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How to Find a Bézier Curve in $\mathbf{E}^3$

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Abstract
"How to find any n^{th} order Bézier curve if we know its first, second, and third derivatives?" Hence we have examined the way to find the Bézier curve based on the control points with matrix form, while derivatives are given in $\mathbf{E}^3$. Further, we examined the control points of a cubic Bézier curve with given derivatives as an example. In this study first we have examined how to find any n^{th} order Bezier curve with known its first, second and third derivatives, which are inherently, the $(n-1)^{th}$ order, the $(n-2)^{th}$ and the $(n-3)^{th}$ Bezier curves in respective order. There is a lot of the number of Bézier curves with known the derivatives with control points. Hence to find a Bézier curve we have to choose any control point of any derivationİn this study we have chosen two special points which are the initial point P_0 and the endpoint P_n.

Keywords: Bézier curves, Cubic Bezier curves, Derivatives of Bezier curve
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1. Introduction and Preliminaries

A Bézier curve is frequently used in computer graphics and related fields, in vector graphics and in animations as a tool to control motions. Especially, in animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline object's behaviors. Users sketch the desired path in Bézier curves, and the application creates the required frames for an object moving along in that given path. For 3D animation, Bézier curves are often used to define 3D paths as well as 2D curves by key-frame interpolation. We have been motivated by the following studies. First, Bézier-curves with curvature and torsion continuity has been examined in [1]. In [2, 3], Bézier curves are outlined for Computer-Aided Geometric Design. Bézier curves and surfaces have been discussed deeply in [4, 5]. Frenet apparatus of both the n^{th} degree Bézier curves have been examined in $\mathbf{E}^3$, in [6]. The Bishop frame and the alternative frame have been associated with the Bézier curves in [7] and [8], respectively. The matrix forms of the cubic Bézier curve and its involute have been examined in [9] and [10], respectively. Cubic Bézier like curves have been studied with different basis in [11]. 5^{th} order Bézier curve and its, first, second, and third derivatives are examined based on the control points of 5^{th} order Bézier Curve in $\mathbf{E}^3$ by [12]. Further, the Bertrand and the Mannheim partner of a cubic Bézier curve based on the control points with matrix form according to Frenet apparatus have been examined in [13, 14]. Some other couples of Bézier curves have been studied in [15].

Generally, a Bézier curve of an n^{th} degree can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ by the following parametrization:

$$\mathbf{B}(t) = \sum_{i=0}^{n} B_i^n(t)\,[P_i]$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ is known to be Bernstein polynomials and $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ are the binomial coefficients. Bézier curves have some specific properties inherited by Bernstein polynomials. Since $B_i^n(t) \geq 0$ for $t \in [0,1]$ and $i = 0, \ldots, n$, and the polynomials have the property of partition of unity that is $\sum_{i=0}^n B_i^n(t) = 1$, the Bézier curves are invariant to affine transformations. Moreover, the curve lies in the convex hull of its control points by two of these properties. The end point interpolation property ensures that any Bézier curve has the first and the last of control points on it, whereas none of others do not touch the curve, necessarily. Moreover, the recursiveness property and the derivatives of the polynomials lead this study, intrinsically.

2. How to find a Bézier curve with known derivatives

Before responding the main question of this paper, we suggest readers to see [9] and [10], where another question that "How to find the control points of a given Bézier curve?" was studied. To solve the latter, we have referred the matrix form of Bézier curves as it is relatively the simplest representation. Further, it is advised to check the matrix representation of 5^{th} and n^{th} order Bézier Curve and derivatives provided in [12] and [16], respectively. Now, let us consider the main argument "How to find a Bézier curve if we know its first derivative ?" with the background of a knowledge on finding the control points of a given Bézier curve.

