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Araştırma Makalesi

Quintic-Septic Nonlinear Schrödinger Equation with a Third-Order Dispersion Term

Irma HACINLIYAN*[1]

[1]*İstanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 34469, İstanbul, Turkey*

**corresponding author e-mail: hacinliy@itu.edu.tr*

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Abstract: In the present study, the quintic-septic nonlinear modulation of a longitudinal wave propagating to contribute the dispersive and higher-order nonlinear effects in a generalized cubically nonlinear elastic medium is considered. In recent work, for the modulation of a longitudinal wave, a cubic nonlinear Schrödinger equation with a third-order dispersive term is obtained by using a multi-scale expansion of quasi-monochromatic wave solutions. The third-quintic-septic longitudinal wave, by choosing specific values of material constants and wave number for which some coefficients of nonlinear terms are disappeared. In this case, a new perturbation expansion is needed to balance nonlinear effects with dispersive effects. As a result, a quintic-septic nonlinear Schrödinger equation with a third-order dispersion term is obtained as a new model that balances quintic-septic nonlinearity with a third-order dispersion term.

Keywords: Nonlinear Schrödinger Equations, Nonlinear Wave Propagation, Generalized Elastic Medium

Üçüncü Mertebe Dispersiyon Terimli Beşli-Yedili Doğrusal Olmayan Schrödinger Denklemi

Öz: Bu çalışmada, genelleştirilmiş, kübik doğrusal olmayan elastik bir ortamda yayılan boyuna bir dalgada, yüksek mertebeden dağılım ve doğrusal olmayan etkilerin katkılarını incelemek için beşli-yedili doğrusal olmayan modülasyonu düşünülmektedir. Son zamandaki çalışmalarda, hemen hemen tek dalga sayılı dalga çözümlerinin çok ölçekli açılımı kullanılarak boyuna bir dalganın modülasyonu için üçüncü mertebeden dispersiyon terimli kübik, doğrusal olmayan Schrödinger denklemi elde edildi. Elde edilen denklemde bazı doğrusal olmayan terimlerin katsayılarının yer almadığı belirli bir malzeme sabiti ve dalga sayısı değerleri seçilirse, boyuna bir dalganın davranışını tanımlamak için, doğrusal olmayan etkileri dağılım etkilerle dengelendiği yeni bir pertürbasyon açılımına ihtiyaç vardır. Sonuç olarak, üçüncü dereceden dağılım terimli beşli-yedili doğrusal olmayan Schrödinger denklemi, beşli-yedili doğrusal olmayan etkinin üçüncü mertebeden dağılım terimiyle dengelendiği yeni bir model olarak elde edilir.

Anahtar kelimeler: Doğrusal Olmayan Schrödinger Denklemleri, Doğrusal Olmayan Dalga Yayılımı, Genelleştirilmiş Elastik Ortam

Irma HACINLIYAN, hacinliy@itu.edu.tr, ORCID: https://orcid.org/0000-0001-7076-2172

1. Introduction

Nonlinear Schrödinger (NLS) type equations describe the long-time behavior of modulated wave propagation in various nonlinear and dispersive media such as fluid, optical, elastic, acoustic, plasma, etc. The NLS equation has solutions called solitons in which the dispersive and nonlinear effects of the medium are balanced [1, 2]. Solitons are localized wave solutions that can retain their shape only by phase shift when they make elastic collisions. This type of wave solution was first observed in 1834 by John Scott Russell in a narrow and shallow channel in Scotland. Due to this special structure of soliton, the derivation of partial differential equations describing nonlinear and dispersive wave propagation in such a medium is one of the important areas in nonlinear scientific research. In this study, the modulation problem for the complex amplitude of (1+1) waves propagating in a homogeneous, infinite, dispersive, and cubic nonlinear elastic medium is considered to examine the contributions of quintic-septic nonlinear and third-order dispersive effects in a wave motion.

Erofeyev and Potapov obtained the governing equations of motion describing the propagation of two transverse waves and one longitudinal wave in an infinite homogeneous micromorphic elastic medium containing high-order displacement gradients and cubic nonlinear effects [3]:

$$\begin{aligned}
&u_{tt}-c_L^2u_{xx}+4c_T^2m^2\left(1+\nu\right)u_{xxxx}=\gamma_1u_xu_{xx}+\gamma_2\left(v_xv_{xx}+w_xw_{xx}\right)+\gamma_3(u_x)^2u_{xx}\\
&\qquad\qquad+\gamma_4\left\{u_{xx}\left((v_x)^2+(w_x)^2\right)+2u_x\left(v_xv_{xx}+w_xw_{xx}\right)\right\},\\
&v_{tt}-c_T^2v_{xx}+4c_T^2m^2v_{xxxx}=\gamma_2\left(u_{xx}v_x+u_xv_{xx}\right)+\gamma_4\left(2v_xu_xu_{xx}+(u_x)^2v_{xx}\right)\\
&\qquad\qquad+\gamma_5\left(3(v_x)^2v_{xx}+2v_xw_xw_{xx}+(w_x)^2v_{xx}\right),\\
&w_{tt}-c_T^2w_{xx}+4c_T^2m^2w_{xxxx}=\gamma_2\left(u_{xx}w_x+u_xw_{xx}\right)+\gamma_4\left(2w_xu_xu_{xx}+(u_x)^2w_{xx}\right)\\
&\qquad\qquad+\gamma_5\left(3(w_x)^2w_{xx}+2v_xw_xv_{xx}+(v_x)^2w_{xx}\right),
\end{aligned}\tag{1}$$

where t is the time variable and x is the spatial variable in the propagation direction. In Equation (1), the function u represents the longitudinal component of the displacement vector with the speed c_L, while v and w represent the transverse components with the velocity c_T. Additionally, the coefficients in Equation (1) are given as

$$\begin{aligned}
&c_L^2=\frac{\lambda+2\mu}{\rho_0},\ c_T^2=\frac{\mu}{\rho_0},\ \gamma_1=\frac{3(\lambda+2\mu)+2(\mathcal{A}_1+3\mathcal{A}_2+\mathcal{A}_3)}{\rho_0},\\
&\gamma_2=\frac{2\lambda+4\mu+\mathcal{A}_1+3\mathcal{A}_2}{\rho_0},\ \gamma_3=\frac{3[\lambda+2\mu+4\mathcal{A}_1+12\mathcal{A}_2+4\mathcal{A}_3+8(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4)]}{2\rho_0},\\
&\gamma_4=\frac{2\lambda+4\mu+5\mathcal{A}_1+14\mathcal{A}_2+4\mathcal{A}_3+6\mathcal{B}_2+4\mathcal{B}_3+8\mathcal{B}_4}{4\rho_0},\ \gamma_5=\frac{\lambda+2\mu+\mathcal{A}_1+2\mathcal{A}_2+2\mathcal{B}_4}{2\rho_0},
\end{aligned}\tag{2}$$

