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New existence results for nonlinear functional hybrid differential equations involving the $\psi-$Caputo fractional derivative

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Abstract

In this manuscript, we are concerned with the existence result of nonlinear hybrid differential equations involving $\psi-$Caputo fractional derivatives of an arbitrary order $\alpha \in (0,1)$. By applying Krasnoselskii fixed point theorem and some fractional analysis techniques, we prove our main result. As application, a nontrivial example is given to demonstrate the effectiveness of our theoretical result.

Keywords: $\psi-$fractional integral $\psi-$Caputo fractional derivative Carathéodory function.
2010 MSC: 34K37, 26A33, 34A08.

1. Introduction

In recent years, fractional differential equations have emerged as a new branch of applied mathematics, due to the evolution of fractional calculus in various scientific disciplines. This theory has been acknowledged as a powerful tool in a variety of engineering and applied science disciplines such as transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, biology, electromagnetic theory, fluid flow phenomena, thermodynamics, mechanics, geology, astrophysics, economics and control theory (see, for instance[2, 11, 15, 17, 18, 19, 25, 26]). Basic issues related to the different fractional derivatives such as Riemann-Liouville [22], Caputo [4], Hilfer [23], Erdelyi-Kober [24] and Hadamard [3]. Fractional differential equations have been of great interest recently such as functional hybrid fractional differential equations which can be employed in modeling and describing non-homogeneous physical phenomena that

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take place in their form. The importance of fractional hybrid differential equations lies in the fact that they include various dynamical systems as particular cases. Furthermore, hybrid differential equations arise from a variety of different areas of applied mathematics and physics, in the deflection of a curved beam having a constant or varying cross section and electromagnetic waves or gravity driven flows. Dhage in [14] proved the existence and uniqueness of extremal solutions for hybrid differential equations by using some fundamental differential inequalities and comparison results. We refer the readers to the articles [1, 5, 8, 9, 12, 16, 28, 29] and references therein for many works on this theory.
Motivated by the above works especially [14], we develop the theory of fractional hybrid differential equations involving ψ-Caputo fractional differential operator of order $\alpha \in (0,1)$. To be more precise, we establish the existence result of the following nonlinear fractional hybrid differential equation:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}\left(\dfrac{u(t)}{\Phi(t,u(t))}\right) = \varphi(t,u(t)), \qquad t \in \Delta = [0,T], \\ \\ u(0) = 0. \end{cases} \tag{1}$$

Where $T > 0$, $\Phi \in C(\Delta \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $\varphi \in \mathbf{C_c}(\Delta \times \mathbb{R}, \mathbb{R})$.

Our manuscript is organized as follows: In Section 2, we give some basic definitions and properties of $\psi-$fractional integral and $\psi-$Caputo fractional derivative which will be used in the rest of our paper. In Section 3, we establish the existence of solutions of the $\psi-$Caputo fractional hybrid differential equation (1) by using some Lipschitz and Carathéodory conditions. As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, see the articles [6, 10, 21] for more details.

Notations

• We denote by $\mathbf{C_c}(\Delta \times \mathbb{R}, \mathbb{R})$ the Caratheódory class of functions on $\Delta \times \mathbb{R}$.
i.e. $\varphi \in \mathbf{C_c}(\Delta \times \mathbb{R}, \mathbb{R})$ if and only if

1) the map $t \mapsto \varphi(t,u(t))$ is measurable for each $u \in \mathbb{R}$, and
2) the map $u \mapsto \varphi(t,u(t))$ is continuous for each $t \in \Delta$.

• We denote by $C(\Delta \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ the space of continuous real-valued functions defined on Δ provided with the topology of the supremum norm

$$\| u \| = \sup_{t \in \Delta} | u(t) | .$$

• We denote by $L^1(\Delta, \mathbb{R})$ the space of Lebesgue integrable real-valued functions on Δ equipped with the following norm

$$\| u \|_{L^1} = \int_\Delta | u(t) | \, dt.$$

Remark 2.1. *Let $u, v \in C(\Delta \times \mathbb{R}, \mathbb{R})$.*
Clearly $C(\Delta \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ is a Banach algebra with the norm $\| \, . \, \|$ and with the multiplication $(u \times v)(t) = u(t) \times v(t)$.

