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THE ALPHA-SKEW HYPERBOLIC SECANT DISTRIBUTION WITH APPLICATIONS TO AN ASTRONOMICAL DATASET

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ABSTRACT

This work demonstrates the attractivity of the alpha-skew hyperbolic secant distribution, a new skewed distribution based on the alpha-skew technique and the hyperbolic secant distribution. In the first part, we determine its main features, including its cumulative distribution function, modality, non-central moments, skewness, kurtosis, moment generating function and characteristic function. The remaining part is devoted to the model applicability in a statistical context. As a first step, the parameters are estimated by maximum likelihood estimates. Then, we perform a data fitting experiment and compare the values of the Akaike and Bayesian information criteriawith those of some other similar distributions. By considering an astronomical dataset and valuable competitors also based on the alpha-skew technique, the alpha-skew hyperbolic secant distribution turns out to be the best.

Keywords: Skewed distribution, Hyperbolic secant distribution, Bimodality, Parametric estimation, Data analysis.

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ASTRONOMİK BİR VERİ KÜMESİ UYGULAMASI İLE ALFA ÇARPIK HİPERBOLİK SEKANT DAĞILIMI

ÖZ

Bu çalışma, alfa-çarpık tekniğine ve hiperbolik sekant dağılımına dayanan yeni bir çarpık dağılım olan alfa-çarpık hiperbolik sekant dağılımının çekiciliğini göstermektedir. Birinci bölümde, kümülatif dağılım fonksiyonu, modalite, merkezi olmayan momentler, çarpıklık, basıklık, moment üreten fonksiyon ve karakteristik fonksiyon dahil olmak üzere ana özellikleri belirlenmiştir. Kalan kısım, istatistiksel bağlamda modelin uygulanabilirliğine ayrılmıştır. İlk adım olarak, parametreler en çok olabilirlik tahminleriyle tahmin edilmiştir. Daha sonra, bir veri uygulaması gerçekleştirilmiş ve Akaike ve Bayesian bilgi kriterlerinin değerlerini diğer bazı benzer dağılımların değerleriyle karşılaştırılmıştır. Alfa-çarpık tekniğine dayanan astronomik bir veri seti ve rakipler göz önüne alındığında, alfa-çarpık hiperbolik sekant dağılımının en iyisi olduğu ortaya çıkmıştır.

Anahtar Kelimeler: Çarpık dağılım, Hiperbolik secant dağılımı, İki modluluk, Parametrik tahmin, Veri analizi

1. INTRODUCTION

Many random events in nature can not be explained by simple unimodal distributons, such as the normal, Laplace and logistic distributions. Asymmetry and bimodality behaviors can be observed in a variety of distributions of data. For example, the time between geyser eruptions, the age of onset of certain pathogens or even the growth estimates of fish species, etc. Therefore, several distributions have been created to allow for the most accurate analysis of these data. For instance, Azzalini (1985) proposed a thorough skewed version of the normal distribution, called the asymmetric normal (AN) distribution. It is defined with the following probability density function (PDF):

$$f_{AN}(x;\lambda) = 2\phi(x)\Phi(\lambda x), \qquad x \in R \tag{1}$$

where $\phi(x)$ and $\Phi(x)$ are the PDF and cumulative distribution function (CDF) of a standard normal variable, respectively, i.e., $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x}\phi(t)dt$, and $\lambda \in R$. It is clear that $f_{AN}(x;0) = \phi(x)$. The additional parameter λ is introduce to produced asymmetrical shapes based on the symmetrical structure of the normal distribution; if the sign of λ changes, the PDF is reflected on the opposite side of the vertical axis, and thus the bell shape of the related model can be skewed to a maximum to accommodate some skewed data. For more technical information and applications, we refer to Gómez *et al.* (2006) and Capitanio (2014).

Motivated by the high level of applicability of the AN distribution, numerous skewed versions of symmetric distributions have been proposed and investigated. See, for example, those in Kim (2005), Elal-Olivero (2010), Asgharzadeh *et al.* (2016), Chesneau *et al.* (2020) and Bakouch *et al.* (2021). In this study, we provide some contributions to the development of the alpha-skew technique as proposed by Elal-Olivero (2010). Thus, a retrospective on this technique is necessary. To begin, the alpha-skew technique was first employed by Elal-Olivero (2010) to create the alpha-skew normal (ASN) distribution defined by the following PDF:and