where λ and μ are linear elastic constants (EC), $\mathcal{A}_i$'s are the second-order EC, $\mathcal{B}_i$'s are third-order EC, and ν, m are the microstructure constants with the mass density ρ_0. When the harmonic wave solutions

$$(u, v, w) = e^{i(kx - \omega t)}(U, V, W), \tag{3}$$

where (U, V, W) represents the complex amplitude vector; ω represents the frequency and k represents the wavenumber, substitute the linear part of Equation (1), the dispersion relations

$$\begin{aligned} D_1(k,\omega) &= \omega^2 - c_L^2 k^2 - 4c_T^2 m^2 (1+\nu) k^4 = 0, \\ D_2(k,\omega) &= \omega^2 - c_T^2 k^2 - 4c_T^2 m^2 k^4 = 0 \end{aligned} \tag{4}$$

are obtained. In Equation (4), D_1 denotes the dispersion relation associated with the longitudinal displacement u, whereas D_2 denotes the dispersion relation associated with the transverse displacements v and w. Due to the dispersion relations (4), it is seen that the longitudinal wave and transverse waves are dispersive.

On the other hand, the cubic NLS (CNLS) equation developed by Erwin Schrödinger in 1927

$$i u_t + p u_{xx} + q |u|^2 u = 0 \tag{5}$$

characterizes the slowly-varying amplitude of a quasi-monochromatic wave that propagates in the weakly nonlinear and weakly dispersive medium [4, 5]. The function u in Equation (5) denotes the envelope of the complex amplitude of the short wave. The CNLS equation was recently re-derived in a study examining the contribution of the high-order dispersive effects on the longitudinal wave modulation emitted in a generalized elastic medium [6]. In [6], under the assumption $D_1(k,\omega) = 0$, the components of the displacement vector were expanded into asymptotic power series of ε. Then $v = 0$, $w = 0$, and the CNLS equation valid at the level ε^3 was obtained. The coefficients in Equation (5) are given by

$$p = \frac{\omega''}{2}, \qquad q = -\frac{k^4}{2\omega}\left(\gamma_3 + \frac{3\gamma_1^2}{2\left(c_{g_L}^2 - c_L^2\right)}\right) \tag{6}$$

with $D_1(2k, 2\omega) = 4k^2\left(c_{g_L}^2 - c_L^2\right)$. In Equation (6), c_{g_L} is the group speed of the longitudinal wave. However, for fairly short pulses, the CNLS equation is not a correct model of the wave motion, so higher-order dispersive effects need to be included in the evolution equation. Therefore, in the same study [6], the evolution equation with third-order dispersive term valid at the level ε^4

$$iU_t + pU_{xx} + i\,\overline{p}\,u_{xxx} + 2q|u|^2 U + qu^2 U^* + i\,r_1 u^2 u_x^* + i\,r_2 |u|^2 u_x = 0, \tag{7}$$

where

$$\begin{aligned} \overline{p} &= -\frac{\omega'''}{6},\; r_1 = -\frac{k^4 c_{g_L}}{2\omega}\left(\gamma_3 + \frac{\gamma_1^2}{2\omega\left(c_{g_L}^2 - c_L^2\right)^2}\left(5\left(c_{g_L}^2 - c_L^2\right) - 48 c_T^2 k^2 m^2 (1+\nu)\right)\right), \\ r_2 &= -\frac{k^3}{\omega^2}\left(\left(k c_{g_L} - 2\omega\right)\gamma_3 + \frac{\gamma_1^2}{2}\left(\left(2k c_{g_L} - 7\omega\right) + \frac{24\left(k c_{g_L} - \omega\right)}{c_{g_L}^2 - c_L^2} c_T^2 k^2 m^2 (1+\nu)\right)\right), \end{aligned} \tag{8}$$

was also found. Then the new depend variable $\phi = \varepsilon u + \varepsilon^2 U$ was defined to combine Equation (5) and Equation (7), which leads to the CNLS equation with third-order dispersive terms:

$$i\phi_t + p\,\phi_{xx} + \overline{q}\,|\phi|^2 \phi + i\varepsilon\left(\overline{p}\,\phi_{xxx} + \overline{r}_1 \phi^2 \phi_x^* + \overline{r}_2 |\phi|^2 \phi_x\right) = \mathrm{O}\left(\varepsilon^3\right), \tag{9}$$

with $(\overline{q}, \overline{r}_1, \overline{r}_2) = (q, r_1, r_2)/\varepsilon^2$.

If the coefficient of the nonlinear term in Equation (5) is zero ($q = 0$), the balance between the nonlinear effect and the dispersive effect in the medium will be disappeared. This situation occurs for the critical wave number k_c that is a solution of the equation

$$\gamma_3 = -\frac{3\gamma_1^2}{2\left(c_{g_L}(k)^2 - c_L^2\right)} \tag{10}$$

A new perturbation expansion is needed to examine the effect of the dropping nonlinearity on wave propagation around the critical wave number k_c. Erbay investigated this problem in a fluid-filled nonlinear elastic tube, and founded the quintic NLS (QNLS) equation [7]:

$$i\,u_t + p\,u_{xx} + q|u|^4 u + i\left(r|u|^2 u_x + s\,u^2 u_x^*\right) = 0. \tag{11}$$

This paper aims to investigate how a balance could be obtained between the third-order dispersion and nonlinear effects on the wave propagating in the generalized elastic medium when the coefficient $\bar{q}$ of the cubic term $|\phi|^2 \phi$ in Equation (9) vanishes. To do this, under the constraint $D_1(2k, 2\omega) = 4k^2 \left(c_{g_L}^2 - c_L^2\right)$ with Equation (10), the new perturbation expansion for the variable u near the critical wave number k_c is needed. A similar problem has been discussed in the literature for waves propagating in different media [8, 9].

In [8], Essamma et al., theoretically characterized the electromagnetic wave propagating in negative-index material by the cubic-quintic NLS (CQNLS) equation with the third-order dispersive term:

$$i u_x + p_1 u_{tt} + i p_2 u_{ttt} + \left(q_1 + q_2 |u|^2\right)|u|^2 u = 0. \tag{12}$$

In addition, on nonlinear optical waves, the CQNLS equation with third- and fourth-order dispersive terms

$$i u_x + p_1 u_{tt} + i p_2 u_{ttt} + p_2 u_{tttt} + \left(q_1 + q_2 |u|^2\right)|u|^2 u + i q_3 \left(|u|^4 u\right)_t = 0 \tag{13}$$

was considered to find soliton solutions and exact solutions [10, 11], and to analyze modulation instability [12].

