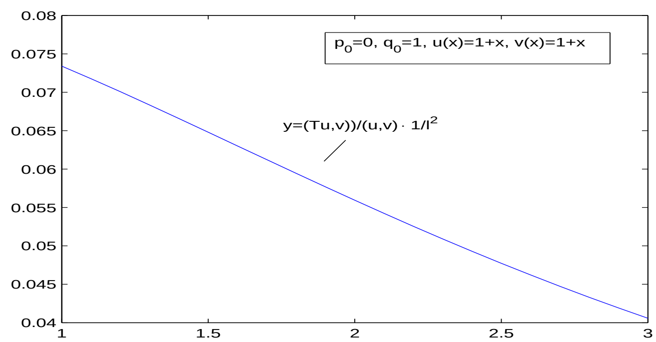

Fig. 10.4: Curve $y = ((Tu,v)/(u,v))/l^2$ for $0 \le l \le 3$

10.4 Computational Aspects

In this subsection, we say something about the used computer equipment, the computational times, and the Matlab numerical integration programs *quadl* and *dblquad*.

(i) As to the *computer equipment*, the following hardware was available: an Intel Core 2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software, for the computations, we used Matlab Version 7.11.

(ii) The *computation time* t of an operation was determined by the command sequence $t_1 = clock; operation; t = etime(clock, t_1)$. It is put out in seconds rounded to four decimal places. For the computation of the values in *Table 10.2*, the computation time was $t = 2.3400\,s$.

(iii) The double integrals $I_1 := (Tu,v)_1 := \int_0^l \int_x^l G_1(x,s)\,u(s)\,v(x)\,ds\,dx$ and $I_2 := (Tu,v)_2 := \int_0^l \int_0^x G_2(x,s)\,u(s)\,v(x)\,ds\,dx$ are computed by the Matlab commands

$$I_1 = dblquad(@(x,s)\,G1uv(x,s).*(x <= s), 0, l, 0, l, [], @quadl);$$

and

$$I_2 = dblquad(@(x,s)\,G2uv(x,s).*(s <= x), 0, l, 0, l, [], @quadl);$$

where

$$y = G1uv(x,s) = G1(x,s)*u(s)*v(x)$$

and

$$y = G2uv(x,s) = G2(x,s) * u(s) * v(x)$$

are defined in corresponding m-files. The quantity (Tu, v) is obtained as the sum of I_1 and I_2. The default absolut tolerance for *quadl* is $tol = 1.0\,e-6$.

The scalar product (u, v) is computed by the Matlab command

$$uv = quadl(@(t), fuv(t), 0, l);$$

where

$$y = fuv(t) = u(t).*v(t);$$

is defined in an associated m-file. Here, it is of interest to note that this command worked correctly for all function pairs u, v in *Table 10.2* except for the function pair $u(x) = 1$, $v(x) = 1$. It does neither work if one replaces $u(x) = 1$, $v(x) = 1$ by $u(x) = x^0$, $v(x) = x^0$, but it works correctly if one choose as replacements $u(x) = x+1-x$, $v(x) = x+1-x$. This is, of course, a shortcoming of the program and should be remedied by the company Mathworks.

10.5 Examples of Buckling Problems in Elastomechanics

In this subsection, we use some verbatim passages from [28].

(i) The Euler Column

As a simple example of a problem from Eleastomechanics, we choose the buckling of a slender elastic bar of length l with hinged ends, also called Euler column, see [28, Section 2.1, pp. 46-49] and [23, Section 7.2, pp. 218-226]. We assume that the bar with constant cross-section is compressed by a centrically applied force F. We further assume that the unloaded bar is exactly straight. When the critical force F_{crit} is applied, besides the undeformed shape, there exists a neighbouring shape with lateral deflection $w \neq 0$, see *Fig. 10.5*.