$$f_{ASN}(x;\alpha) = \frac{(1-\alpha x)^2+1}{2+\alpha^2}\phi(x), \qquad x,\alpha \in R, \tag{2}$$

with $\alpha \in R$. It can also be written as $f_{ASN}(x;\alpha) = c_\alpha w(x;\alpha)\phi(x)$, where $c_\alpha = 1/(2+\alpha^2)$ and $w(x;\alpha) = (1-\alpha x)^2 + 1$. The weight function $w(x;\alpha)$ characterizes the alpha-skewed technique; it modulates the functionalities of the pdf $\phi(x)$ thanks to α, which itself modulates the effect of the polynomial term in this weight function. It is worth noting that $w(x;0) = 2$. The constant c_α is just a "normalization constant" which is evaluated to make the integral of $c_\alpha w(x;\alpha)\phi(x)$ over R equal to one. The main advantage of the ASN distribution is that it has both unimodal and bimodal behavior, contrary to the AN distribution, for instance. Thus, it is more appropriate for data whose distribution presents such characteristics. By applying this alpha-skew technique to the Laplace distribution, Harandi and Alamatsaz (2013) proposed the alpha-skew Laplace (ASLa) distribution. It is defined by the PDF given as

$$f_{ASLa}(x;\alpha) = \frac{(1-\alpha x)^2+1}{2+2\alpha^2}\psi(x), \qquad x,\alpha \in R, \tag{3}$$

where $\psi(x)$ denotes the PDF of the Laplace distribution, i.e., $\psi(x) = 2^{-1}e^{-|x|}$. It is worth noting that $f_{ASLa}(x;\alpha) = d_\alpha w(x;\alpha)\psi(x)$, where $d_\alpha = 1/(2+2\alpha^2)$ is the normalization constant adjusted to the Laplace distribution under consideration. In practice, the ASLa distribution reveals itself to be a suitable alternative to the ASN distribution, presenting differences mainly in the kurtosis and tails features. Similarly, Hazarika and Chakraborty

(2014) has developed the alpha-skew logistic (ASLo) distribution, which consists of applying the alpha-skew technique to the logistic distribution. It is defined by the following PDF:

$$f_{ASLo}(x;\alpha) = 3\frac{(1-\alpha x)^2+1}{6+\pi^2\alpha^2}\upsilon(x), \qquad x,\alpha \in R \tag{4}$$

where $\upsilon(x)$ denotes the PDF of the logistic distribution, i.e., $\upsilon(x) = e^{-x}/(1+e^{-x})^2$. It has proved to be an interesting competitor to the ASN and ASLa distributions. Previous research has demonstrated that the alpha-skew technique is ideal for introducing a manageable unimodal or bimodal skewed effect in any symmetric distribution defined by R.

On the topic of the alpha-skew technique, there are, however, some unexplored directions of research on this topic. In particular, to the best our knowledge, its application to the famous hyperbolic secant (HS) distribution has never been explored. This work aims to fill this gap. To begin, the HS distribution is a symmetric distribution defined on R, and introduced by Baten (1934) and Talacko (1956). It is defined by the following PDF:

$$f_{HS}(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi x}{2}\right), \qquad x \in R \tag{5}$$

where "sech" denotes the hyperbolic secant function defined by $\operatorname{sech}(x) = 1/\cosh(x) = 2/(e^x + e^{-x})$. Among its properties, it exhibits far greater leptokurtosis than the normal and logistic distributions. Furthermore, moments of all order exist, as well as the function that generates them, exist. As a matter of fact, if $\mathrm{f}_{HS}(x)$ is properly weighted, we can modify its symmetric shapes on the real line (i.e., add the skewness and increase or decrease the kurtosis). Generalizations of the hyperbolic secant distribution can be found in the book by Fischer (2013). They are quite competitive with the existing distribution on the modeling plan. Based on previous research, the goal of this paper is to combine the alpha-skew technique with the hyperbolic secant distribution to create a new “R distribution” that can accommodate both unimodal and bimodal forms. It is naturally called the alpha-skew hyperbolic secant (ASHS) distribution. We highlight its main theoretical features and show how it can outperform the ASN, ASLa and ASLo distributions in concrete data analysis scenarios. An astronomical dataset is considered in this regard.

The paper will be structured as follows. We characterize the ASHS distribution in Section 2 and present several interesting moment properties in Section 3. In Section 4, we will present our parametric estimation technique, and the results of our study on a real dataset. Finally, Section 6 will conclude the paper.