In [9], the cubic-quintic-septic NLS (CQSNLS) equation with third- and fourth-order dispersive term

$$\begin{aligned} &i u_x + p_1 u_{tt} + i p_2 u_{ttt} + p_2 u_{tttt} \\ &\quad + \left(q_1 + q_2 |u|^2 + q_3 |u|^4\right)|u|^2 u + i\left(\left(q_4 + q_5 |u|^2 + q_6 |u|^4\right)|u|^2 u\right)_t = 0 \end{aligned} \tag{14}$$

was obtained as the model for the propagation of femtosecond optical pulses through fiber Bragg grating structure. Solitary wave solutions and dipole solution solutions were also presented in [9] and in [13], respectively.

2. Material and Method

In nonlinear wave theory, various asymptotic methods are used for the formal derivation of the evolution equations [14-17]. In this study, the reductive perturbation method [14,15] will be applied to obtain the quintic-septic NLS (QSNLS) equation with the third-order dispersive term in a generalized elastic medium. The main principle of this method is based on understanding how the amplitudes of harmonic waves are modulated by a nonlinear system. Therefore, two harmonic waves

$$u_1(x,t) = U\, e^{i\left[(k+\Delta k)x-(\omega+\Delta\omega)t\right]},$$
$$u_2(x,t) = U\, e^{i\left[(k-\Delta k)x-(\omega-\Delta\omega)t\right]} \tag{15}$$

are considered in the linear system. The waves $u_1(x,\, t)$ and $u_2(x,\, t)$ have the same amplitude U, but their wavenumbers are very close to each other. Superposition of the harmonic waves (15) simulated in Figure 1,

$$u(x,\, t) = u_1(x,\, t) + u_2(x,\, t) = 2U\cos(\Delta k\, x - \Delta\omega t)e^{i(k\,x-\omega t)}, \tag{16}$$

divides the harmonic waves into two parts: a slow-varying part $\cos(\Delta k\, x - \Delta\omega t)$ and a fast-varying part $e^{i(k\,x-\omega t)}$. That means the amplitude of the wave (16) is modulated.

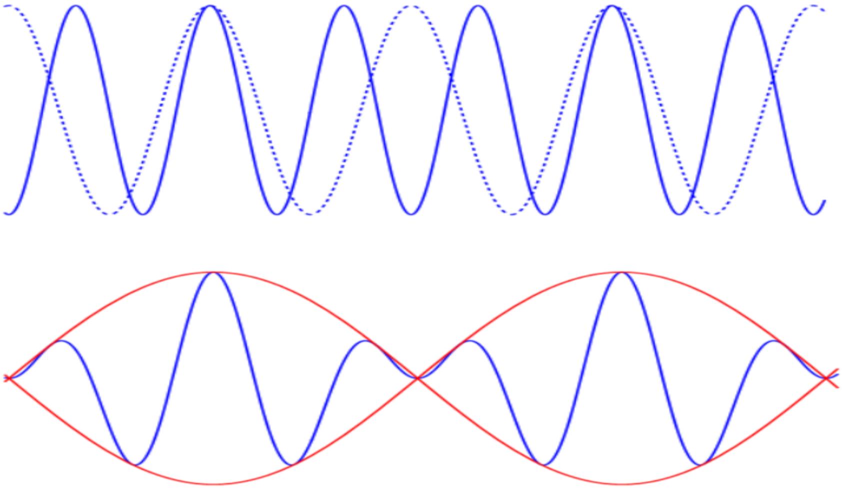

Figure 1. Superposition of two harmonic waves

At this stage, under the assumptions $\Delta k/k \ll 1$ and $\Delta\omega/\omega \ll 1$, the phase of the amplitude is expressed as

$$\Delta k\, x - \Delta\omega t \;=\; \Delta k\left(x - \frac{d\omega}{dk}t\right) - \frac{1}{2}\frac{d^2\omega}{dk^2}\left(\Delta k\right)^2\, t + \ldots \tag{17}$$

Thanks to Equation (17), the slow variables

$$\xi = \varepsilon\left(x - c_{g_L} t\right), \quad \tau = \varepsilon^2 t \tag{18}$$

are introduced with $\varepsilon = \Delta k$ is a small parameter and $c_{g_L} = d\omega/dk$. In this way, the parameter ε appears to be a tool for measuring the weakness of dispersion. On the other hand, ε will also measure the weakness of nonlinearity if the components of the displacement vector are written in asymptotic power series of ε.

This study aims to derive the evolution equation that re-establishes the balance between nonlinear and dispersive effects of the longitudinal wave propagating in the generalized elastic medium around the critical wave number k_c that provides Equation (10). Considering that the coefficient $\bar{q}$ of the cubic nonlinear term in Equation (9) disappears around k_c, nonlinearity should be taken as $\varepsilon^{1/2}$ instead of ε (for simplification, the subscript c will be omitted for the rest part of the paper.). For this purpose, the solution u of the governing equations (1) will be taken into account as the asymptotic series given by

$$\begin{aligned}
u(\xi,\tau) &= u_1^{(0)}(\xi,\tau)+\varepsilon^{\frac{1}{2}}[u_1^{(1)}(\xi,\tau)e^{i\theta}+c.c]+\varepsilon[u_2^{(0)}(\xi,\tau)+u_2^{(2)}(\xi,\tau)e^{2i\theta}+c.c] \\
&+\varepsilon^{\frac{3}{2}}[u_2^{(1)}(\xi,\tau)e^{i\theta}+u_2^{(3)}(\xi,\tau)e^{3i\theta}+c.c] \\
&+\varepsilon^{2}[u_3^{(0)}(\xi,\tau)+u_3^{(2)}(\xi,\tau)e^{2i\theta}+u_3^{(4)}(\xi,\tau)e^{4i\theta}+c.c] \\
&+\varepsilon^{\frac{5}{2}}[u_3^{(3)}(\xi,\tau)e^{3i\theta}+u_3^{(5)}(\xi,\tau)e^{5i\theta}+c.c] \\
&+\varepsilon^{3}[u_4^{(2)}(\xi,\tau)e^{2i\theta}+u_4^{(4)}(\xi,\tau)e^{4i\theta}+u_4^{(6)}(\xi,\tau)e^{3i\theta}+c.c] \\
&+\varepsilon^{\frac{7}{2}}[u_4^{(3)}(\xi,\tau)e^{3i\theta}+u_4^{(5)}(\xi,\tau)e^{5i\theta}+u_4^{(7)}(\xi,\tau)e^{7i\theta}+c.c] \\
&+\varepsilon^{4}[u_5^{(2)}(\xi,\tau)e^{2i\theta}+u_5^{(4)}(\xi,\tau)e^{4i\theta}+u_5^{(6)}(\xi,\tau)e^{6i\theta}+u_5^{(8)}(\xi,\tau)e^{8i\theta}+c.c]+\ldots,
\end{aligned} \tag{19}$$