Definition 2.2. *[7] Let $q > 0$, $g \in L^1([\Delta, \mathbb{R})$ and $\psi \in C^n(\Delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \Delta$. The ψ-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{q-1} g(s) ds. \tag{2}$$

Remark 2.3. *Note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation (2) is reduced to the Riemann-Liouville and Hadamard fractional integrals respectively.*

Definition 2.4. *[7] Let $q > 0$, $g \in C^{n-1}(\Delta, \mathbb{R})$ and $\psi \in C^n(\Delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \Delta$. The $\psi-$Caputo fractional derivative at order q of the function g is given by*

$$^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1} g_\psi^{[n]}(s) ds, \tag{3}$$

where

$$g_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n g(s) \quad and \quad n = [q] + 1,$$

and $[q]$ denotes the integer part of the real number q.

Remark 2.5. *In particular, note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation (3) is reduced to the the Caputo fractional derivative and Caputo-Hadamard fractional derivative respectively.*

Remark 2.6. *If $q \in (0, 1)$, then the equation (3) can be written as follows*

$$^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} g'(s) ds.$$

In another way, we have

$$^C D_{0^+}^{q,\psi} g(t) = I_{0^+}^{1-q,\psi} \left(\frac{g'(t)}{\psi'(t)} \right).$$

Proposition 2.7. *[7] Let $q > 0$, if $g \in C^{n-1}(\Delta, \mathbb{R})$, then we have*

1) $^C D_{0^+}^{q,\psi} I_{0^+}^{q,\psi} g(t) = g(t)$.
2) $I_{0^+}^{q,\psi}\, {}^C D_{0^+}^{q,\psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k$.
3) $I_{0^+}^{q,\psi}$ *is linear and bounded from $C(\Delta, \mathbb{R})$ to $C(\Delta, \mathbb{R})$.*

Proposition 2.8. *[7] Let $t > 0$ and $q, \alpha, \beta \geq 0$, then we have*

1) $I_{0^+}^{q,\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(0))^{\alpha+\beta-1}$.
2) $^C D_{0^+}^{q,\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(0))^{\alpha-\beta-1}$.
3) $^C D_{0^+}^{q,\psi} (\psi(t) - \psi(0))^n = 0, \quad$ *for all $n \in \mathbb{N}$.*

Lemma 2.9. *(See [13, 27]). Let $\boldsymbol{C}$ be a non-empty, closed convex and bounded subset of a Banach algebra X and Let $\Lambda : \boldsymbol{C} \to X$ and $\mathcal{B} : \boldsymbol{C} \to X$ be two operators such that*

1. *Λ is Lipschitzian with a Lipschitz constant λ,*
2. *$\mathcal{B}$ is completely continuous,*
3. *$u = \Lambda u \mathcal{B} v \Rightarrow u \in \boldsymbol{C}$ for all $v \in \boldsymbol{C}$, and*
4. *$\lambda M < 1$ where $M = \| \mathcal{B}(\boldsymbol{C})) \| = Sup\{\mathcal{B}(v) \mid v \in \boldsymbol{C}\}$.*
   Then the equation $\Lambda u \mathcal{B} u = u$ has a solution in $\boldsymbol{C}$.

We assume the following assumptions throughout the rest of our paper.

(H_1) The function $u \mapsto \varphi(t,u)$ is increasing on $\mathbb{R}$ almost every where fort $t \in \Delta$.

(H_2) There exists a constant $k > 0$ such that

$$\mid \varphi(t,u) - \varphi(t,v) \mid \leq k \mid u - v \mid \quad for \quad all \quad t \in \Delta \quad and \quad u, v \in \mathbb{R}.$$

(H_3) There exists a function $g \in L^1(\Delta, \mathbb{R})$

$$\mid \Phi(t,u) \mid \leq g(t) \quad a.e.\ t \in \Delta, \quad for \ \ all \ \ u \in \mathbb{R}.$$

3. Main results

In this section, before we give the existence result of the fractional initial value problem (1), we need to prove the following fundamental lemma.

Lemma 3.1. *Suppose that hypothesis (H_1) holds, then the function $u(t) \in C(\Delta, \mathbb{R})$ is a solution of the fractional hybrid differential equation (1) if and only if u satisfies the fractional hybrid integrale equation*

$$u(t) = \frac{\Phi(t,u)}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, u(s)) ds. \tag{4}$$