Theorem 2.1. *For $t \in [0,1]$, $i \in \mathbf{N_0}$ and $P_i \in \mathbf{E}^3$, a Bézier curve of n^{th} order defined by $\mathbf{B}(t) = \sum_{i=0}^n B_i^n(t)[P_i]$ has the following control points by means of the given its first derivative and the initial point P_0*

$$P_i = P_0 + \frac{Q_0 + Q_1 + Q_2 + \ldots + Q_{i-1}}{n}, \; 1 \leq i \leq n$$
$$P_k = P_{k-1} + \frac{Q_{k-1}}{n}.$$

Proof. The derivative of the any Bézier curve $\mathbf{B}(t)$ is

$$\mathbf{B}'(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1} Q_i$$

where Q_0, $Q_1, \ldots, Q_{n-1}$ are the control points. The first derivative of a n^{th} order Bézier curve has the following matrix representation

$$\alpha'(t) = \begin{bmatrix} t^{n-1} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T [B'] \begin{bmatrix} Q_0 \\ Q_1 \\ . \\ . \\ . \\ Q_{n-1} \end{bmatrix}$$

where $[B']$ is the coefficient matrix of the $(n-1)^{th}$ order Bezier curve which is the derivative of the n^{th} order Bezier curve and the control points Q_0, $Q_1, \ldots, Q_{n-1}$ are

$$\begin{aligned} Q_0 &= n(P_1 - P_0) \\ Q_1 &= n(P_2 - P_1) \\ &\ldots \\ Q_{n-1} &= n(P_n - P_{n-1}). \end{aligned}$$

For more detail see in [16] .There are a lot of number Bézier curves with the first dervatives have these control points. Then we have to choose any initial point. In this study we choose first two special points which are the initial point P_0 and the end point P_n, Let the n^{th} order Bézier curve pass through from a given the initial point P_0, then

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{n} \\
P_2 &= P_1 + \frac{Q_1}{n} \\
P_3 &= P_2 + \frac{Q_2}{n} \\
&\dots \\
P_n &= P_{n-1} + \frac{Q_{n-1}}{n}
\end{aligned}$$

if replace we get all the control points based on the Q_i,

$$\begin{aligned}
P_1 &= P_0 + \frac{Q_0}{n} \\
P_2 &= P_0 + \frac{Q_0}{n} + \frac{Q_1}{n} \\
P_3 &= P_0 + \frac{Q_0}{n} + \frac{Q_1}{n} + \frac{Q_2}{n} \\
&\dots. \\
P_n &= P_0 + \frac{Q_0}{n} + \frac{Q_1}{n} + \frac{Q_2}{n} + \dots + \frac{Q_{n-1}}{n}.
\end{aligned}$$

This complete the proof. □

Corollary 2.2. *The derivative of n^{th} order Bézier curve can not has the origin $(0,0,0)$ as a control point.*

Proof. Let first derivative of n^{th} order Bézier curve has the origin $Q_i = (0,0,0)$

$$\begin{aligned}
Q_i &= n(P_{i+1} - P_i) = (0,0,0) \\
P_{i+1} &= P_i.
\end{aligned}$$

Hence Bézier curve has n control points and cant be n^{th} order Bézier curve. So derivative of n^{th} order Bézier curve cannot has the origin $Q_i(0,0,0)$ as a control point. □

Corollary 2.3. *If the first derivative of n^{th} order Bézier curve with given control points $Q_i, 0 < i < n-1$, is given and n^{th} order Bézier curve has initial point $P_0 = (0,0,0)$,has the following control points*

$$P_i = \frac{Q_0 + Q_1 + Q_2 + \dots + Q_{i-1}}{n}, \ 1 \leq i \leq n.$$

Proof. Since $P_i = P_0 + \frac{Q_0+Q_1+Q_2+\dots+Q_{i-1}}{n}$, $1 \leq i \leq n$, and $P_0 = (0,0,0)$, it is clear that

$$P_i = \frac{Q_0 + Q_1 + Q_2 + \dots + Q_{i-1}}{n}, 1 \leq i \leq n.$$

□

Corollary 2.4. *n^{th} order Bézier curve with given the first derivative and the initial point P_0, under the condition $P_0 = Q_0$, has the following control points*

$$\begin{aligned}
P_i &= \frac{(n+1)P_0}{n} + \frac{Q_1 + Q_2 + \dots + Q_{i-1}}{n}, \ 1 \leq i \leq n \\
P_i &= P_{i-1} + \frac{Q_{i-1}}{n}.
\end{aligned}$$