2. ASHS DISTRIBUTION

The ASHS distribution is presented in this section, along with the shape properties of the related PDF.

Definition 2.1 A random variable X is said to follow the ASHS distribution with skewness parameter $\alpha \in R$, if its PDF is

$$f_{ASHS}(x;\alpha) = \frac{(1-\alpha x)^2+1}{2+\alpha^2} \mathrm{f}_{\mathrm{HS}}(x), \qquad x \in R \tag{6}$$

with $\mathrm{f}_{\mathrm{HS}}(x) = (1/2)\,\mathrm{sech}(\pi x/2)$. That is,

$$f_{ASHS}(x;\alpha) = \frac{1}{2}\frac{(1-\alpha x)^2+1}{2+\alpha^2}\,\mathrm{sech}\left(\frac{\pi x}{2}\right), \qquad x \in R \tag{7}$$

In the definition above, the alpha-skew technique is applied to the HS distribution in the following sense: we have weighted the PDF of the HS distribution by the alpha-skew weight function $w(x;\alpha) = (1-\alpha x)^2 + 1$, in such as way that $f_{ASHS}(x;\alpha) = e_\alpha w(x;\alpha) f_{HS}(x)$, where $e_\alpha = 1/(2+\alpha^2)$ is the normalization constant, i.e., such that $\int_{-\infty}^{+\infty} f_{ASHS}(x)dx = 1$. This constant has been calculated by using the following well-known integral results: $\int_{-\infty}^{+\infty} x f_{HS}(x)dx = 0$ and $\int_{-\infty}^{+\infty} x^2 f_{HS}(x)dx = 1$. Note that this normalizaton constant corresponds to the one of the ASN distribution.

The elements of the set $M = argmax_{x\in R} f_{ASHS}(x;\alpha)$ are the modes of the ASHS distribution. They can be determined via the study of the derivative of $f_{ASHS}(x;\alpha)$ given by

$$f'_{ASHS}(x;\alpha) = -\frac{\pi\left(\alpha^2x^2-2\alpha x+2\right)\tanh(\pi x/2)+4\alpha(1-\alpha x)}{4(2+\alpha^2)}\,\mathrm{sech}\left(\frac{\pi x}{2}\right) \tag{8}$$

As a result, we have $M \in \{0, x_1, x_2\}$, where x_1 and x_2 are possible roots of the following equation: $\pi(\alpha^2x^2 - 2\alpha x + 2)\tanh(\pi x/2) + 4\alpha(1-\alpha x) = 0$. There is no analytical expression for x_1 and x_2; we need to determine them numerically, but, clearly, the unimodality and bimodality of the ASHS distribution depend on the choice of α. For this reason, we chose to complete our mode analysis by a graphical study.

Figures 1 and 2 show some plots of $f_{ASHS}(x;\alpha)$ for different choices of the parameter α; Figure 1 considers the unimodal case, whereas Figure 2 considers the bimodal cases, with the determination of x_1 and x_2 (and $x = 0$ becomes a minimum point).

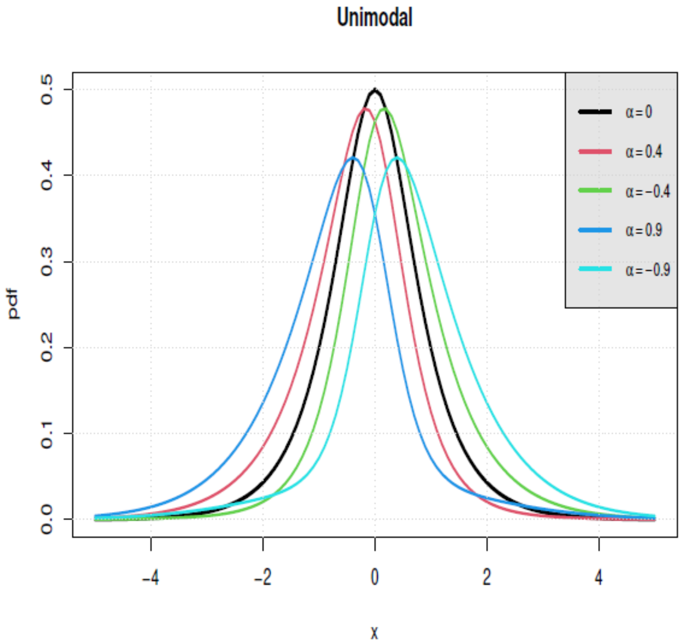

Figure 1. Some unimodal shapes of the PDF of the ASHS distribution for small values of α

In Figure 1, shows how, as α approaches zero, the ASHS distribution becomes unimodal. Furthermore, the tails of distributions are thicker and decrease more abruptly. When $\alpha < 0$, the bell shape spreads to the left and spreads to the right when $\alpha > 0$. Therefore, the ASHS model can serve to analyze data with such skewness properties in their distribution.

