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Research Article

Approximation by statistical convergence with respect to power series methods

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Abstract

In the present work, using statistical convergence with respect to power series methods, we obtain various Korovkin-type approximation theorems for linear operators defined on derivatives of functions. Then we give an example satisfying our approximation theorem. We study certain rate of convergence related to this method. In the final section we summarize these results to emphasize the importance of the study.

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1. Introduction

Statistical convergence which is a regular non-matrix summability method is effective in "summing" non-convergent sequences ([12,23]). Recently, its use in approximation theory has been considered in [13]. Gadjiev and Orhan proved a Korovkin type theorem by considering statistical convergence instead of ordinary convergence in that work. By relaxing the positivity condition on linear operators, various approximation theorems have also been obtained. For instance, Duman and Anastassiou [2,3] relaxed the positivity condition of linear operators in the Korovkin-type approximation theory via the concept of statistical convergence. Following these studies many authors have given several approximation results via summability theory and convergence methods (see, e.g., [4,5,9–11,17,18,28,29]).

Korovkin type approximation theory deals with the convergence of the sequences of the linear operators to the identity operator ([1,14]). If the classical and some other convergence methods do not work, then it would be beneficial to use $P-$statistical convergence. Ünver and Orhan [30] have been investigated the relationship not only between these concepts but also between the concepts of statistical convergence defined by power series methods and defined by classical methods and showed that $P-$statistical convergence methods and statistical convergence are imcompatible (see, also [6–8,16,20,21,24]).

In the present work the main aim is to study some Korovkin-type approximation theorems for linear operators defined on derivatives of functions via $P-$statistical convergence

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method. By giving an appropriate application, we provide some graphs in order to illustrate the efficiency of our result when it is compared with other results in the literature. It should also be noted that we study the rate of convergence. In the final section we summarize the results obtained in present paper.

We pause to collect some basic concepts and notations:

Definition 1.1. Let (p_j) be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{j=0}^{\infty} p_j t^j$$

has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j = L$$

exists then we say that $x = (x_j)$ is convergent in the sense of power series method ([15,22]). It is worthwhile to point out that the method is regular if and only if $\lim_{0<t\to R^-} \frac{p_j t^j}{p(t)} = 0$ for every j.

This convergence method is a general version of Abel and Borel summability methods. Korovkin type theorems related to these methods can be found in [19, 25–27].

Definition 1.2. [30] Let P be a regular power series method and let $E \subset \mathbb{N}_0$. If the limit

$$\delta_P(E) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in E} p_j t^j$$

exists then $\delta_P(E)$ is called the $P-$density of E.

Note that from the definitions of power series method and $P-$density it is obvious that $0 \leq \delta_P(E) \leq 1$ whenever it exists (see also [30]).

Definition 1.3. [30] Let $x = (x_j)$ be a real sequence and let P be a regular power series method. Then x is said to be $P-$statistically convergent to L if for any $\varepsilon > 0$

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in E_\varepsilon} p_j t^j = 0$$

where $E_\varepsilon = \{j \in \mathbb{N}_0 : |x_j - L| \geq \varepsilon\}$ that is, $\delta_P(E_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case we write $st_P - \lim x = L$.

2. Approximation properties via $P-$statistical convergence

Let k be a nonnegative integer. By $C^k[0,1]$, we denote the space of the $k-$times continuously differentiable functions on $[0,1]$ endowed with the sup-norm $\|.\|$. Then throughout the paper we consider the following function spaces:

$$\begin{aligned}
C_+^1 &= \left\{ f \in C^1[0,1] : f' \geq 0 \right\} & C_+ &= \{ f \in C[0,1] : f \geq 0 \} \\
C_+^2 &= \left\{ f \in C^2[0,1] : f'' \geq 0 \right\} & C_{+,1} &= \{ f \in C^1[0,1] : f \geq 0 \} \\
C_-^1 &= \left\{ f \in C^1[0,1] : f' \leq 0 \right\} & C_{+,2} &= \{ f \in C^2[0,1] : f \geq 0 \} \\
C_-^2 &= \left\{ f \in C^2[0,1] : f'' \leq 0 \right\} & &
\end{aligned}$$

We assume here and throughout the paper that the power series method is regular and the test functions are

$$f_i(x) = x^i, \ i = 0, 1, 2, 3, 4.$$

Also, we denote the value of $T(f)$ at a point $x \in [0,1]$ by $T(f(y); x)$ or, briefly, $T(f; x)$.

