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Araştırma Makalesi/Research Article

D-Recurrent Kropina Spaces With Generalized Douglas Metric

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Abstract- **In this paper, we obtain the necessary and sufficient condition for a D- recurrent Kropina space to be a generalized Douglas space. Further we prove equivalent conditions for a D- recurrent Kropina space with weak Berwald metric.**

Keywords: **D- recurrent Finsler spaces,Douglas metrics, Generalized Douglas metrics, Kropina spaces, Weak Berwald metrics.**

I. INTRODUCTION

Every finsler metric F and G_F^i, the spray coefficent of F, induce a spray $G_F = y^i \frac{\partial}{\partial x^i} - 2G_F^i \frac{\partial}{\partial y^i}$ which determines the geodesics, where $G_F^i = G_F^i(x,y)$ is called the spray coefficients of Finsler metric F. If the spray coefficients G_F^i are in the form

$$G_F^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i \tag{1}$$

then the Finsler metric F is called a Douglas metric. Douglas metrics are Finsler metrics with vanishing Douglas curvature tensor. An $n-$ dimensional Finsler space F_n with Douglas metric is called a Douglas space.

The Douglas space was first introduced by S. Bácsó and M. Matsumoto in [1]. In [2], M. Matsumoto extended theory of finsler spaces with $(\alpha,\beta)-$metric given by [3, 4] and he obtained the necessary and sufficient conditions for some Finsler spaces with an $(\alpha,\beta)-$metric to be a Douglas metric.

In [5], we studied Kropina metric which is a special class of $(\alpha,\beta)-$metric and characterized Kropina spaces with generalized Douglas-Weyl metric.

In this paper, we consider $D-$recurrent Kropina spaces and investigate the conditions for a $D-$recurrent Kropina space to be a generalized Douglas space.

II.PRELIMINARIES

A Minkowski norm on a vector space V is a nonnegative function $F: V \to [0,\infty)$ with following properties:

[1.] $F(y) \geq 0$ for any $y \in V$, and $F(y) = 0 \Leftrightarrow y = 0$,

[2.] $F(\lambda y) = \lambda F(y)$, for any $\lambda > 0$ and any $y \in V$, i.e. F is positivelyn $y-$homogeneous of degree one,

[3.] $F(\mathrm{y})$ is on $V \setminus \{0\}$ such that the matrix

$$g_{ij}(y) = \frac{1}{2}\left[F^2\right]_{y^i y^j}(y) \tag{2}$$

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is positive definite. On an $n-$dimesional manifold M, a Finsler metric F is a C^∞ function on $TM_0 = TM \setminus \{0\}$ such that $F_x = F_{T_xM}$ is a Minkowski norm on T_xM for any $x \in M$.

Let F be a Finsler metric on M. The pair (M, F) is called a Finsler space. In a Finsler space, the metric tensor is given by

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} F^2(x,y), \tag{3}$$

where $x = x^i$ denotes the coordinates of $p \in M$ and $(x,y) = (x^i, y^i)$ denotes the local coordinates of $y \in T_pM$ [6, 7].

In Finsler geometry, Finsler metrics are separated into several classes according to their geometric properties. In this work, we concerned with a Finsler space with Kropina metric.

For a positive C^∞ function on $\phi(s)$ satisfying the condition

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ \left(|s| \le b_0 \text{ on } (-b_0, b_0)\right), \tag{4}$$

where $b^2 = a^{ij}(\mathrm{x}) b_i b_j = \mathrm{b_i b^i}$, $b_i = b_i(\mathrm{x})$. If $\|\beta_x\|_\alpha = \sqrt{a^{ij} b_i b_j} < b_0$ for any $x \in M$ then the function defined by

$$F = \alpha\,\phi(s), \quad s = \frac{\beta}{\alpha}, \quad \left(\alpha = \alpha(x,y), \beta = \beta(x,y)\right), \tag{5}$$

is a metric and it is called a $(\alpha, \beta)-$metric, where $\alpha(x,y) = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric, $\beta(x,y) = b_i(x) y^i$ is a $1-$form[3].

Kropina metric is a special class of $(\alpha, \beta)-$metrics having the form $F = \dfrac{\alpha^2}{\beta}$.

Lemma 2.1. *The relation between the spray coefficients of a Kropina metric G_F^i and the spray coefficients of α Riemannian metric G_α^i is*

$$G_F^i = G_\alpha^i - \frac{\alpha^2}{2\beta} s_0^i - \frac{\alpha^2 s_0 + \beta r_{00}}{\alpha^2 b^2} y^i + \frac{\alpha^2 s_0 + \beta r_{00}}{2\beta b^2} b^i, \tag{6}$$

where $r_{ij} = \frac{1}{2}\left(b_{i;j} + b_{j;i}\right)$, $s_{ij} = \frac{1}{2}\left(b_{i;j} - b_{j;i}\right)$, $s_j^i = a^{ir} s_{rj}$, $s_0^i = s_j^i y^j$, $s_j = s_{ij} b^i$, $s_0 = s_i y^i$, $r_{00} = r_{ij} y^i y^j$,

$b^2 = a^{ij} b_i b_j$ and $b_{i;j}$ denotes the covariant derivatives of b_i with respect to α [5].