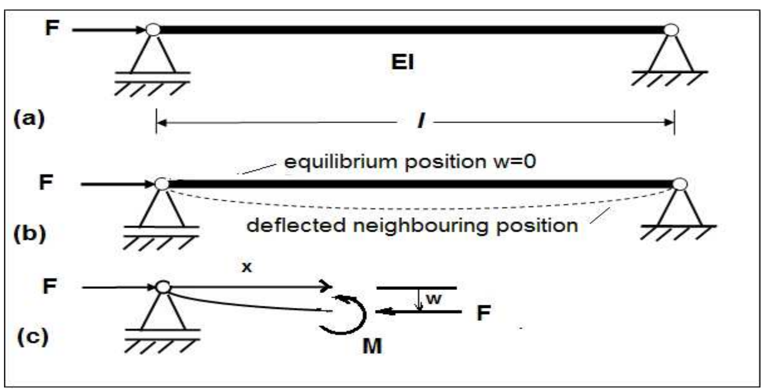

Fig. 10.5: Euler Buckling Column

In order to determine F_{crit}, it is necessary to set up the equilibrium conditions for the deflected shape, i.e., for the deformed bar. (Hereby, the change of the length can be neglected.) If one cuts the bar at the place x (*Fig.10.5 (c)*), then from the equilibrium of the bending moment about the left end taken counterclockwise for the deformed bar, one obtains

$$\overset{\curvearrowleft}{0}: \quad M - F\,w = 0. \tag{10.28}$$

Here, we have taken into account that, under horizontal force, there is no vertical bearing reaction. Substituting this in the law of elasticity $-EI\,w'' = M$ for the shearless bending bar leads to

$$-EI\,w'' = F\,w. \tag{10.29}$$

With the abbreviation

$$\mu = \frac{F}{EI}, \tag{10.30}$$

the *buckling equation* reads

$$-w'' = \mu\, w. \tag{10.31}$$

The boundary conditions for the hinges at the ends have the form

$$w(0) = w(l) = 0. \tag{10.32}$$

The BEVP consisting of (10.31) and (10.32) has the eigenvalues

$$\mu = \mu_j = j^2\, \frac{\pi^2}{l^2},\; j = 1,2,\ldots \tag{10.33}$$

and the eigenfunctions read

$$\chi_j(x) = \psi_j(x) = \varphi_j(x) = \sqrt{\frac{2}{l}}\,\sin j\,\pi\,\frac{x}{l},\; j = 1,2,\ldots. \tag{10.34}$$

As already found in Subsection 10.1, the Green's function $G(x,s) = G(x,s,p_0 \to 0, q_0 \to 0)$ turn into

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{x\,(l-s)}{l},\; 0 \le x \le s \le l, \\[2ex] G_2(x,s) = \dfrac{s\,(l-x)}{l},\; 0 \le s \le x \le l, \end{cases}$$

so that the largest eigenvalue is

$$\lambda_1 = \frac{l^2}{\pi^2}.$$

In [23, p.223, Fig. 7/5], the critical forces for other boundary conditions such as clamped end - hinged end can be found.

(ii) Some References to Other Problems of Mathematical Physics and Engineering

Many examples for Eigenvalue Problems that can be treated by the methods of the paper may be found in the classical books [4],[5], [20], and [28].

In [4, Chapter I, pp. 5-39], one finds examples from the area of Engineering Mechanics. Further, there is a list of examples at the end of this book, cf. pages 406-456.

In [5, Chapter V, pp. 234-343], one finds vibratory and eigenvalue problems of Mathematical Physics.

The book [20, Chapter V, pp. 168-221] contains eigenvalue problems with many examples from Elastomechanics.

Books on the Theory of Elastic Stability such as [28] written primarily for engineers are full of examples from this field.

11. Changes for Other Arrangements of the Eigenvalues

(i) Changes for the Real Parts of the Eigenvalues

An arrangement of the eigenvalues as in (4.1) is possible, for instance, when all real parts are positive. However, such an arrangement is not possible if there are infinitely many eigenvalues with negative real parts and infinitely many eigenvalues with positive real parts.

In the general case that contains the last-mentioned one we proceed similarly as in [26, Section 15] for symmetric compact operators in Hilbert space: So, the sequence of eigenvalues and eigenvectors will be numbered such that eigenvalues with positive real parts have positive indices and eigenvalues with negative real parts have negative indices. Accordingly, there are sequences of numbers $J_+,\, J_-$ whereby the finite resp. infinite sequence of eigenvalues can be arranged in the form

$$Re\,\lambda_{-1} \le Re\,\lambda_{-2} \le \ldots Re\,\lambda_{-k} \le \ldots < 0 \le \ldots \le Re\,\lambda_j \ldots \le Re\,\lambda_2 \le Re\,\lambda_1 \tag{11.1}$$