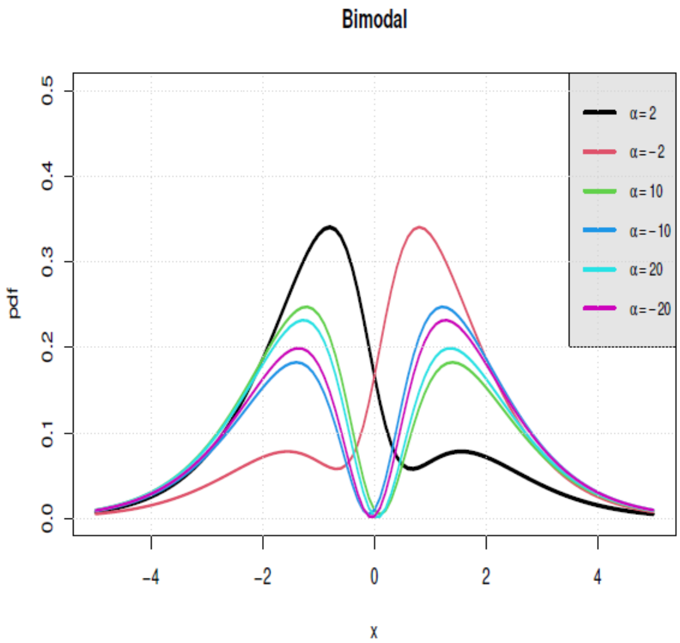

Figure 2. Some bimodal shapes of the PDF of the ASHS distribution for large values of α

In Figure 2, for some large values of α, we observe that the ASHS distribution has the two modes x_1 and x_2. Thus, the ASHS model is ideal for the analysis of bimodal distribution-type data, which are often encountered in applications.

3. MOMENT ANALYSIS

A moment analysis of the ASHS distribution is now performed. Hereafter, we designate by X a random variable with the ASHS distribution as described in Definition 2.1.

Proposition 3.1 *The moment generating function of* X *is given by*

$$M(t) = E(e^{tX}) = \frac{1}{2+\alpha^2}[2\sec(t) - 2\alpha\tan(t)\sec(t) + \alpha^2\sec(t)\tan^2(t) + \alpha^2\sec^3(t)], \quad |t| < 1,$$

where $\sec(t) = 1/\cos(t)$.

Proof: To begin, we can decompose $f_{ASHS}(x;\alpha)$ as

$$f_{ASHS}(x;\alpha) = \frac{1}{2+\alpha^2}[2f_{SHS}(x;\alpha) - 2\alpha x f_{SHS}(x;\alpha) + \alpha^2 x^2 f_{SHS}(x;\alpha)].$$

Therefore, by introducing a random variable Y with the HS distribution, we have

$$\boldsymbol{M}(t) = \int_{-\infty}^{+\infty} e^{tx} f_{ASHS}(x;\alpha)dx$$

$$= \frac{1}{2+\alpha^2}\left[2\int_{-\infty}^{+\infty} e^{tx} f_{SHS}(x;\alpha)dx - 2\alpha\int_{-\infty}^{+\infty} xe^{tx} f_{SHS}(x;\alpha)dx + \alpha^2\int_{-\infty}^{+\infty} x^2e^{tx} f_{SHS}(x;\alpha)dx\right]$$

$$= \frac{1}{2+\alpha^2}[2E(e^{tY}) - 2\alpha E(Ye^{tY}) + \alpha^2 E(Y^2e^{tY})].$$

It follows from Fischer (2013) that $E(e^{tY}) = \sec(t), \quad |t| < 1$, from which we deduce that

$$E(Ye^{tY}) = \frac{\partial E(e^{tY})}{\partial t} = \tan(t)\sec(t)$$

and

$$E(Y^2e^{tY}) = \frac{\partial^2 E(e^{tY})}{\partial t^2} = \sec(t)\tan^2(t) + \sec^3(t).$$

By substitution, we get

$$\boldsymbol{M}(t) = \frac{1}{2+\alpha^2}[2\sec(t) - 2\alpha\tan(t)\sec(t) + \alpha^2\sec(t)\tan^2(t) + \alpha^2\sec^3(t)].$$

The stated result is obtained. □

With a similar approach to Proposition 3.1, we can define the characteristic function of X. This is formulated in the next result.