Proof. Let u be a solution of the problem (1), then we apply the $\psi-$fractional integral $I_{0^+}^{\alpha,\psi}$ on both sides of (1)we have

$$I_{0^+}^{q,\psi}\ {}^C D_{0^+}^{\alpha,\psi} \left(\frac{u(t)}{\Phi(t,u(t))} \right) = I_{0^+}^{\alpha,\psi} \varphi(t,u(t)),$$

from Proposition 2.7 we obtain

$$\frac{u(t)}{\Phi(t,u(t))} - \frac{u(0)}{\Phi(t,u(0))} = I_{0^+}^{\alpha,\psi} \varphi(t,u(t)),$$

$$\frac{u(t)}{\Phi(t,u(t))} = \frac{0}{\Phi(t,0)} + I_{0^+}^{\alpha,\psi} \varphi(t,u(t)),$$

$$\frac{u(t)}{\Phi(t,u(t))} = 0 + I_{0^+}^{\alpha,\psi} \varphi(t,u(t)),$$

thus

$$u(t) = \frac{\Phi(t,u)}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, u(s)) ds.$$

Hence equation (4) holds.

Conversely, it is clear that if $u(t)$ satisfies the equation (4), then we divide by $\Phi(t,u)$ and we apply the $\psi-$Caputo fractional derivative ${}^C D_{0^+}^{\alpha,\psi}$ to both sides of equation (4) and we use Proposition 2.7, we obtain

$${}^C D_{0^+}^{\alpha,\psi} \left(\frac{u(t)}{\Phi(t,u)} \right) = {}^C D_{0^+}^{\alpha,\psi} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, u(s)) ds \right),$$

$${}^C D_{0^+}^{\alpha,\psi} \left(\frac{u(t)}{\Phi(t,u)} \right) =^C D_{0^+}^{q,\psi} I_{0^+}^{q,\psi} \varphi(t,u) = \varphi(t,u),$$

thus

$${}^C D_{0^+}^{\alpha,\psi} \left(\frac{u(t)}{\Phi(t,u(t))} \right) = \varphi(t,u(t)).$$

Finally, we need to verify that the condition $u(0) = 0$ in th equation (1) also holds. For this purpose, we substitute $t = 0$ in (4), we obtain

$$u(0) = \frac{\Phi(0,u)}{\Gamma(\alpha)} \int_0^0 \psi'(s)(\psi(0) - \psi(s))^{\alpha-1}\varphi(s,u(s))ds = 0.$$

$$u(0) = 0.$$

Since the map $u \mapsto \frac{u(t)}{\Phi(t,u)}$ is increasing on $\mathbb{R}$ almost every where for $t \in \Delta$, then the map $u \mapsto \frac{u(t)}{\Phi(0,u)}$ is injective on $\mathbb{R}$ and $u(0) = 0$. This completes the proof.

□

Theorem 3.2. *Assume that hypotheses $(H_1) - (H_3)$ hold. Further, if*

$$k\left(\frac{(\psi(T) - \psi(0))^\alpha \parallel g \parallel_{L^1}}{\Gamma(\alpha+1)}\right) < 1,$$

then the fractional hybrid differential equation (1) has a solution defined on Δ.

Proof. Let $E = C(\Delta, \mathbb{R})$ and let C_b be a subset of the space E defined by

$$C_b = \{u \in E : \parallel u \parallel \leq b\}.$$

where

$$b = \frac{\Phi_0(\psi(T) - \psi(0))^\alpha \parallel g \parallel_{L^1}}{\Gamma(\alpha+1) - k(\psi(T) - \psi(0))^\alpha \parallel g \parallel_{L^1}},$$

and

$$\Phi_0 = \sup_{t\in\Delta} \Phi(t,0).$$

It is easy te see that C_b is a closed, convex and bounded subset of the Banach space E. By using Lemma 3.1, the fractional hybrid differential equation (1) is equivalent to the following nonlinear fractional hybrid integral equation

$$u(t) = \frac{\Phi(t,u)}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\varphi(s,u(s))ds. \tag{5}$$

Let $\Lambda : E \to E$ and $\mathcal{B} : C_b \to E$ be two operators defined by

$$\Lambda u(t) = \Phi(t, u(t)),$$

and

$$\mathcal{B}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\varphi(s,u(s))ds.$$

We can transforme the nonlinear fractional hybrid integral equation (5) into the operator equation as

$$\Lambda u(t)\mathcal{B}u(t) = u(t), \quad t \in \Delta. \tag{6}$$

Now, we will show that the operators Λ and $\mathcal{B}$ satisfy all the conditions of Lemma 2.9.
First, we prove that Λ is a Lipschitz operator on E with the Lipschitz constant k.
Let $u, v \in E$, then by hypothesis (H_2)

$$\mid \Lambda u(t) - \Lambda v(t) \mid = \mid \varphi(t,u(t)) - \varphi(t,v(t)) \mid \leq k \mid u(t) - v(t) \mid \quad for \quad all \quad t \in \Delta,$$