where $\theta = kx - \omega t$ and $c.c.$ denotes the complex conjugate of the preceding expression. The functions $u_1^{(1)}$ and $u_2^{(1)}$ in Equation (19) are called the first-order longitudinal mode and the second-order longitudinal mode, respectively. As the longitudinal wave motion is considered, the transverse components v and w are assumed to be a linear function in time t. In addition, $D_1(k,\omega)=0$, $D_1(2k,2\omega)=4k^2\left(c_{g_L}^2-c_L^2\right)\neq 0,$ $D_1(nk,n\omega)\neq 0$, $(n=2,3,\ldots)$, and $D_2(lk,l\omega)\neq 0,\ (l=1,2,\ldots)$ will be taken.

Putting the new coordinates (18) and the expansion (19) into Equation (1) leads to a hierarchy of equations in half power of ε. By equating the coefficients of the same powers, perturbation equations are obtained and then are solved to find the evolution equations describing the long-time behavior of the wave motion.

3. Results

In this section, the QSNLS equation with the third-order dispersive term involving the first- and the second-order longitudinal modes is derived. For this purpose, perturbation equations are first obtained by plugging the transformations (18) together with the asymptotic series solution (19) into the governing equation (1). Then, these perturbation equations are solved to drive the evolution equations for the first- and second-order components. Here are only presented the solutions of the perturbation equation required to find the evolution equations. Finally, by properly combining the first- and second-order modes, an evolution equation containing quintic-septic nonlinear and third-order dispersive terms is obtained.

For $O\left(\varepsilon^{1/2}\right)$, the perturbation equation is $D_1(k,\omega)u_1^{(1)}=0$. Recalling that $D_1(k,\omega)=0$, the first-order mode $u_1^{(1)}$ is an arbitrary function of the slow variables ξ and τ. For order ε, the term $u_2^{(2)}$ is expressed by the first-order mode $u_1^{(1)}$, as follows:

$$u_2^{(2)}=\frac{i\,k\gamma_1}{4\left(c_{g_L}^2-c_L^2\right)}\left(u_1^{(1)}\right)^2. \tag{20}$$

In the next order, $O\left(\varepsilon^{3/2}\right)$, the equations

$$\begin{aligned}&2i\left(c_{g_L}\omega-c_L^2k-8c_T^2k^3m^2\left(1+\nu\right)\right)u_{1,\xi}^{(1)}-D_1(k,\omega)u_2^{(1)}+k^2\gamma_1u_{1,\xi}^{(0)}u_1^{(1)}\\&\qquad-2ik^3\gamma_1u_2^{(2)}u_1^{(1)*}+k^4\gamma_3\left|u_1^{(1)}\right|^2u_1^{(1)}=0,\\&D_1(3k,3\omega)u_2^{(3)}+k^4\gamma_3\left(u_1^{(1)}\right)^3-6ik^3\gamma_1u_2^{(2)}u_1^{(1)}=0\end{aligned} \tag{21}$$

are obtained. By Equation (21), the second-order mode $u_2^{(1)}$ remains an arbitrary function, and $c_{g_L}=\omega'$ is verified, that is, c_{g_L} is the group velocity of the longitudinal waves. Meanwhile, imposing Equation (10) and Equation (20) in Equation (21) implies

$$u_{1,\xi}^{(0)}=\frac{k^2\gamma_1}{c_{g_L}^2-c_L^2}\left|u_1^{(1)}\right|^2,\ u_2^{(3)}=0. \tag{22}$$

With similar calculations, for order ε^2, the functions $u_3^{(2)}$ and $u_3^{(4)}$ express in terms of the first-order and second-order modes:

$$\begin{aligned}u_3^{(2)}&=\frac{i\,k\gamma_1}{2(c_{g_L}^2-c_L^2)}u_2^{(1)}u_1^{(1)}+\frac{3\gamma_1\left(c_{g_L}^2-c_L^2+16c_T^2k^2m(1+\nu)\right)}{4(c_{g_L}^2-c_L^2)^2}u_{1,\xi}^{(1)}u_1^{(1)}\\&\quad-\frac{5ik^3\gamma_1^3}{4\left(c_{g_L}^2-c_L^2\right)^3}\left|u_1^{(1)}\right|^2\left(u_1^{(1)}\right)^2,\ u_3^{(4)}=\frac{5\,i\,k^4\gamma_1^3}{2D_1(4k,4\omega)(c_{g_L}^2-c_L^2)^2}\left(u_1^{(1)}\right)^4.\end{aligned} \tag{23}$$

For $O(\varepsilon^{5/2})$, the equation for the first-order longitudinal mode is founded by using the solutions of lower perturbation equations:

$$
\begin{aligned}
&\mathrm{i}u_{1,\tau}^{(1)} + pu_{1,\xi\xi}^{(1)} + \frac{12ik^5m^2(1+\nu)c_T^2\gamma_1^2}{\omega\left(c_{g_L}^2 - c_L^2\right)^2}\left|u_1^{(1)}\right|^2 u_{1,\xi}^{(1)} + \frac{\mathrm{i}k^3\gamma_1^2}{2\omega\left(c_{g_L}^2 - c_L^2\right)}\left(u_1^{(1)}\right)^2 u_{1,\xi}^{(1)*} \\
&+ \frac{k^4\gamma_1^2}{2\omega\left(c_{g_L}^2 - c_L^2\right)}\left(u_2^{(1)*}u_1^{(1)} + u_2^{(1)}u_1^{(1)*}\right)u_1^{(1)} + \frac{25k^6\gamma_1^4}{8\omega\left(c_{g_L}^2 - c_L^2\right)^3}\left|u_1^{(1)}\right|^4 u_1^{(1)} - \frac{k^2\gamma_1}{2\omega}u_{2,\xi}^{(0)}u_1^{(1)} = 0,
\end{aligned}
\tag{24}
$$

where $p = \dfrac{c_L^2 - c_{g_L}^2 + 24c_T^2k^2m^2(1+\nu)}{2\omega} = \dfrac{\omega''}{2}$. In addition, the solution $u_3^{(3)}$ expresses explicitly in terms of the first-order mode.