Proposition 3.2 *The characteristic function of X is given by*

$$\varphi(t) = E\left(e^{itX}\right) = \frac{1}{2+\alpha^2}[2\operatorname{sech}(t) - 2i\alpha\tanh(t)\operatorname{sech}(t) - \alpha^2\operatorname{sech}(t)\tanh^2(t) + \alpha^2\operatorname{sech}^3(t)],$$

where i is the complex number such that $i^2 = -1$, and $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$.

Proof: To begin, without loss of generality, we can link $\varphi(t)$ and $\boldsymbol{M}(t)$ by the following equation: $\varphi(t) = \boldsymbol{M}(it), t \in R$. It follows from Proposition 3.1 that

$$\varphi(t) = \boldsymbol{M}(it) = \frac{1}{2+\alpha^2}[2\sec(it) - 2\alpha\tan(it)\sec(it) + \alpha^2\sec(it)\tan^2(it) + \alpha^2\sec^3(it)].$$

Since $\sec(it) = \operatorname{sech}(t)$ and $\tan(it) = itanh(t)$, we have

$$\varphi(t) = \frac{1}{2+\alpha^2}[2\operatorname{sech}(t) - 2i\alpha\tanh(t)\operatorname{sech}(t) - \alpha^2\operatorname{sech}(t)\tanh^2(t) + \alpha^2\operatorname{sech}^3(t)].$$

This ends the proof of Proposition 3.2. □

The function $\varphi(t)$ caracterizes completely the ASHS distribution, and can be used for further distributional developments, such as theorem limits.

Proposition 3.3 *The mean and variance of X are specified by*

$$E(X) = -\frac{2\alpha}{2+\alpha^2}$$

and

$$V(X) = \frac{5\alpha^4 + 8\alpha^2 + 4}{(2+\alpha^2)^2},$$

respectively.

Proof: The moments of X can be derived from $\boldsymbol{M}(t)$. More precisely, the r-th moment of X is obtained by the following formula: $E(X^r) = \partial^r\boldsymbol{M}(t)/\partial t^r|_{t=0}$. By virtue of Proposition 3.1, we have

$$\frac{\partial\boldsymbol{M}(t)}{\partial t} = \frac{1}{2+\alpha^2}[\alpha\sec^3(t)(5\alpha\tan(t) - 2) + \tan(t)\sec(t)(\alpha\tan(t)(\alpha\tan(t) - 2) + 2)]$$

and

$$\frac{\partial^2\boldsymbol{M}(t)}{\partial t^2} = \frac{1}{2+\alpha^2}\sec(t)\,[\alpha^2 + 24\alpha^2\sec^4(t) - 4\sec^2(t)\,(5\alpha^2 + 3\alpha\tan(t) - 1) + 2\alpha\tan(t) - 2].$$

Since $\sec(0) = 1$ and $\tan(0) = 0$, we immediately get

$$E(X) = \left.\frac{\partial \boldsymbol{M}(t)}{\partial t}\right|_{t=0} = -\frac{2\alpha}{2+\alpha^2}$$

and

$$E(X^2) = \left.\frac{\partial^2 \boldsymbol{M}(t)}{\partial t^2}\right|_{t=0} = \frac{1}{2+\alpha^2}[\alpha^2 + 24\alpha^2 - 4(5\alpha^2 - 1) - 2] = \frac{2+5\alpha^2}{2+\alpha^2}$$

Therefore,

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2+5\alpha^2}{2+\alpha^2} - \left(-\frac{2\alpha}{2+\alpha^2}\right)^2 = \frac{5\alpha^4 + 8\alpha^2 + 4}{(2+\alpha^2)^2}.$$

The desired formula is obtained. □

From Proposition 3.3, we remark that $E(X) = 0$ when $\alpha = 0$ and $E(X)$ tends to 0 when $|\alpha| \to +\infty$. Also, $E(X)$ is maximal when $\alpha = -\sqrt{2}$ with $E(X) = 1/\sqrt{2}$, and minimal when $\alpha = \sqrt{2}$ with $E(X) = 1/\sqrt{2}$. For the variance, we have $V(X) \in [1{,}5[$ with $V(X) = 1$ if and only if $\alpha = 0$, and $V(X)$ tends to 5 when $|\alpha| \to +\infty$. Also, $|V(X)|$ increases as $|\alpha|$ oncreases.