Theorem 2.5. *The Bézier curve based on the control points a n^{th} order Bézier curve with given the first derivative and the end point P_n, has the following control points as in the following ways*

$$P_{i-1} = P_n - \frac{Q_0 + Q_1 + Q_2 + ... + Q_{i-1}}{n}, \ 1 \leq i \leq n$$
$$P_{i-1} = P_n - \frac{Q_{i-1}}{n}.$$

Proof. If the first derivative of n^{th} order Bézier curve is given,

$$\alpha'(t) = \begin{bmatrix} t^{n-1} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B' \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ . \\ . \\ . \\ Q_{n-1} \end{bmatrix}$$

then the control points $Q_0, Q_1, ..., Q_{n-1}$ are given , where

$$\begin{aligned} Q_0 &= n(P_1 - P_0) \\ Q_1 &= n(P_2 - P_1) \\ &... \\ Q_{n-1} &= n(P_n - P_{n-1}). \end{aligned}$$

Let the n^{th} order Bézier curve passing through the end point P_n, then

$$\begin{aligned} &P_n \text{ is given} \\ P_{n-1} &= P_n - \frac{Q_{n-1}}{n} \\ P_{n-2} &= P_{n-1} - \frac{Q_{n-2}}{n} \\ P_{n-3} &= P_{n-2} - \frac{Q_{n-3}}{n} \\ &... \\ P_2 &= P_3 - \frac{Q_2}{n} \\ P_1 &= P_0 - \frac{Q_1}{n} \\ P_0 &= P_1 - \frac{Q_0}{n} \end{aligned}$$

if replace we get all the control points based on the Q_i,

$$\begin{aligned}
&P_n \text{ is given}\\
P_{n-1} &= P_n - \frac{Q_{n-1}}{n}\\
P_{n-2} &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n}\\
P_{n-3} &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n}\\
&.\\
&.\\
&.\\
P_2 &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n} - \ldots - \frac{Q_2}{n}\\
P_1 &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n} - \ldots - \frac{Q_2}{n} - \frac{Q_1}{n}\\
P_0 &= P_n - \frac{Q_{n-1}}{n} - \frac{Q_{n-2}}{n} - \frac{Q_{n-3}}{n} - \ldots - \frac{Q_2}{n} - \frac{Q_1}{n} - \frac{Q_0}{n}
\end{aligned}$$

This completes the proof. □

Corollary 2.6. *The n^{th} order Bézier curve with given the first derivative and the end point $P_n = (0,0,0)$ has the following control points as in the following ways*

$$P_{i-1} = -\frac{Q_0 + Q_1 + Q_2 + \ldots + Q_{i-1}}{n}, 1 \le i \le n.$$

Corollary 2.7. *The Bézier curve based on the control points a n^{th} order Bézier curve with given the first derivative and the end point P_n, under the condition $P_n = Q_0$, has the following control points as in the following ways*

$$\begin{aligned}
P_{i-1} &= \frac{(n-1)}{n} P_n - \frac{Q_1 + Q_2 + \ldots + Q_{i-1}}{n},\ 1 \le i \le n\\
P_{k-1} &= P_n - \frac{Q_{k-1}}{n}.
\end{aligned}$$