$$
\begin{aligned}
u_3^{(3)} = &\frac{3\mathrm{i}k^3\gamma_1^2\left(c_{g_L}^2 - c_L^2 + 24c_T^2k^2m^2(1+\nu)\right)}{D_1(3k,3\omega)\left(c_{g_L}^2 - c_L^2\right)^2}\left(u_1^{(1)}\right)^2 u_{1,\xi}^{(1)} \\
&+ \frac{15k^6\gamma_1^4\left(7D_1(4k,4\omega) + 16k^2\left(c_{g_L}^2 - c_L^2\right)\right)}{8\left(c_{g_L}^2 - c_L^2\right)^3 D_1(3k,3\omega)D_1(4k,4\omega)}\left(u_1^{(1)}\right)^3\left|u_1^{(1)}\right|^2
\end{aligned}
\tag{25}
$$

To present compact form of the evolution equation (24) for the first-order mode, $u_{2,\xi}^{(0)}$ is calculated from the higher-order perturbation equation in the order of ε^3:

$$
\begin{aligned}
u_{2,\xi}^{(0)} = &\frac{k\gamma_1}{c_{g_L}^2 - c_L^2}\left(i\left(u_{1,\xi}^{(1)*}u_1^{(1)} - u_{1,\xi}^{(1)}u_1^{(1)*}\right) + k\left(u_2^{(1)}u_1^{(1)*} + u_2^{(1)*}u_1^{(1)}\right) - \frac{15k^3\gamma_1^2}{4\left(c_{g_L}^2 - c_L^2\right)^2}\left|u_1^{(1)}\right|^4\right) \\
&+ \frac{2k^2c_{g_L}\gamma_1}{\left(c_{g_L}^2 - c_L^2\right)^2}\int_{-\infty}^{\xi}\left(\left|u_1^{(1)}\right|^2\right)_\tau d\xi.
\end{aligned}
\tag{26}
$$

The integrand of all integral is assumed to be rapidly decreasing functions of ξ as ξ approaches to $-\infty$. Now, substituting Equation (26) to Equation (24), the integrodifferential equation involving the first-order mode $u_1^{(1)}$ is obtained:

$$
\mathrm{i}u_{1,\tau}^{(1)} + pu_{1,\xi\xi}^{(1)} + \left(iq_1u_{1,\xi}^{(1)} + q_2\left|u_1^{(1)}\right|^2 u_1^{(1)}\right)\left|u_1^{(1)}\right|^2 + q_3u_1^{(1)}\int_{-\infty}^{\xi}\left(\left|u_1^{(1)}\right|^2\right)_\tau d\xi = 0, \tag{27}
$$

where the coefficients in Equation (27) are given by

$$q_1 = \frac{k^3\gamma_1^2\left(c_{g_L}^2 - c_L^2 + 24c_T^2k^2m^2(1+\nu)\right)}{2\left(c_{g_L}^2 - c_L^2\right)^2\omega},\ q_2 = \frac{5k^6\gamma_1^4}{\left(c_{g_L}^2 - c_L^2\right)^3\omega},\ q_3 = -\frac{c_{g_L}k^2\gamma_1^2}{\left(c_{g_L}^2 - c_L^2\right)^2\omega}. \tag{28}$$

At this stage, the integral term should be eliminated from Equation (27). Therefore, subtracting the $u_1^{(1)*}$ multiplied of Equation (27) from its complex conjugate gives

$$\left(\left|u_1^{(1)}\right|^2\right)_\tau = -\left(i\,p\left(u_{1,\xi}^{(1)*}u_1^{(1)} - u_{1,\xi}^{(1)}u_1^{(1)*}\right) + \frac{q_1}{2}\left|u_1^{(1)}\right|^4\right)_\xi. \tag{29}$$

When plugging Equation (29) to Equation (27) leads to the QNLS equation describing the propagation of the first-order longitudinal wave near the critical wavenumber k_c:

$$\mathrm{i}u_{1,\tau}^{(1)} + pu_{1,\xi\xi}^{(1)} + i\left(\left(q_1 + pq_3\right)u_{1,\xi}^{(1)}u_1^{(1)*} - pq_3u_{1,\xi}^{(1)*}u_1^{(1)}\right)u_1^{(1)} + \left(q_2 - \frac{q_1q_3}{2}\right)\left|u_1^{(1)}\right|^4 u_1^{(1)} = 0. \tag{30}$$

In addition, for $O(\varepsilon^3)$, the solution $u_4^{(2)}$ is given by

$$\begin{aligned} u_4^{(2)} = {} & i\left(A_1\left|u_1^{(1)}\right|^2u_1^{(1)} + A_2u_2^{(1)} + A_3u_{1,\xi}^{(1)}\right)\left|u_1^{(1)}\right|^2u_1^{(1)} + \left(\frac{iA_2}{3}u_2^{(1)*} + A_4u_{1,\xi}^{(1)*}\right)\left(u_1^{(1)}\right)^3 \\ & + i\,A_5\left(u_2^{(1)}\right)^2 + A_6\left(u_2^{(1)}u_1^{(1)}\right)_\xi + i\,A_7\left(\left(u_{1,\xi}^{(1)}\right)^2 - 3u_1^{(1)}u_{1,\xi\xi}^{(1)}\right), \end{aligned} \tag{31}$$

with the coefficients

$$\begin{aligned} A_1 = {} & \frac{k^6\gamma_1^5c_{g_L}\left(49054c_L^6 - 100327c_L^4c_{g_L}^2 + 66620c_L^2c_{g_L}^4 - 14888c_{g_L}^6\right)}{3840\omega\left(c_{g_L}^2 - c_L^2\right)^8} \\ & - \frac{k^5\gamma_1^5c_L^2\left(49055c_L^4 - 83812c_L^2c_{g_L}^2 + 35216c_{g_L}^4\right)}{3840\left(c_{g_L}^2 - c_L^2\right)^8},\ A_2 = -\frac{15k^3\gamma_1^3}{4\left(c_{g_L}^2 - c_L^2\right)^3}, \\ A_3 = {} & \frac{k^2\gamma_1^3\left(9kc_{g_L}\left(278c_L^4 - 289c_L^2c_{g_L}^2 + 56c_{g_L}^4\right) - \omega\left(541c_L^4 + 1636c_L^2c_{g_L}^2 - 1772c_{g_L}^4\right)\right)}{192\omega\left(c_{g_L}^2 - c_L^2\right)^5}, \\ A_4 = {} & \frac{k^2\gamma_1^3\left(25\omega + 18kc_{g_L}\right)}{8\omega\left(c_{g_L}^2 - c_L^2\right)^3},\ A_5 = \frac{k\gamma_1}{4\left(c_{g_L}^2 - c_L^2\right)},\ A_6 = \frac{-\gamma_1}{4\left(c_{g_L}^2 - c_L^2\right)},\ A_7 = \frac{\gamma_1}{8k\left(c_{g_L}^2 - c_L^2\right)}. \end{aligned} \tag{32}$$