A Douglas metric is a Finsler metric having the spray coefficients in the form

$$G_F^i = G_F^i(\mathrm{x}, \mathrm{y}) = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k + P(x,y) y^i. \tag{7}$$

In [8], it is shown that a Finsler space is of Douglas metric if and only if the Douglas tensor defined by

$$D^h_{ijk} = G^h_{Fijk} - \frac{1}{n+1}\left(y^h \frac{\partial G_{Fij}}{\partial y^k} + \delta^h_i G_{Fjk} + \delta^h_k G_{Fij} + \delta^h_j G_{Fki} \right) \tag{8}$$

vanishes identically.

A Finsler metric F is said to be a generalized Douglas metric, if F satisfies the condition

$$h_i^p D^i_{jkl|m} y^m = 0, \tag{9}$$

where $h_i^p = \delta_i^p - l^p l_i, l^p = \dfrac{y^p}{F}, l_i = \dfrac{\partial F}{\partial y^i} = F_{y^i}$ and "|" denotes covariant derivative with respect to the Berwald connection $B\Gamma(G^i_{Fjk}, G^i_{Fj}, 0)$.

The geometric meaning of the above identity is that the rate of change Douglas curvature D^i_{jkl} along a geodesic is tangent to the geodesic[9].

It is clear that we have the following results for $D^i_{jkl} = 0$ or G^i_{jk} are quadratic.

Corollary 2.1. *Every Douglas space is a generalized Douglas space.*
Corollary 2.2. *Every Berwald space is a generalized Douglas space.*

III. D-RECURRENT KROPINA SPACES

Let $\phi(x, y)$ is a positively homegeneous function of degree one in y. If the Douglas tensor of a Finsler space F_n satisfies the condition

$$D^i_{jkl|m}\, y^m = \phi(x, y)\, D^i_{jkl}\,. \tag{10}$$

F_n is called $D-$recurrent space[10].

Suppose that $F_n(n > 2)$ is a $D-$recurrent Kropina space with generalized Douglas metric. Then, we have

$$h_i^p\, D^i_{jkl|m}\, y^m = h_i^p\, \phi(x, y)\, D^i_{jkl} = 0 \tag{11}$$

$$h_i^p D^i_{jkl} = 0 \tag{12}$$

for $\phi(\mathrm{x}, \mathrm{y}) \neq 0$.

Substituting (6) into (8), the Douglas tensor D^i_{jkl} of Kropina space reduces to

$$D^i_{jkl} = A^i_m(y^m F_{jkl} + F_{jk}\delta^m_l + F_{jl}\delta^m_k + F_{kl}\delta^m_j) \tag{13}$$

where

$$A^i_m(x) = \frac{s_m b^i - b^2 s^i_m}{2b^2},\ A^i_0 = A^i_m y^m = \frac{s_0 b^i - b^2 s^i_0}{2b^2},$$

$$F_{jk} = 2\beta^{-2}\left\{a_{jk}\beta + \alpha^2\beta^{-1}b_k b_j - (y_j b_k + y_k b_j)\right\}$$

and

$$F_{jkl} = -2\beta^{-3}\left\{(a_{jk}b_l + a_{lj}b_k + a_{kl}b_j)\beta + 3\alpha^2\beta^{-1}b_j b_k b_l - 2(y_j b_k b_l + y_l b_j b_k + y_k b_l b_j)\right\}$$

Contracting (12) with y_p and using (13) we get

$$(s_m\beta + b^2 s_{m0})\left(y^m F_{jkl} + \delta^m_j F_{kl} + \delta^m_k F_{jl} + \delta^m_l F_{kj}\right) = 0$$

which yields the polynomial equation

$$\beta^3(A_0) + \beta^2(A_1) + \beta(\mathrm{A}_2) + (\beta B_0 + B_1)\alpha^2 = 0 \tag{14}$$

where,

$$A_0 = s_j a_{kl} + s_l a_{jk} + s_k a_{lj}$$

$$A_1 = -s_0(a_{jk}b_l + a_{lj}b_k + b_j a_{kl}) - s_l(y_j b_k + y_k b_j) - s_k(y_l b_j + y_j b_l) - s_j(y_l b_k + y_k b_l)$$
$$+ b^2(s_{j0}a_{kl} + s_{l0}a_{jk} + s_{k0}a_{lj})$$

$$A_2 = -b^2\left(s_{l0}(y_j b_k + y_k b_j) + s_{k0}(y_l b_j + y_j b_l) + s_{j0}(y_l b_k + y_k b_l)\right) + 2s_0(y_j b_l b_k + y_k b_l b_j + y_l b_j b_k)$$

$$B_0 = s_l b_j b_k + s_k b_l b_j + s_j b_k b_l$$

$$B_1 = b^2(s_{l0}b_j b_k + s_{k0}b_l b_j + s_{j0}b_k b_l) - 3s_0 b_j b_k b_l .$$