Theorem 2.8. *The n^{th} order Bézier curve with given the first derivative and any point P_k ,$0 < k < n$, is given, has the following control points $P_{k+1}, P_{k+2}, \ldots, P_n$ and $P_0, P_1, \ldots, P_{k-1}$*

$$\begin{aligned}
P_{k+1} &= P_k + \frac{Q_k}{n}\\
P_{k+2} &= P_k + \frac{Q_k}{n} + \frac{Q_{k+1}}{n}\\
&\ldots\\
P_n &= P_k + \frac{Q_k}{n} + \frac{Q_{k+1}}{n} + \ldots \frac{Q_{n-1}}{n}
\end{aligned}$$

$$\begin{aligned}
P_{k-1} &= P_k - \frac{Q_{k-1}}{n}\\
P_{k-2} &= P_{k-} - \frac{Q_{k-1}}{n} - \frac{Q_{k-2}}{n}\\
&\ldots\\
P_0 &= P_k - \frac{Q_{k-1}}{n} - \frac{Q_{k-2}}{n} - \frac{Q_2 + Q_1 + Q_0}{n}.
\end{aligned}$$

Second, lets find the answer of ”How to find a Bézier curve if we know the second derivative ? ”

Theorem 2.9. *The n^{th} order Bézier curve with given the initial point P_0, the initial point Q_0 of the first derivative and the control points $R_0, R_1, ..., R_{n-2}$ of the second derivation, has the following control points as in the following ways*

$$P_1 = P_0 + \frac{Q_0}{n}$$
$$P_i = P_0 + i\frac{Q_0}{n} + \frac{(i-1)R_0}{n(n-1)} + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + 1\frac{R_{i-2}}{n(n-1)},\ 2 \le i \le n.$$

Proof. The second derivative of n^{th} order Bézier curve by using matrix representation is

$$\alpha''(t) = \begin{bmatrix} t^{n-2} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B'' \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ . \\ . \\ . \\ R_{n-2} \end{bmatrix}$$

$$\alpha''(t) = \begin{bmatrix} t^{n-2} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B'' \end{bmatrix} \begin{bmatrix} (n-1)(Q_1 - Q_0) \\ (n-1)(Q_2 - Q_1) \\ . \\ . \\ . \\ (n-1)(Q_{n-1} - Q_{n-2}) \end{bmatrix}$$

Control points $R_0, R_1, ..., R_{n-2}$, and Q_0 are given, we can easily find the $Q_1, Q_2, ..., Q_{n-1}$.

Q_0 is given

$$Q_1 = Q_0 + \frac{R_0}{n-1}$$
$$Q_2 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1}$$
$$Q_3 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + \frac{R_2}{n-1}$$

.

.

.

$$Q_{n-1} = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + ...\frac{R_{n-2}}{n-1}.$$

Also if the initial control point P_0 is given we can find easly control points of n^{th} order Bézier curve

P_0 and Q_0 are given

$$P_1 = P_0 + \frac{Q_0}{n}$$
$$P_2 = P_0 + \frac{2Q_0}{n} + \frac{R_0}{n(n-1)}$$
$$P_3 = P_0 + \frac{3Q_0}{n} + \frac{2R_0}{n(n-1)} + \frac{R_1}{n(n-1)}$$
$$P_4 = P_0 + \frac{4Q_0}{n} + \frac{3R_0}{n(n-1)} + \frac{2R_1}{n(n-1)} + \frac{1R_2}{n(n-1)}$$

...

$$P_i = P_0 + \frac{iQ_0}{n} + \frac{(i-1)R_0}{n(n-1)} + \frac{(i-2)R_1}{n(n-1)} + \frac{(i-3)R_2}{n(n-1)} + ... + \frac{R_{n-2}}{n(n-1)}.$$

□

Corollary 2.10. *The n^{th} order Bézier curve with given the initial point P_0, the initial point Q_0 of the first derivative and the control points $R_0, R_1, ..., R_{n-2}$ of the second derivation, under the condition $P_0 = Q_0 = R_0$, has the following control points as in the following ways*

$$P_i = \frac{(in + n(n-1) - 1)}{n(n-1)} P_0 + \frac{(i-2) R_1}{n(n-1)} + \frac{(i-3) R_2}{n(n-1)} + ... + 1 \frac{R_{i-2}}{n(n-1)}, 2 \leq i \leq n.$$