To be able to observe the contribution of higher-order dispersion and higher nonlinear impacts, the behavior of the mode $u_2^{(1)}$ is determined on the order of $\varepsilon^{7/2}$:

$$
\begin{aligned}
&\mathrm{i}\, u_{2,\tau}^{(1)} + p u_{2,\xi\xi}^{(1)} + \mathrm{i} r u_{1,\xi\xi\xi}^{(1)} + iB_1\left(\left(u_1^{(1)}\right)^2 u_{2,\xi}^{(1)*} - \left(\left(u_1^{(1)}\right)^2\right)_\xi u_2^{(1)*}\right) + B_2\left|u_2^{(1)}\right|^2 u_1^{(1)} \\
&+iB_3\left(u_{1,\xi}^{(1)}u_1^{(1)*} - u_{1,\xi}^{(1)*}u_1^{(1)}\right)u_2^{(1)} - \left(2iB_1 u_{2,\xi}^{(1)} - B_4\left|u_1^{(1)}\right|^2 u_2^{(1)} - B_5\left(u_1^{(1)}\right)^2 u_2^{(1)*}\right)\left|u_1^{(1)}\right|^2 \\
&+\left(B_6\left|u_{1,\xi}^{(1)}\right|^2 + B_7 u_{1,\xi\xi}^{(1)}u_1^{(1)*} + B_8 u_{1,\xi\xi}^{(1)*}u_1^{(1)}\right)u_1^{(1)} + B_9\left(u_{1,\xi}^{(1)}\right)^2 u_1^{(1)*} \\
&+\left(i\left(B_{10}u_{1,\xi}^{(1)*}u_1^{(1)} + B_{11}u_{1,\xi}^{(1)}u_1^{(1)*}\right) + B_{12}\left|u_1^{(1)}\right|^4\right)\left|u_1^{(1)}\right|^2 u_1^{(1)} + B_{13}u_{3,\xi}^{(0)}u_1^{(1)} = 0,
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
&r = \frac{c_{g_L}\left(c_L^2 - c_{g_L}^2\right) - 8km^2c_T^2\left(1+\nu\right)\left(2\omega - 3kc_{g_L}\right)}{2\omega^2} = -\frac{\omega'''}{6}, B_1 = \frac{k^3\gamma_1^2}{2\omega\left(c_{g_L}^2 - c_L^2\right)}, \\
&B_2 = kB_1, B_3 = \frac{k^3\gamma_1^2\left(3kc_{g_L} - \omega\right)}{2\omega^2\left(c_{g_L}^2 - c_L^2\right)}, B_4 = \frac{k^6\gamma_1^4\left(45\omega - kc_{g_L}\right)}{4\omega^2\left(c_{g_L}^2 - c_L^2\right)^3}, B_5 = \frac{25k^6\gamma_1^4}{4\omega\left(c_{g_L}^2 - c_L^2\right)^3}, \\
&B_6 = \frac{k^3\gamma_1^2\left(11kc_L^2 + 2c_{g_L}\left(3\omega - 7kc_{g_L}\right)\right)}{6\omega^3\left(c_L^2 - c_{g_L}^2\right)}, B_7 = \frac{k^4\gamma_1^2\left(142c_L^4 + 137c_L^2c_{g_L}^2 - 99c_{g_L}^4\right)}{36\omega^3\left(c_{g_L}^2 - c_L^2\right)^2}, \\
&B_8 = \frac{3k^3c_{g_L}\gamma_1^2\left(kc_{g_L} - \omega\right)}{2\omega^3\left(c_{g_L}^2 - c_L^2\right)}, B_9 = \frac{k^4\gamma_1^2\left(9c_L^2 - 7c_{g_L}^2\right)}{2\omega^3\left(c_{g_L}^2 - c_L^2\right)}, B_{10} = \frac{k^7\gamma_1^4\left(338c_L^2 - 149c_{g_L}^2\right)}{12\omega^3\left(c_{g_L}^2 - c_L^2\right)^3}, \\
&B_{11} = \frac{k^7\gamma_1^4\left(11c_{g_L}^2 - 164c_L^2\right)}{12\omega^3\left(c_{g_L}^2 - c_L^2\right)^3} + \frac{k^9c_{g_L}^2\gamma_1^4\left(25c_L^2 - 7c_{g_L}^2\right)}{2\omega^3\left(c_{g_L}^2 - c_L^2\right)^3 D_1\left(3k, 3w\right)}, \\
&B_{12} = \frac{k^8\gamma_1^6\left(14975c_L^6 - 24805c_L^4c_{g_L}^2 + 11399c_L^2c_{g_L}^4 - 12c_{g_L}^6\right)}{480\omega\left(c_{g_L}^2 - c_L^2\right)^8} \\
&\quad - \frac{k^9c_{g_L}\gamma_1^6\left(40645c_L^6 - 87798c_L^4c_{g_L}^2 + 65349c_L^2c_{g_L}^4 - 15082c_{g_L}^6\right)}{960\omega^2\left(c_{g_L}^2 - c_L^2\right)^8}, B_{13} = -\frac{k^2\gamma_1}{2\omega}.
\end{aligned}
\tag{34}
$$

The term $u_{3,\xi}^{(0)}$ in Equation (33) is found from the solution of the equation for $O(\varepsilon^4)$ and the expression of $u_{3,\xi}^{(0)}$ in terms of the first-mode $u_1^{(1)}$ and the second-mode $u_2^{(1)}$ is as follows:

$$
\begin{aligned}
u_{3,\xi}^{(0)} &= C_1\left|u_2^{(1)}\right|^2 + iC_2\left(u_{2,\xi}^{(1)*}u_1^{(1)} + u_{1,\xi}^{(1)*}u_2^{(1)} - u_{1,\xi}^{(1)}u_2^{(1)*} - u_{2,\xi}^{(1)}u_1^{(1)*}\right) \\
&+ C_3\left(u_2^{(1)*}u_1^{(1)} + u_2^{(1)}u_1^{(1)*}\right)\left|u_1^{(1)}\right|^2 + C_4\int_{-\infty}^{\xi}\left(u_2^{(1)*}u_1^{(1)} + u_2^{(1)}u_1^{(1)*}\right)_\tau d\xi \\
&+ C_5\left(u_{1,\xi\xi}^{(1)*}u_1^{(1)} + u_{1,\xi\xi}^{(1)}u_1^{(1)*}\right) + C_6\left|u_{1,\xi}^{(1)}\right|^2 - iC_7\left(u_{1,\xi}^{(1)*}u_1^{(1)} - u_{1,\xi}^{(1)}u_1^{(1)*}\right)\left|u_1^{(1)}\right|^2 \\
&+ C_8\left|u_1^{(1)}\right|^6 + iC_9\int_{-\infty}^{\xi}\left(\left(u_{1,\xi}^{(1)}u_1^{(1)*}\right)^2 - \left(u_{1,\xi}^{(1)*}u_1^{(1)}\right)^2\right)d\xi.
\end{aligned} \tag{35}
$$