Taking supremum over t, we obtain

$$\parallel \Lambda u - \Lambda v \parallel \leq k \parallel u - v \parallel, \quad for \quad all \quad u, v \in E.$$

Secondly, we show the operator $\mathcal{B}$ is completely continuous.
For this purpose, it is enough to prove that the operator $\mathcal{B}$ is continuous and $\mathcal{B}(C_b)$ is uniformly bounded and equicontinuous.
Let us show that the operator $\mathcal{B}$ is continuous.
Let u_n be a sequence in C_b converging to $u \in C_b$, then by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\mapsto+\infty} \mathcal{B}u_n(t) = \lim_{n\mapsto+\infty} \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,u_n(s))ds,$$

$$\lim_{n\mapsto+\infty} \mathcal{B}u_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \lim_{n\mapsto+\infty}\varphi(s,u_n(s))ds,$$

$$\lim_{n\mapsto+\infty} \mathcal{B}u_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds,$$

$$\lim_{n\mapsto+\infty} \mathcal{B}u_n(t) = \mathcal{B}u(t), \quad for \quad all \quad t\in\Delta.$$

Wich shows that $\mathcal{B}$ is a continuous operator on C_b.

Next we show that $\mathcal{B}(C_b)$ is a uniformly bounded.
Let $u \in C_b$, then we have

$$|\mathcal{B}u(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds\right|,$$

$$|\mathcal{B}u(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(s))^{\alpha-1} \times \left|\psi'(s)\right| \times |\varphi(s,u(s))|\, ds,$$

by using (H_3) we obtain

$$|\mathcal{B}u(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \times |g(s))|\, ds,$$

$$|\mathcal{B}u(t)| \leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \parallel g \parallel \quad for \quad all \quad t\in\Delta,$$

taking supremum over t, we obtain

$$\parallel \mathcal{B}u \parallel \leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \parallel g \parallel \quad for \quad all \quad u\in C_b.$$

This shows that $\mathcal{B}$ is uniformly bounded on C_b.

Now, let us also show that $\mathcal{B}(C_b)$ is equicontinuous on Δ.

Let $u \in \mathcal{B}(C_b)$ and $t_1, t_2 \in \Delta$ such that $t_1 < t_2$, then we have

$$|\mathcal{B}u(t_1) - \mathcal{B}u(t_2)| = \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds$$

$$-\frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds \Big|,$$

$$|\mathcal{B}u(t_1) - \mathcal{B}u(t_2)| \leq \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds|$$
$$+|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds|,$$

$$|\mathcal{B}u(t_1)-\mathcal{B}u(t_2)|\leq\frac{\parallel g\parallel_{L^1}}{\Gamma(\alpha+1)}\left(|\psi^\alpha(t_2)-\psi^\alpha(t_1)-(\psi(t_2)-\psi(t_1))^\alpha|-(\psi(t_2)-\psi(t_1))^\alpha\right),$$

since ψ is a continuous function, then we obatin

$$\lim_{t_1\to t_2}|\mathcal{B}u(t_1)-\mathcal{B}u(t_2)|=0.$$

Which shows that $\mathcal{B}(C_b)$ is equicontinuous.
Now the set $\mathcal{B}(C_b)$ is uniformly bounded and equicontinuous and by using Arzelà–Ascoli Theorem [20] we deduce that $\mathcal{B}(C_b)$ is compact, wich implies that the operator $\mathcal{B}$ is completely continuous.
Now it remains to show that the third assumption in Lemma 2.9 is verified.
Let $u\in E$ and $v\in C_b$ be arbitrary such that $u=\Lambda u\mathcal{B}v$, then by hypothesis (H_2), we have

$$|u(t)|=|\Lambda u(t)|\,|\mathcal{B}v(t)|=\left|\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds\right|,$$
$$|u(t)|=|\Phi(t,u(t))|\left|\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,u(s))ds\right|,$$
$$|u(t)|\leq(|\Phi(t,u(t))-\Phi(t,0)|+|\Phi(t,0)|)\,\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,|\varphi(s,u(s))|\,ds,$$
$$|u(t)|\leq(|ku(t)|+|\Phi_0|)\left(\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)ds\right),$$
$$|u(t)|\leq(|ku(t)|+|\Phi_0|)\,\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\parallel g\parallel_{L^1},$$

wich implies that

$$|u(t)|\leq(|ku(t)|)\,\frac{\Phi_0(\psi(T)-\psi(0))^\alpha\parallel g\parallel_{L^1}}{\Gamma(\alpha+1)-k(\psi(T)-\psi(0))^\alpha\parallel g\parallel_{L^1}},$$

taking supremum over t, we obtain

$$\parallel u\parallel\leq\frac{\Phi_0(\psi(T)-\psi(0))^\alpha\parallel g\parallel_{L^1}}{\Gamma(\alpha+1)-k(\psi(T)-\psi(0))^\alpha\parallel g\parallel_{L^1}}:=b.$$