Transvection (14) by $b^j b^k$ gives

$$b^2(\alpha^2 b^2 - \beta^2)\left(b^2 s_{l0} - s_0 b_l + s_l \beta\right) = 0 \qquad (15)$$

It is clear that $b^2 = 0$ and $n = 2$ in case of $\alpha^2 b^2 - \beta^2 = 0$ [11, 12]. Then $\alpha^2 b^2 - \beta^2 \neq 0$ for $n > 2$. In this case for $n > 2$ or $b^2 \neq 0$ from (15). We get

$$b^2 s_{l0} - s_0 b_l + s_l \beta = 0$$

or

$$s_{kl} = \frac{1}{b^2}(s_l b_k - s_k b_l). \qquad (16)$$

On the other hand, the condition (16) is the necessary and sufficient condition for a Kropina space to be a Douglas space [2]. Since every Douglas space is a generalized Douglas space we have the equation

$$h_i^p D^i_{jkl|m}\, y^m = 0$$

under the condition (16). Thus, we prove that,

Theorem 3.1. *Every $D-$recurrent Kropina space $F_n(n > 2)$ has a generalized Douglas metric if and only if the condition*

$$s_{ij} = \frac{1}{b^2}\left(b_i s_j - b_j s_i\right) \qquad (17)$$

holds.

According to [13], we have the following lemmas.

Lemma 3.1. *Every Kropina space $F_n(n > 2)$ is a weak Berwald space if and only if $r_{00} = c(x)\alpha^2$, where $c = c(x)$ is a scalar function in x.*

Lemma 3.2. *Every Kropina space $F_n(n > 2)$ is a Berwald space if and only if the conditions*

$$r_{00} = c(x)\alpha^2 \text{ and } s_{ij} = \frac{s_j b_i - s_i b_j}{b^2}$$

hold.

Lemma 3.1. and 3.2. lead to the following results

Corollary 3.1. *Every Berwald space with Kropina metric is a D-recurrent Finsler space.*

Corollary 3.2. *$D-$recurrent Kropina spaces are a subclass of Douglas spaces.*

Theorem 3.2. For a $D-$recurrent Kropina space $F_n(n > 2)$ with weak Berwald metric, the followings are equivalent:

[1.] F_n is a Berwald space,

[2.] F_n is a generalized Douglas space, [3.] For F_n, $s_{ij} = \dfrac{s_j b_i - s_i b_j}{b^2}$,

[4.] F_n is a Douglas space.

Proof.

$[1.] \Leftrightarrow [2.]$. Assume [1.] holds. Every Berwald space is a Douglas space and a generalized Douglas space. Conversely, if F_n is a generalized Douglas space. From Corollary 2.2, it is clear that, F_n is a Berwald space.

$[2.] \Leftrightarrow [3.]$. Assume [2.] holds. According to the Theorem 3.1. the condition given in [3.] is satisfied. Conversely, if a Kropina space F_n has $s_{ij} = \dfrac{s_j b_i - s_i b_j}{b^2}$, according to M. Matsumoto[2] and Corollary 2.1, F_n is a Douglas space and so, F_n is a generalized Douglas space .

$[3.] \Leftrightarrow [4.]$. Assume [3.] holds. According to [2], a Kropina space is a Douglas space under the condition $s_{ij} = \dfrac{s_j b_i - s_i b_j}{b^2}$. Conversely, if a Kropina space F_n is a Douglas space, according to [2], for F_n, the condition $s_{ij} = \dfrac{s_j b_i - s_i b_j}{b^2}$ is hold.

$[4.] \Leftrightarrow [1.]$. Assume [4.] holds. According to hypothesis, since $r_{00} = c(x)\alpha^2$ to Lemma 3.2, space is Berwald space. Conversely, if F_n is a Berwald space, since every Berwald space is a Douglas space, F_n is a Douglas space. (Corollary 2.1). Therefore we have proved.

IV. CONCLUSIONS

In this work, we found necessary and sufficient condition for which a $D-$recurrent Kropina space is generalized Douglas metric, and proved. In addition to this, it was shown that, $F_n(n > 2)$ Kropina space with $D-$recurrent and weak Berwald metric satisfies equivalent states.

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