The coefficients in Equation (35) are given by

$$
\begin{aligned}
C_1 &= \frac{k^2\gamma_1}{c_{g_L}^2 - c_L^2}, C_2 = \frac{k\gamma_1}{c_{g_L}^2 - c_L^2}, C_3 = \frac{15k^4\gamma_1^3}{2\left(c_L^2 - c_{g_L}^2\right)^3}, C_4 = \frac{2k^2 c_{g_L}\gamma_1}{\left(c_{g_L}^2 - c_L^2\right)^2}, C_5 = \frac{\gamma_1}{3\left(c_{g_L}^2 - c_L^2\right)}, \\
C_6 &= \frac{k^2\gamma_1\left(321c_L^4 - 118c_L^2c_{g_L}^2 - 167c_{g_L}^4\right)}{18\omega^2\left(c_{g_L}^2 - c_L^2\right)^2},\ C_7 = \frac{k^3\gamma_1^3}{\left(c_L^2 - c_{g_L}^2\right)^3}\left(\frac{13}{2} + \frac{6kc_{g_L}}{\omega}\right), \\
C_8 &= \frac{k^8\gamma_1^5\left(2233c_L^6 - 4248c_L^4c_{g_L}^2 + 1428c_L^2c_{g_L}^4 + 8c_{g_L}^6\right)}{72\omega^2\left(c_L^2 - c_{g_L}^2\right)^7} \\
&\quad + \frac{k^9 c_{g_L}\gamma_1^5\left(563c_L^6 + 2025c_L^4c_{g_L}^2 - 2785c_L^2c_{g_L}^4 + 776c_{g_L}^6\right)}{72\omega^3\left(c_L^2 - c_{g_L}^2\right)^7}, \\
C_9 &= \frac{k^5\gamma_1^3}{4\omega^2\left(c_{g_L}^2 - c_L^2\right)^4}\left(3\left(49c_L^4 - 74c_L^2c_{g_L}^2 + 26c_{g_L}^4\right) - \frac{kc_{g_L}}{\omega}\left(48c_L^4 - c_L^2c_{g_L}^2 - 44c_{g_L}^4\right)\right).
\end{aligned} \tag{36}
$$

Now, the integrodifferential equation involving the second-order mode $u_2^{(1)}$ given as below is calculated by substituting Equation (35) to Equation (33):

$$
\begin{aligned}
&\mathrm{i}\, u_{2,\tau}^{(1)} + pu_{2,\xi\xi}^{(1)} + \mathrm{i}ru_{1,\xi\xi\xi}^{(1)} - \frac{i}{2}\left(2B_1 + B_{13}C_2\right)\left(\left(\left(u_1^{(1)}\right)^2\right)_\xi u_2^{(1)*} + 2\left|u_1^{(1)}\right|^2 u_{2,\xi}^{(1)}\right) \\
&+i\left(B_3u_{1,\xi}^{(1)}u_1^{(1)*} - \left(B_3 - B_{13}C_2\right)u_{1,\xi}^{(1)*}u_1^{(1)}\right)u_2^{(1)} + B_{13}C_4u_1^{(1)}\int_{-\infty}^{\xi}\left(u_2^{(1)*}u_1^{(1)} + u_2^{(1)}u_1^{(1)*}\right)_\tau d\xi \\
&+\left(\left(B_4 + B_{13}C_3\right)u_2^{(1)}u_1^{(1)*} + \left(B_5 + B_{13}C_3\right)u_2^{(1)*}u_1^{(1)}\right)\left|u_1^{(1)}\right|^2 u_1^{(1)} + \left(B_6 + B_{13}C_6\right)\left|u_{1,\xi}^{(1)}\right|^2 u_1^{(1)} \\
&+\left(\left(B_7 + B_{13}C_5\right)u_{1,\xi\xi}^{(1)}u_1^{(1)*} + \left(B_8 + B_{13}C_5\right)u_{1,\xi\xi}^{(1)*}u_1^{(1)}\right)u_1^{(1)} + B_9\left(u_{1,\xi}^{(1)}\right)^2 u_1^{(1)*} \\
&+\left(i\left(B_{10} - B_{13}C_7\right)u_{1,\xi}^{(1)*}u_1^{(1)} + i\left(B_{11} + B_{13}C_7\right)u_{1,\xi}^{(1)}u_1^{(1)*} + \left(B_{12} + B_{13}C_8\right)\left|u_1^{(1)}\right|^4\right)\left|u_1^{(1)}\right|^2 u_1^{(1)} \\
&+iB_{13}C_9u_1^{(1)}\int_{-\infty}^{\xi}\left(\left(u_{1,\xi}^{(1)}u_1^{(1)*}\right)^2 - \left(u_{1,\xi}^{(1)*}u_1^{(1)}\right)^2\right)d\xi = 0.
\end{aligned} \tag{37}
$$

To eliminate the integral terms in (37), the real part of $u_2^{(1)*}$ times Equation (27) is added to the real part of $u_1^{(1)*}$ times Equation (37). As a result,

$$\begin{aligned}
&\int_{-\infty}^{\xi}\left(u_2^{(1)*}u_1^{(1)}+u_2^{(1)}u_1^{(1)*}\right)_{\tau}d\xi=i\left(B_8+B_9-B_7\right)\int_{-\infty}^{\xi}\left(\left(u_{1,\xi}^{(1)*}u_1^{(1)}\right)^2-\left(u_{1,\xi}^{(1)}u_1^{(1)*}\right)^2\right)d\xi\\
&+ip\left(u_{2,\xi}^{(1)}u_1^{(1)*}+u_{1,\xi}^{(1)}u_2^{(1)*}-u_{2,\xi}^{(1)*}u_1^{(1)}-u_{1,\xi}^{(1)*}u_2^{(1)}\right)+r\left|u_{1,\xi}^{(1)}\right|^2\\
&+\left(\left(pq_3-B_3\right)\left(u_2^{(1)*}u_1^{(1)}+u_2^{(1)}u_1^{(1)*}\right)-\left(B_{10}+B_{11}\right)\left|u_1^{(1)}\right|^4/3\right)\left|u_1^{(1)}\right|^2
\end{aligned}\tag{38}$$

is founded. Then substituting Equation (38) into Equation (37) with the constraint