Proof. Since

$$P_i = P_0 + i\frac{P_0}{n} + \frac{(i-1) P_0}{n(n-1)} + \frac{(i-2) R_1}{n(n-1)} + \frac{(i-3) R_2}{n(n-1)} + ... + 1 \frac{R_{i-2}}{n(n-1)},$$

$$P_i = \frac{n(n-1) P_0 + i(n-1) P_0 + (i-1) P_0}{n(n-1)} + \frac{(i-2) R_1}{n(n-1)} + \frac{(i-3) R_2}{n(n-1)} + ... + 1 \frac{R_{i-2}}{n(n-1)}$$

$$P_i = \frac{n(n-1) + i(n-1) + (i-1)}{n(n-1)} P_0 + \frac{(i-2) R_1}{n(n-1)} + \frac{(i-3) R_2}{n(n-1)} + ... + 1 \frac{R_{i-2}}{n(n-1)}$$

it is clear. □

Now, let us find the answer to "How to find a Bézier curve if we know its third derivative ? "

Theorem 2.11. *The n^{th} order Bézier curve with given the initial point P_0 , the initial point Q_0 of the first derivative , the initial point R_0 of the second derivative and the control points $S_0, S_1, ..., S_{n-3}$ of the third derivative has the following control points as in the following ways*

$$P_i = P_0 + i\frac{Q_0}{n} + \frac{((i-1) + ... + 3 + 2 + 1) R_0}{n(n-1)} + \frac{((i-2) + ... + 3 + 2 + 1) S_0}{n(n-1)(n-2)}$$
$$+ \frac{((i-3) + ... + 3 + 2 + 1) S_1}{n(n-1)(n-2)} + ... + \frac{3 S_{n-4}}{n(n-1)(n-2)} + \frac{1 S_{n-3}}{n(n-1)(n-2)}.$$

Proof. The third derivative of n^{th} order Bézier curve by using matrix representation is

$$\alpha'''(t) = \begin{bmatrix} t^{n-3} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B''' \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ . \\ . \\ . \\ S_{n-3} \end{bmatrix},$$
$$= \begin{bmatrix} t^{n-3} \\ . \\ . \\ . \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B''' \end{bmatrix} \begin{bmatrix} (n-2)(R_1 - R_0) \\ (n-2)(R_2 - R_1) \\ . \\ . \\ . \\ (n-2)(R_{n-2} - R_{n-3}) \end{bmatrix}$$

Control points $S_0, S_1, ..., S_{n-3}$, and R_0 are given, hence solving the following system

R_0 is given,

$$R_1 = R_0 + \frac{S_0}{(n-2)},$$

$$R_2 = R_0 + \frac{S_0}{(n-2)} + \frac{S_1}{(n-2)},$$

$$R_3 = R_0 + \frac{S_0}{(n-2)} + \frac{S_1}{(n-2)} + \frac{S_2}{(n-2)},$$

...

$$R_{n-2} = R_0 + \frac{S_0}{(n-2)} + \frac{S_1}{(n-2)} + ... + \frac{S_{n-3}}{(n-2)}.$$

We can easily find the $R_1, R_2, ..., R_{n-2}$. Also if the initial control point Q_0 of first derivative is given we can find easily Q_i control points of n^{th} order Bézier curve