for $j \in J_+,\, k \in J_-$. For the index sequences J_+, J_-, it may happen that $J_+ = \emptyset$, $J_+ = (1,2,\ldots,m^+)$, or $J_+ = (1,2,\ldots)$ and $J_- = \emptyset$, $J_- = (1,2,\ldots,m^-)$, or $J_- = (1,2,\ldots)$, depending on whether no, finitely many, or infinitely many eigenvalues of T with positive real resp. negative real parts exist. Herewith, the formula (3.2) turns into

$$Tu = \sum_{j\in J_+} \lambda_j\,(u,\psi_j)\,\chi_j + \sum_{k\in J_-} \lambda_{-k}\,(u,\psi_{-k})\,\chi_{-k} \tag{11.2}$$

and the formula (3.3) into

$$Pu = \sum_{j\in J_+} (u,\psi_j)\,\chi_j + \sum_{k\in J_-} (u,\psi_{-k})\,\chi_{-k} \tag{11.3}$$

Further, the formulas (3.14), (3.15), (3.17) respectively become

$$(Tu,v) = \sum_{j\in J_+} \lambda_j (u,\psi_j)(\chi_j,v) + \sum_{k\in J_-} \lambda_{-k}(u,\psi_{-k})(\chi_{-k},v), \tag{11.4}$$

$u, v \in H$,

$$(u,v) = (Pu,v) = \sum_{j\in J_+} (u,\psi_j)(\chi_j,v) + \sum_{k\in J_-} (u,\psi_{-k})(\chi_{-k},v), \tag{11.5}$$

$u, v \in H$,

$$Re\,(Tu,v) = \sum_{j\in J_+} Re\,\lambda_j (u,\psi_j)(\chi_j,v) + \sum_{k\in J_-} Re\,\lambda_{-k}(u,\psi_{-k})(\chi_{-k},v), \tag{11.6}$$

$u \in N_{\chi,\mathbb{R}}$, $v \in N_{\psi,\mathbb{R}}$.

At this point, we make the important remark that the eigenvalues of $-T$ are obtained by multiplying the eigenvalues of T by -1. Therefore, it is sufficient to characterize the positive real parts of the eigenvalues by extremal principles since the corresponding statements on the negative real parts of the eigenvalues are obtained by applying the formulas for the operator $-T$ resp. the pertinent expression $\frac{Re\,(-Tu,v)}{(u,v)}$.

It is left to the reader to show that the formulas in Theorems 4.1 - 4.4 remain valid for J_+ instead of J for the arrangement (11.1).

(ii) Changes for the Imaginary Parts of the Eigenvalues

As to the imaginary parts of the eigenvalues, considerations similar to those in (i) have to be taken into account.

(iii) Moduli of the Eigenvalues

It is not necessary to make any changes in the arrangement (6.1) for the moduli of the eigenvalues.

12. Conclusion and Outlook to Future Work

In this paper, it could be shown that generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of simple eigenvalues of nonsymmetric compact operators can be derived that resemble corresponding results for diagonalizable matrices. Since the underlying Hilbert space is assumed to be infinite-dimensional, the proofs differ, in part, significantly from those in the finite-dimensional case of matrices. For instance, in the proof of Theorem 4.2, the denotation of codimension of a subspace of a Hilbert space was necessary that can be avoided in the finite-dimensional case, cf. [26, Section 15, pp.84-85].

In a subsequent paper, the results of this paper will be extended to defective, more precisely, to not necessarily simple eigenvalues of nonsymmetric compact operators.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] N.I. Achieser, I.M. Glasmann, Theorie der linearen Operatoren im Hilbert-Raum (Theory of Linear Operators in Hilbert Space; German Translation of the Russian Original), Akademie-Verlag, Berlin, 1968.

[2] W.E. Boyce, R.C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 8th Edition, John Wiley & Sons, Inc., Hoboke, 2005.

[3] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, New York Toronto London, 1955.

[4] L. Collatz, Eigenwertaufgaben mit technischen Anwendungen (Eigenvalue Problems with Applications in Engineering), Geest & Portig K.-G., Leipzig, 1949.

[5] R. Courant, D. Hilbert, Mathematische Methoden der Physik, Band 1 (Methods of Mathematical Physics, Vol. 1), Springer-Verlag, Berlin, Heidelberg, New York, 1968.

[6] R.D. Grigorieff, Diskrete Approximation von Eigenwertproblemen. III: Asymptotische Entwicklungen (Discrete Approximation of Eigenvalue Problems. III: Asymptotic Expansions), Num. Math. 25(1975)79-97.