Similarly, but with more mathematical efforts, we can express the moment skewness and kurtosis of X, defined by

$$SK = E\left[\left(\frac{X - E(X)}{\sqrt{V(X)}}\right)^3\right]$$

and

$$KU = E\left[\left(\frac{X - E(X)}{\sqrt{V(X)}}\right)^4\right],$$

respectively. Since they are, however, very fastidious to express analytically, we propose to analyze them with a numerical approach.

Table 1 shows the mean, variance, moment skewness and moment kurtosis of X for some values of the parameter α.

Table 1. The mean, variance, skewness and kurtosis values of X for some selected values of α

α	$E(X)$	$V(X)$	SK	KU
-5000	0.00004	5	-0.0003577628	2.44
-25	0.07974482	4.980882	-0.07137116	2.45410
-9	0.2168675	4.856583	-0.1948623	2.54753
-3	0.5454545	3.975207	-0.4971015	3.30519
0	0	1	0	5
1	-0.6666667	1.888889	0.285336	5.71972
12	-0.1643836	4.918184	0.1474212	2.50085
27	-0.0738714	4.983599	0.06610857	2.45209
5000	-0.00004	5	0.0003577628	2.44

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From Table 1, we see that the skewness is negative or positive according to the fact that α is negative or positive, respectively. Thus, the skewness of the ASHS distribution is completely modulated by the sign and value of α. We can see that the kurtosis is mostly in $[2,5]$, implying that the ASHS distribution could be platykurtic (corresponding to $KU < 3$), almost symmetric (corresponding to $KU \approx 3$), or leptokurtic (corresponding to $KU > 3$).

In practice, as with any one parameter continuous distribution, the ASHS distribution is not flexible enough to capture all the distributional properties behind the data. For this reason, it is natural to consider a scale-location version defined by the distribution of the following random variable:

$$Y = \mu + \sigma X \tag{9}$$

with $\mu \in R$ and $\sigma > 0$. The related PDF is obtained as

$$f_{ASHS}(x; \alpha, \mu, \sigma) = \frac{1}{2\sigma} \frac{(1-\alpha(x-\mu)/\sigma)^2+1}{2+\alpha^2} \operatorname{sech}\left[\frac{\pi}{2}\left(\frac{x-\mu}{\sigma}\right)\right], \qquad x \in R \tag{10}$$

It can be considered as the three-parameter version of the ASHS distribution. For the sake of continuity, we will, however, keep the name of ASHS distribution.

4. ESTIMATION AND APPLICATIONS

This section is devoted to the practice of the ASHS distribution in statistics.

4.1. Estimation

Based on data and the the three-parameter ASHS distribution, we propose to estimate the parameters by the maximum likelihood method. Thus, by denoting $x_1, \ldots, x_n$ the data, assuming that they are all independent realizations of X, we define the likelihood function as

$$L(x_1, \dots, x_n; \alpha, \mu, \sigma) = \prod_{i=1}^{n} f_{ASHS}(x_i; \alpha, \mu, \sigma)$$

$$= \prod_{i=1}^{n} \frac{1}{2\sigma} \frac{(1-\alpha(x_i-\mu)/\sigma)^2+1}{2+\alpha^2} \operatorname{sech}\left[\frac{\pi}{2}\left(\frac{x_i-\mu}{\sigma}\right)\right]. \quad (11)$$

Then, the maximum likelihood estimates (MLEs) of α, μ and σ are obtained as

$$(\hat{\alpha}, \hat{\mu}, \hat{\sigma}) = argmax_{(\alpha,\mu,\sigma)\in R^2 \times (0,+\infty)} L(x_1, \dots, x_n; \alpha, \mu, \sigma). \quad (12)$$

Instead of the maximum likelihood function, the logarithm transformation of the likelihood function can be considered without loss of generality. Due to the complexity of the likelihood function, there is no closed form expression for the MLEs. However, as an optimisation problem, we can still approximate the values of $\hat{\alpha}$, $\hat{\mu}$ and $\hat{\sigma}$ accurately using efficient iterative algorithms, such as the Newton-Raphson algorithm or the Gauss-Newton algorithm. Under certain regularity assumptions, it can be shown that the underlying random estimators are asymptotically efficient, consistent and asymptotically normal. These properties make possible the construction of confidence intervals and statistical tests. We may refer to the book of Casella and Berger (1990) for more details in this regard.