$$
\begin{aligned}
&Q_0 \text{ is given,}\\
&Q_1 = Q_0 + \frac{R_0}{n-1},\\
&Q_2 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1},\\
&Q_3 = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + \frac{R_2}{n-1},\\
&Q_{n-1} = Q_0 + \frac{R_0}{n-1} + \frac{R_1}{n-1} + ...\frac{R_{n-2}}{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
&Q_0, R_0 \text{ are given,}\\
&Q_1 = Q_0 + \frac{R_0}{(n-1)},\\
&Q_2 = Q_0 + 2\frac{R_0}{(n-1)} + 1\frac{S_0}{(n-1)(n-2)},\\
&Q_3 = Q_0 + 3\frac{R_0}{n-1} + \frac{2S_0}{(n-1)(n-2)} + \frac{S_1}{(n-1)(n-2)},\\
&Q_4 = Q_0 + \frac{4R_0}{n-1} + \frac{3S_0}{(n-1)(n-2)} + \frac{2S_1}{(n-1)(n-2)} + 1\frac{S_2}{(n-1)(n-2)},\\
&\quad ...\\
&Q_{i-1} = Q_0 + \frac{(i-1)R_0}{n-1} + \frac{(i-2)S_0}{(n-1)(n-2)} + \frac{(i-3)S_1}{(n-1)(n-2)}\\
&\quad + ... + \frac{2S_{i-4}}{(n-1)(n-2)} + \frac{1S_{i-3}}{(n-1)(n-2)}
\end{aligned}
$$

the n^{th} order Bézier curve pass through from a given initial point P_0, then

$$
\begin{aligned}
&P_1 = P_0 + \frac{Q_0}{n},\\
&P_2 = P_0 + \frac{2Q_0}{n} + \frac{R_0}{n(n-1)},\\
&P_3 = P_0 + \frac{3Q_0}{n} + \frac{3R_0}{n(n-1)} + \frac{S_0}{n(n-1)(n-2)},\\
&P_4 = P_0 + \frac{4Q_0}{n} + \frac{6R_0}{n(n-1)} + \frac{3S_0}{n(n-1)(n-2)} + \frac{S_1}{n(n-1)(n-2)},\\
&\quad ...\\
&P_n = P_0 + n\frac{Q_0}{n} + \frac{((i-1)+...+3+2+1)R_0}{n(n-1)} + \frac{((i-2)+...+3+2+1)S_0}{n(n-1)(n-2)} + ...\\
&\quad + \frac{3S_{n-4}}{n(n-1)(n-2)} + 1\frac{S_{n-3}}{n(n-1)(n-2)}.
\end{aligned}
$$

□

3. How to find a cubic Bézier curve with known derivatives

In this section as an application we will study on cubic Bézier curves which are defined in $\mathbf{E}^3$. For more detail see in [3] .

Definition 3.1. *A cubic Bézier curve is a special Bézier curve has only four control points P_0, P_1, P_2 and P_3, with the parametrization*

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$$

and matrix form of its the cubic Bézier curve with control points P_0, P_1, P_2, P_3, is

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

Also using the derivatives of a cubic Bézier curve Frenet apparatus $\{T, N, B, \kappa, \tau\}$ have already been given as in the [9]. The first derivative of a cubic Bézier curve by using matrix representation is given by

$$\alpha'(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

where $Q_0 = 3(P_1 - P_0), Q_1 = 3(P_2 - P_1), Q_2 = 3(P_3 - P_2)$ are control points. The second derivative of a cubic Bézier curve in matrix representation is

$$\alpha''(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix}$$

where $R_0 = 6(P_2 - 2P_1 + P_0)$, $R_1 = 6(P_3 - 2P_2 + P_1)$ are control points.

Theorem 3.2. *The cubic Bézier curve with given the first derivative and the initial point P_0, has the following control points*

$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + \frac{Q_0 + Q_1}{3},$$
$$P_3 = P_0 + \frac{Q_0 + Q_1 + Q_2}{3}.$$

Corollary 3.3. *The cubic Bézier curve with given the first derivative and the initial point P_0,under the condition $P_0 = Q_0$,has the following control points*

$$P_1 = P_0 + \frac{P_0}{3} = \frac{4P_0}{3},$$
$$P_2 = P_0 + \frac{P_0 + Q_1}{3} = \frac{4P_0}{3} + \frac{Q_1}{3},$$
$$P_3 = P_0 + \frac{P_0 + Q_1 + Q_2}{3} = \frac{4P_0}{3} + \frac{Q_1 + Q_2}{3}.$$