[7] H. Heuser, Funktionalanalysis (Functional Analysis), B.G. Teubner, Stuttgart, 1975.

[8] L.V. Kantorovich, G.P. Akilov, Functional Analysis in Normed Spaces; English Translation of the Russian Original), Pergamon Press, 1964.

[9] L.W.Kantorowitsch, G.P. Akilow, Funktionalanalysis in normierten Räumen (Functional Analysis in Normed Spaces; German Translation of the Russian Original), Akademie-Verlag Berlin, 1965.

[10] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.

[11] K. Knopp, Theory of Functions, Parts I and II (English Translation of the German Original), Dover Publications, 1975.

[12] L. Kohaupt, Construction of a Biorthogonal System of Principal Vectors for Matrices A and A^* with Applications to $\dot{x} = Ax$, $x(t_0) = x_0$, Journal of Computational Mathematics and Optimization 3(3)(2007)163-192.

[13] L. Kohaupt, Biorthogonalization of the Principal Vectors for the Matrices A and A^* with Applications to the Computation of the Explicit Representation of the Solution $x(t)$ of $\dot{x} = Ax$, $x(t_0) = x_0$, Applied Mathematical Sciences 2(20)(2008)961-974.

[14] L. Kohaupt, Introduction to a Gram-Schmidt-type Biorthogonalization Method, Rocky Mountain Journal of Mathematics 44(4)(2014)1265-1279.

[15] L. Kohaupt, Generalized Rayleigh-quotient formulas for the eigenvalues of self-adjoint matrices, Journal of Algebra and Applied Mathematics 14(1)(2016)1-26.

[16] L. Kohaupt, Generalized Rayleigh-quotient formulas for the eigenvalues of diagonalizable matrices, Journal of Advances in Mathematics 14(2)(2018)7702-7728.

[17] L. Kohaupt, Generalized Rayleigh-quotient formulas for the eigenvalues of general matrices, Universal Journal of Mathematics and Applications 4(1)(2021)9-25.

[18] L. Kohaupt, Eigenvalue Expansion of Nonsymmetric Linear Compact Operators in Hilbert Space, CAMS IV(2)(2021)55-74.

[19] W. Luther, K. Niederdrenk, F. Reutter, H. Yserentant, Gewöhnliche Differentialgleichungen, Analytische und numerische Behandlung (Ordinary Differential Equations, Analytic and Numerical Treatment), Vieweg, Braunschweig Wiesbaden, 1987.

[20] S.G. Michlin, Variationsmethoden der Mathematischen Physik (Variational Methods of Mathematical Physics; German Translation of the Russian Original), Akademie-Verlag, Berlin, 1962.

[21] B.N. Parlett, D.R. Taylor, Z.A.Liu, A look-ahead Lanczos algorithm for unsymmetric matrices, Mathematics of Computation 44(169)(1985)105-124.

[22] E.C. Pestel, F.A. Leckie, Matrix Methods in Elastomechanics, McGraw-Hill Book Company, Inc., New York San Francisco Toronto London, 1963.

[23] W. Schnell, D. Gross, W. Hauger, Technische Mechanik, Band 2: Elastostatik (Mechanics for Engineers, Vol. 2: Elastostatics), Springer-Verlag, Berlin Heidelberg New York Tokyo, 1985.

[24] F. Stummel, Diskrete Konvergenz linearer Operatoren II (Discrete Convergence of Linear Operators, Part II), Mathematische Zeitschrift 120 (1971)231-264.

[25] F. Stummel, K. Hainer, Introduction to Numerical Analysis, Scottish Academic Press, Edinburgh, 1980.

[26] F. Stummel, L. Kohaupt, Eigenwertaufgaben in Hilbertschen Räumen. Mit Aufgaben und vollständigen Lösungen (Eigenvalue Problems in Hilbert Spaces. With Exercises and Complete Solutions), Logos Verlag, Berlin, 2021.

[27] A.E. Taylor, Introduction to Functional Analysis, John Wiley & Sons, New York London, 1958.

[28] St. P. Timoshenko, J. M. Gere, Theory of Elastic Stability, McGraw-Hill Kogakusha, Ltd., Tokyo et al., 1961.

[29] W. Walter, Gewöhnliche Differentialgleichungen. Eine Einführung. (Ordinary Differential Equations. An Introduction), Springer, Berlin et al., 2000.

[30] D. Werner, Funktionalanalysis (Functional Analysis), 8th Edition, Springer Spektrum, 2018.