4.2. Methodology

In order to examine the efficiency of the ASHS distribution's modeling for a given data, we will compare it to thee competitors also based on the alpha-skew technique: the three-parameter ASN distribution by Elal-Olivero (2010), three-parameter ASLa distribution by Harandi and Alamatsaz (2013) and three-parameter ASLo distribution by Hazarika and Chakraborty (2014). The three-parameter ASN distribution is defined by the following PDF:

$$f_{ASN}(x; \alpha, \mu, \sigma) = \frac{(1-\alpha(x-\mu)/\sigma)^2+1}{2+\alpha^2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in R, \quad (13)$$

with $\alpha, \mu \in R$ and $\sigma > 0$.

The three-parameter ASLa distributon is defined by the following PDF:

$$f_{ASL_a}(x; \alpha, \mu, \sigma) = \frac{(1-\alpha(x-\mu)/\sigma)^2+1}{2+2\alpha^2} \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}, \qquad x \in R, \quad (14)$$

with $\alpha, \mu \in R$ and $\sigma > 0$.

The three-parameter ASLo distributon is defined by the following PDF:

$$f_{ASL_o}(x; \alpha, \mu, \sigma) = 3\frac{(1-\alpha(x-\mu)/\sigma)^2+1}{6+\pi^2\alpha^2} \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma\left(1+e^{-\frac{x-\mu}{\sigma}}\right)^2}, \qquad x \in R, \quad (15)$$

with $\alpha, \mu \in R$ and $\sigma > 0$.

The involved parameters are supposed to be unknown; we estimate them by the maximum likelihood method. From the obtained estimates, we derive the maximum log-likelihood value denoted by $\hat{\ell}$. Then, we calculate the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) for comparison. The technical formulas for the AIC and BIC are $AIC = 2k - 2\hat{\ell}$ and $BIC = klog(n) - 2\hat{\ell}$ respectively, where k denotes the number of parameters to be estimated and $\hat{\ell}$ denotes the estimated log-likelihood function. In the case of the ASHS distribution, we have $k = 3$ and $\hat{\ell} = \log[L(x_1, \dots, x_n; \hat{\alpha}, \hat{\mu}, \hat{\sigma})]$. The numerical rule is simple; the smaller the values of the AIC and BIC are, the better the distribution fits the data.

The codes will be made with the function optim of the R software (see R Development Core Team (2005)).

4.3. Application to an Astronomical Dataset

The considered dataset was first described in Roeder (1990). It is made up of the velocities of 82 distant galaxies that diverge from our own. The following link contains the dataset:

http://www.stats.bris.ac.uk/peter/mixdata

The data are: (9.172, 9.350, 9.483, 9.558, 9.775, 10.227, 10.406, 16.084, 16.170, 18.419, 18.552, 18.600, 18.927, 19.052, 19.070, 19.330, 19.343, 19.349, 19.440, 19.473, 19.529, 19.541, 19.547, 19.663, 19.846, 19.856, 19.863, 19.914, 19.918, 19.973, 19.989, 20.166, 20.175, 20.179, 20.196, 20.215, 20.221, 20.415, 20.629, 20.795, 20.821, 20.846, 20.875, 20.986, 21.137, 21.492, 21.701, 21.814, 21.921, 21.960, 22.185, 22.209, 22.242, 22.249, 22.314, 22.374, 22.495, 22.746, 22.747, 22.888, 22.914, 23.206, 23.241, 23.263, 23.484, 23.538, 23.542, 23.666, 23.706, 23.711, 24.129, 24.285, 24.289, 24.366, 24.717, 24.990, 25.633, 26.960, 26.995, 32.065, 32.789, 34.279)

The descriptive statistics of this dataset are described in Table 2.

values of the parameters of the TSGHS distribution. We fixed $\mu = 0$and $\sigma = 1$

Table 2. Descriptive statistics of the astronomical data

Mean	Median	Standard deviation	Variance	Skewness	Kurtosis	Minimum	Maximum
20.83146	20.8335	4.568135	20.86785	-0.4309618	5.259059	9.172	34.279

From Table 2, we see that the data are left-skewed and leptokurtic with a variance of 20.86. These distributional data properties are covered by the functional capability of the ASHS distribution. Table 3 shows the estimation and criterion results for the considered distributions.