Theorem 3.4. *The cubic Bézier curve with given the first derivative and the end point P_3, has the following control points*

$$P_2 = P_3 - \frac{Q_2}{3},$$
$$P_1 = P_3 - \frac{Q_2}{3} - \frac{Q_1}{3},$$
$$P_0 = P_3 - \frac{Q_2 + Q_1 + Q_0}{3}.$$

Corollary 3.5. *The cubic Bézier curve with given the first derivative and the end point P_3, under the condition $P_3 = Q_2$ has the following control points*

$$P_2 = P_3 - \frac{P_3}{3} = \frac{2P_3}{3},$$
$$P_1 = \frac{2P_3}{3} - \frac{Q_1}{3},$$
$$P_0 = \frac{2P_3}{3} - \frac{Q_1 + Q_0}{3}.$$

For an example, let us consider the cubic Bézier curve $\alpha(t) = (3t^3 - 6t^2 + 3t, -3t^3 + 3t^2, t^3)$ with the control points $P_0 = (0,0,0), P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (0,0,1)$. (See, Figure 3.1)

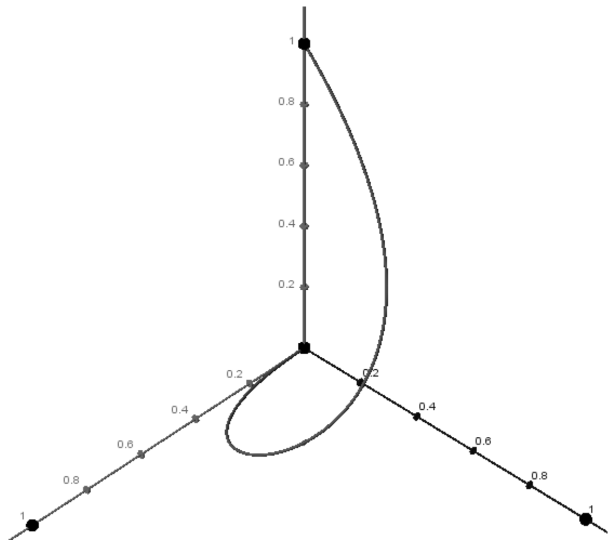

Figure 3.1.
The 3^{rd} order cubic Bézier curve $\alpha(t) = (3t^3 - 6t^2 + 3t, -3t^3 + 3t^2, t^3)$

Example 3.6. *If the first derivative of the cubic Bézier curve is $\alpha'(t) = (9t^2 - 12t + 3, -9t^2 + 6t, 3t^2)$ given. It's matrix representation is*

$$\alpha'(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}.$$

We can find the control points Q_i, $0 < i < 2$ as in the following way easily

$$\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 9 & -9 & 3 \\ -12 & 6 & 0 \\ 3 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -3 & 3 \end{bmatrix}.$$

There are a lot of number Bézier curves with the first dervatives have these control points. Then we have to choose any initial point. To make the correction our example, let the initial point be $P_0 = (0,0,0)$ with $Q_0 = (3,0,0), Q_1 = (-3,3,0), Q_2 = (0,-3,3)$. Since

$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + \frac{Q_0 + Q_1}{3},$$
$$P_3 = P_0 + \frac{Q_0 + Q_1 + Q_2}{3},$$

we get

$$P_1 = (0,0,0) + \frac{(3,0,0)}{3} = (1,0,0),$$
$$P_2 = P_0 + \frac{(3,0,0) + (-3,3,0)}{3} = (0,1,0),$$
$$P_3 = P_0 + \frac{(3,0,0) + (-3,3,0) + (0,-3,3)}{3} = (0,0,1).$$

Now we can write the cubic Bézier curve

$$\begin{aligned}
\alpha(t) &= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \\
&= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3t^3 - 6t^2 + 3t & 3t^2 - 3t^3 & t^3 \end{bmatrix}.
\end{aligned}$$