$$C_9=C_4\left(B_7-B_8-B_9\right)\tag{39}$$

leads to the partial differential equation in terms of the second-order mode $u_2^{(1)}$:

$$\begin{aligned}
&\mathrm{i}\,u_{2,\tau}^{(1)}+pu_{2,\xi\xi}^{(1)}+\mathrm{i}ru_{1,\xi\xi\xi}^{(1)}-\frac{i}{2}\left(2B_1+B_{13}\left(C_2-pC_4\right)\right)\left(\left(\left(u_1^{(1)}\right)^2\right)_{\xi}u_2^{(1)*}+2\left|u_1^{(1)}\right|^2u_{2,\xi}^{(1)}\right)\\
&+i\left(B_3u_{1,\xi}^{(1)}u_1^{(1)*}-\left(B_3+B_{13}\left(pC_4-C_2\right)\right)u_{1,\xi}^{(1)*}u_1^{(1)}\right)u_2^{(1)}-i\,pB_{13}C_4\left(u_1^{(1)}\right)^2u_{2,\xi}^{(1)*}\\
&+\left(B_5+B_{13}\left(C_3+C_4\left(pq_3-B_3\right)\right)\right)\left|u_1^{(1)}\right|^2\left(u_1^{(1)}\right)^2u_2^{(1)*}\\
&+\left(B_4+B_{13}\left(C_3+C_4\left(pq_3-B_3\right)\right)\right)\left|u_1^{(1)}\right|^4u_2^{(1)}+\left(B_6+B_{13}\left(C_6+rC_4\right)\right)\left|u_{1,\xi}^{(1)}\right|^2u_1^{(1)}\\
&+\left(\left(B_7+B_{13}C_5\right)u_{1,\xi\xi}^{(1)}u_1^{(1)*}+\left(B_8+B_{13}C_5\right)u_{1,\xi\xi}^{(1)*}u_1^{(1)}\right)u_1^{(1)}+B_9\left(u_{1,\xi}^{(1)}\right)^2u_1^{(1)*}\\
&+i\left(\left(B_{10}-B_{13}C_7\right)u_{1,\xi}^{(1)*}u_1^{(1)}+\left(B_{11}+B_{13}C_7\right)u_{1,\xi}^{(1)}u_1^{(1)*}\right)\left|u_1^{(1)}\right|^2u_1^{(1)}\\
&+\left(B_{12}+B_{13}\left(C_8-C_4\left(B_{10}+B_{11}\right)/3\right)\right)\left|u_1^{(1)}\right|^6u_1^{(1)}=0.
\end{aligned}\tag{40}$$

On the other hand, the constraint given in Equation (39) causes relationships between some coefficients in Equation (30) and Equation (40)., i.e.,

$$\begin{aligned}
&q_1+p\,q_3=-\left(2B_1+B_{13}\left(C_2-p\,C_4\right)\right)=B_3,\\
&p\,q_3=p\,B_{13}C_4=\left(B_3+B_{13}\left(p\,C_4-C_2\right)\right)/2,\\
&q_2-\frac{q_1\,q_3}{2}=\frac{\left(B_4+B_{13}\left(C_7+C_4\left(p\,q_3-B_3\right)\right)\right)}{3}=\frac{\left(B_5+B_{13}\left(C_3+C_4\left(p\,q_3-B_3\right)\right)\right)}{2}.
\end{aligned}\tag{41}$$

One can easily see that that Equation (40) is linear in terms of the second-order mode $u_2^{(1)}$, however, it is nonlinear in the first-order mode $u_1^{(1)}$. As a result, defining the dependent variable transformation

$$\phi = \varepsilon u_1^{(1)} + \varepsilon^2 u_2^{(1)}, \tag{42}$$

Equation (30) and Equation (40) can be conveniently combined, that is, adding ε times Equation (30) to ε times Equation (40) implies the equation

$$\begin{aligned} &\mathrm{i}\phi_\tau + p\phi_{\xi\xi} + i\left(\alpha_1\phi_\xi\phi^* + \alpha_2\phi_\xi^*\phi\right)\phi + \beta_1|\phi|^4\phi + \mathrm{i}\varepsilon\left(r\phi_{\xi\xi\xi} + \left(\beta_2\phi_\xi\phi^* + \beta_3\phi_\xi^*\phi\right)|\phi|^2\phi\right) \\ &\quad + \varepsilon\left(\alpha_3|\phi_\xi|^2\phi + \alpha_4\left(\phi_\xi\right)^2\phi^* + \alpha_5|\phi|^2\phi_{\xi\xi} + \alpha_6\phi^2\phi_{\xi\xi}^* + \chi|\phi|^6\phi\right) = 0. \end{aligned} \tag{43}$$

Equation (43) is valid for $O(\varepsilon^3)$ with the coefficients

$$\begin{aligned} &\alpha_1 = B_3/\varepsilon^2,\ \alpha_2 = -pB_{13}C_4/\varepsilon^2,\ \alpha_3 = \left(B_6 + B_{13}\left(C_6 + rC_4\right)\right)/\varepsilon^2,\ \alpha_4 = B_9/\varepsilon^2, \\ &\alpha_5 = \left(B_7 + B_{13}C_5\right)/\varepsilon^2,\ \alpha_6 = \left(B_8 + B_{13}C_5\right)/\varepsilon^2,\ \beta_1 = \left(q_2 - q_1q_3/2\right)/\varepsilon^4, \\ &\beta_2 = \left(B_{11} + B_{13}C_7\right)/\varepsilon^4,\ \beta_3 = \left(B_{10} - B_{13}C_7\right)/\varepsilon^4, \\ &\chi = \left(B_{12} + B_{13}\left(C_8 - C_4\left(B_{10} + B_{11}\right)/3\right)\right)/\varepsilon^6. \end{aligned} \tag{44}$$

4. Conclusion and Comment

In this study, by a perturbation approach, a nonlinear evolution equation, which may be said to be the quintic-septic NLS equation with a third-order dispersion term, is obtained to propagate the longitudinal wave in a dispersive and nonlinearly elastic medium. Similar evolution equations balancing the septic nonlinear effect with the third- and fourth-order dispersive effect are found to descript the nonlinear wave motion propagating in different media [8, 9]. However, for the generalized elastic medium, this balance could be achieved between septic nonlinearity and third-order dispersibility. The balance problem of fourth-order dispersive effect and higher-order nonlinearity can be studied in the future.

Author Statement

Irma Hacinliyan: Investigation, Formal Analysis, Validation, Original Draft Writing, Review, and Editing.

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As the author of this study, I declare that I do not have any support and thank you statement.

Conflict of Interest

As the author of this study, I declare that I do not have any conflict of interest statement.

Ethics Committee Approval and Informed Consent

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

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