Table 3. MLEs, $\hat{\ell}$, AIC and BIC related to the considered distributions for the astronomical data

	ASHS	ASN	ASLa	ASLo
$\hat{\alpha}$	-1.375106	0.8475302	0.5950219	0.4970658
$\hat{\mu}$	18.97209	23.80885	22.888	23.21376
$\hat{\sigma}$	2.712994	4.391553	2.250925	1.849911
$\hat{\ell}$	-223.3841	-239.6859	-227.9634	-230.0986
AIC	452.7681	485.3719	461.9268	466.1973
BIC	459.9883	492.5921	469.1469	473.4175

Based on the AIC and BIC, Table 3 indicates that the ASHS distribution is better than the ASN, ASLa and ASLo distributions, with $AIC = 452.7681$ and $BIC = 459.9883$.

Figure 3 illustrates that by showing the graphical representations of thee estimated PDFs of the distributions over the histogram of the data.

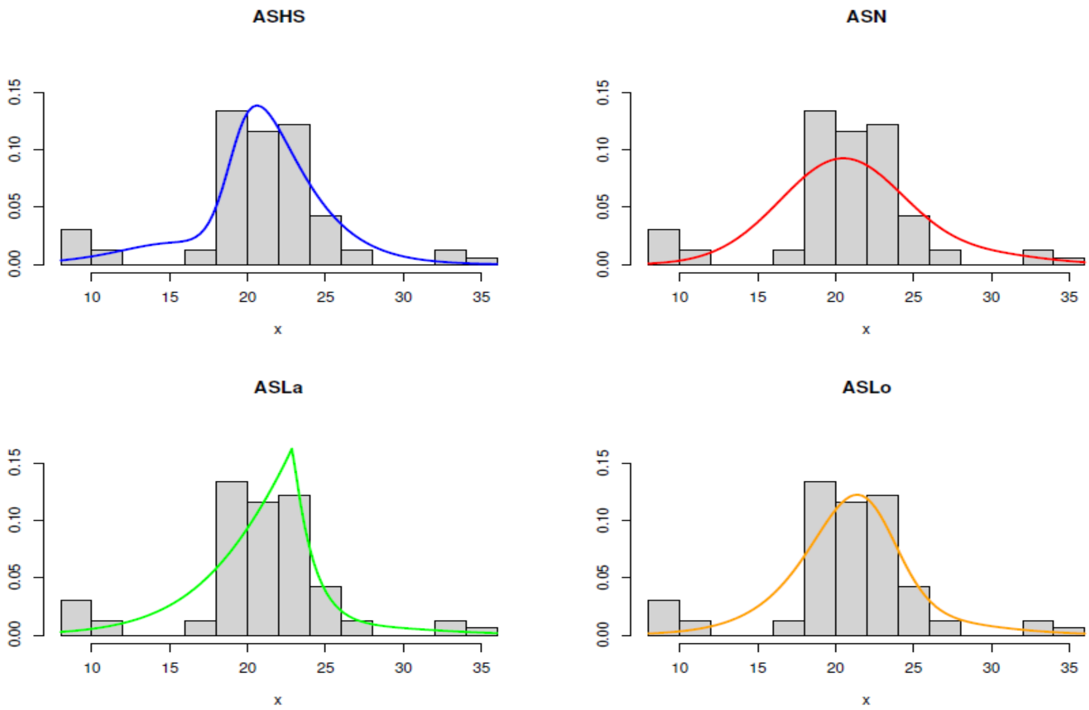

Figure 3. The estimated PDFs of the considered distributions in color over the histogram for the astronomical data

The ASHS distribution matches the data better than the ASN, ASLa and ASLo distributions in Figure 3. In comparison to rival models, fitting with ASHS contains more information, especially on the skewness to the left and the considerations of the tails.

5. CONCLUSION

In this paper, the alpha-skew hyperbolic secant (ASHS) distribution is introduced. It has the advantage of presenting both unimodal and bimodal behavior, as well as various kurtosis levels. Various of its properties are investigated, including characteristic function, moments, and other basic properties. An application of the ASHS distribution to an astronomical dataset is provided to illustrate that this distribution may give a better fit than the alpha-skew normal, alpha-skew-Laplace and alpha-skew logistic distributions in terms of standard criteria. Multivariate versions of this distribution, regression-type models or clustering methods are possible perspectives of this work.

ETHICAL DECLARATION

In the writing process of the study titled “The Alpha-Skew Hyperbolic Secant Distribution With Applications to an Astronomical Dataset”, there were followed the scientific, ethical and the citation rules; was not made any falsification on the collected data and this study was not sent to any other academic media for evaluation.

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