Let the end point be $P_3 = (0,0,1)$ with $Q_0 = (3,0,0), Q_1 = (-3,3,0), Q_2 = (0,-3,3)$. Since

$$\begin{aligned}
P_2 &= P_3 - \frac{Q_2}{3}, \\
P_1 &= P_3 - \frac{Q_2}{3} - \frac{Q_1}{3}, \\
P_0 &= P_3 - \frac{Q_2 + Q_1 + Q_0}{3},
\end{aligned}$$

$$\begin{aligned}
P_2 &= (0,0,1) - \frac{(0,-3,3)}{3} = (0,1,0), \\
P_1 &= (0,0,1) - \frac{(0,-3,3)}{3} - \frac{(-3,3,0)}{3} = (1,0,0), \\
P_0 &= (0,0,1) - \frac{(0,-3,3) + (-3,3,0) + (3,0,0)}{3} = (0,0,0).
\end{aligned}$$

Let the any point except the initial or the end point be $P_2 = (0,1,0)$ with $Q_0 = (3,0,0), Q_1 = (-3,3,0), Q_2 = (0,-3,3)$ are given. Since

$$\begin{aligned}
P_3 &= P_2 + \frac{Q_2}{3}, \\
&P_2 \text{ is given}, \\
P_1 &= P_2 + \frac{Q_2}{3} - \frac{Q_2}{3} - \frac{Q_1}{3} \\
&= P_2 - \frac{Q_1}{3}, \\
P_0 &= P_2 + \frac{Q_2}{3} - \frac{Q_2 + Q_1 + Q_0}{3} \\
&= P_2 - \frac{Q_1 + Q_0}{3},
\end{aligned}$$

$$\begin{aligned}
P_3 &= (0,1,0) + \frac{(0,-3,3)}{3} = (0,0,1), \\
&P_2 \text{ is given}, \\
P_1 &= P_2 + \frac{Q_2}{3} - \frac{Q_2}{3} - \frac{Q_1}{3} \\
&= (0,1,0) - \frac{(-3,3,0)}{3} = (1,0,0), \\
P_0 &= P_2 + \frac{Q_2}{3} - \frac{Q_2 + Q_1 + Q_0}{3}, \\
&= (0,1,0) - \frac{(-3,3,0) + (3,0,0)}{3} = (0,0,0)
\end{aligned}$$

To find cubic Bézier curve with given second derivative we have the following theorem;

Theorem 3.7. *The cubic Bézier curve with given the second derivative,the initial point Q_0 and the initial point P_0, has the following control points*

$$P_0 \text{ and } Q_0 \text{ are given},$$
$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + 2\frac{Q_0}{3} + \frac{R_0}{6},$$
$$P_3 = P_0 + 3\frac{Q_0}{3} + 2\frac{R_0}{6} + \frac{R_1}{6}.$$

Example 3.8. *The second derivative $\alpha''(t) = (18t-12, -18t+6, 6t)$ of a cubic Bézier curve in matrix representation is*

$$\alpha''(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} 18 & -18 & 6 \\ -12 & 6 & 0 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix}$$

where $R_0 = \begin{pmatrix} -12, & 6, & 0 \end{pmatrix}$ and $R_1 = \begin{pmatrix} 6, & -12, & 6 \end{pmatrix}$ are the control points

$$P_0 = (0,0,0) \text{ and } Q_0 = (3,0,0) \text{are given},$$
$$P_1 = P_0 + \frac{Q_0}{3},$$
$$P_2 = P_0 + 2\frac{Q_0}{3} + \frac{R_0}{6},$$
$$P_3 = P_0 + Q_0 + \frac{R_0}{3} + \frac{R_1}{6},$$

$$P_0 = (0,0,0) \text{ and } Q_0 = (3,0,0) \text{ are given},$$
$$P_1 = (0,0,0) + \frac{(3,0,0)}{3} = (1,0,0),$$
$$P_2 = (0,0,0) + 2\frac{(3,0,0)}{3} + \frac{(-12,6,0)}{6} = (0,1,0),$$
$$P_3 = (0,0,0) + (3,0,0) + \frac{(-12,6,0)}{3} + \frac{(6,-12,6)}{6} = (0,0,1).$$

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