Since

$$M=\parallel\mathcal{B}(C_b)\parallel=Sup\{\mathcal{B}(u):u\in C_b\}\leq\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\parallel g\parallel_{L^1},$$

then we get

$$\lambda M\leq k\left(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\parallel g\parallel_{L^1}\right)<1.$$

Finally, all conditions of Lemma 2.9 are satisfied for the operators Λ and $\mathcal{B}$. Hence the fractional hybrid differential equation (1) has a solution defined on Δ.

□

4. An illustrative example

In this section we give a nontrivial example to illustrate our main result.
Consider the following hybrid fractional differential equation:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{2}{3},t^2}\left(\frac{u(t)}{\Phi(t,u(t))}\right) = \varphi(t,u(t)), & t\in\Delta=[0,1], \\ u(0)=0. \end{cases} \tag{7}$$

where $\alpha=\frac{2}{3}$, $T=1$, $\psi(t)=t^2$, $\varphi(t,u(t))=\dfrac{t^2}{12}sin(u(t))$ and
$\Phi(t,u(t))=\dfrac{e^{-t}}{9+e^t}\left(\dfrac{|u(t)|}{1+|u(t)|}\right)$.
It is clear that the assumption (H_1) is satisfied.
To prove the assumption (H_2), let $t\in\Delta$ and $u,v\in\mathbb{R}$, then we have

$$|\Phi(t,u(t))-\Phi(t,v(t))| = \left|\frac{e^{-t}}{9+e^t}\left(\frac{|u(t)|}{1+|u(t)|}\right)-\frac{e^{-t}}{9+e^t}\left(\frac{|v(t)|}{1+|v(t)|}\right)\right|,$$

$$|\Phi(t,u(t))-\Phi(t,v(t))| \leq \left|\frac{e^{-t}}{9+e^t}\right|\left|\left(\frac{|u(t)|}{1+|u(t)|}\right)-\left(\frac{|v(t)|}{1+|v(t)|}\right)\right|,$$

$$|\Phi(t,u(t))-\Phi(t,v(t))| \leq \frac{1}{10}\left|\frac{u(t)-v(t)}{(1+|u(t)|)(1+|v(t)|)}\right|,$$

$$|\Phi(t,u(t))-\Phi(t,v(t))| \leq \frac{1}{10}\left|u(t)-v(t)\right|,$$

Thus, the assumption (H_2) in holds true with $k=\frac{1}{10}$.

It remains to verify the assumption (H_3). Let $t\in\Delta$ and $u\in\mathbb{R}$, then we have

$$|\varphi(t,u(t))| = \left|\frac{t^2}{12}sin(u(t))\right|,$$

$$|\varphi(t,u(t))| \leq \frac{t^2}{12}\left|sin(u(t))\right|,$$

$$|\varphi(t,u(t))| \leq \frac{t^2}{12}.$$

Wich implies that the assumption (H_3) is verified with $g(t)=\frac{t^2}{12}$.
Moreover, we have

$$k\left(\frac{(\psi(T)-\psi(0))^{\alpha}\parallel g\parallel_{L^1}}{\Gamma(\alpha+1)}\right)=\frac{1}{10}\times\left(\frac{(1^2-0^2)^{\frac{1}{2}}\times\frac{1}{36}}{\Gamma(\frac{5}{3})}\right)=\frac{1}{360}\times\frac{1}{\Gamma(\frac{5}{3})}=0.31\times 10^{-3}<1.$$

Finally, all the conditions of Theorem 3.2 are satisfied, thus the fractional hybrid problem (7) has a solution defined on $[0,1]$.

5. Conclusion

In the current paper, we gave the definition of solutions for fractional hybrid initial value problem using the $\psi-$Caputo fractional derivative of order $\alpha\in(0,1)$. In addition, we discussed the existence of at least one solution for this problem by using some Lipschitz and Carathéodory conditions. Finally, a nontrivial example is presented to illustrate the applicability of our obtained result.

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