Theorem 2.1. *Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself. Assume that*

$$\delta_P\left(\left\{j\in\mathbb{N}_0 : T_j\left(C_{+,2}\cap C_+^2\right)\subset C_{+,2}\right\}\right)=1. \tag{2.1}$$

Then

$$st_P-\lim\|T_j(f_i)-f_i\|=0 \text{ for } i=0,1,2 \tag{2.2}$$

if and only if

$$st_P-\lim\|T_j(f)-f\|=0 \text{ for all } f\in C^2[0,1]. \tag{2.3}$$

Proof. First we assume that $T_j(f)$ is P-statistically convergent to f for every $f\in C^2[0,1]$. Since $f_i\in C^2[0,1]$, $i=0,1,2$, $T_j(f_i)$ is P-statistically convergent to f_i for each $i=0,1,2$. Therefore, only the sufficiency part does really require a proof. Let $x\in[0,1]$ be fixed and let $f\in C^2[0,1]$. Since f is bounded and uniformly continuous on $[0,1]$, for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$-\varepsilon-\frac{2M_1\beta}{\delta^2}\varphi_x(y)\le f(y)-f(x)\le\varepsilon+\frac{2M_1\beta}{\delta^2}\varphi_x(y) \tag{2.4}$$

holds for all $y\in[0,1]$ and for any $\beta\ge1$ where $M_1=\|f\|$ and $\varphi_x(y)=(y-x)^2$.

Then by (2.4) we get that

$$h_{1,\beta}(y):=\varepsilon+\frac{2M_1\beta}{\delta^2}\varphi_x(y)+f(y)-f(x)\ge0, \tag{2.5}$$

$$h_{2,\beta}(y):=\varepsilon+\frac{2M_1\beta}{\delta^2}\varphi_x(y)-f(y)+f(x)\ge0. \tag{2.6}$$

Also, for all $y\in[0,1]$,

$$h''_{1,\beta}(y):=\frac{4M_1\beta}{\delta^2}+f''(y) \text{ and } h''_{2,\beta}(y):=\frac{4M_1\beta}{\delta^2}-f''(y).$$

Because of f'' is bounded on $[0,1]$, we can choose $\beta\ge1$ so that $h''_{1,\beta}(y)\ge0$, $h''_{2,\beta}(y)\ge0$, for each $y\in[0,1]$. Hence $h_{1,\beta},\ h_{2,\beta}\in C_{+,2}\cap C_+^2$ and let

$$E_1:=\left\{j\in\mathbb{N}_0 : T_j\left(C_{+,2}\cap C_+^2\right)\subset C_{+,2}\right\}.$$

By (2.1) it is clear that $\delta_P(E_1)=1$ and so $\delta_P(\mathbb{N}_0\setminus E_1)=0$. Then we can write

$$T_j(h_{i,\beta};x)\ge0, \text{ for every } j\in E_1,\ i=1,2. \tag{2.7}$$

From (2.5) – (2.7) and by using similar arguments as in the proof of Theorem 2.1 in [2]; for every $\varepsilon>0$, we get

$$\|T_j(f)-f\|\le\varepsilon+A_1\sum_{i=0}^{2}\|T_j(f_i)-f_i\| \tag{2.8}$$

where $A_1=\max\left\{\varepsilon+M_1+\frac{2M_1\beta}{\delta^2},\frac{4M_2\beta}{\delta^2}\right\}$. Now for a given $r>0$, choose an $\varepsilon>0$ such that $\varepsilon<r$, and consider the following sets:

$$F:=\{j\in\mathbb{N}_0 : \|T_j(f)-f\|\ge r\},$$

$$F_i:=\left\{j\in\mathbb{N}_0 : \|T_j(f_i)-f_i\|\ge\frac{r-\varepsilon}{3A_1}\right\},\ i=0,1,2.$$

Then it follows from (2.8) that

$$F\cap E_1\subset\bigcup_{i=0}^{2}(F_i\cap E_1),$$

which gives that

$$\frac{1}{p(t)}\sum_{j\in F\cap E_1}p_jt^j\leq\frac{1}{p(t)}\sum_{i=0}^{2}\left(\sum_{j\in F_i\cap E_1}p_jt^j\right)\leq\frac{1}{p(t)}\sum_{i=0}^{2}\left(\sum_{j\in F_i}p_jt^j\right). \tag{2.9}$$

Now letting $0<t\to R^-$ in the both sides of (2.9) and using (2.2), we immediately get that

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{j\in F\cap E_1}p_jt^j=0. \tag{2.10}$$

Furthermore since

$$\begin{aligned}\frac{1}{p(t)}\sum_{j\in F}p_jt^j &= \frac{1}{p(t)}\sum_{j\in F\cap E_1}p_jt^j+\frac{1}{p(t)}\sum_{j\in F\cap(\mathbb{N}_0\backslash E_1)}p_jt^j\\ &\leq \frac{1}{p(t)}\sum_{j\in F\cap E_1}p_jt^j+\frac{1}{p(t)}\sum_{j\in(\mathbb{N}_0\backslash E_1)}p_jt^j\end{aligned}$$

holds for every $j\in\mathbb{N}_0$, taking again limit $0<t\to R^-$ in the last inequality and hence it follows from hypothesis and the inequailty (2.10) that

$$st_P-\lim\|T_j(f)-f\|=0.$$

□

Theorem 2.2. *Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself. Assume that*

$$\delta_P\left(\left\{j\in\mathbb{N}_0:T_j\left(C_{+,2}\cap C_-^2\right)\subset C_-^2\right\}\right)=1. \tag{2.11}$$

Then

$$st_P-\lim\left\|[T_j(f_i)]''-f_i''\right\|=0,\ \textit{for } i=0,1,2,3,4 \tag{2.12}$$

if and only if

$$st_P-\lim\left\|[T_j(f)]''-f''\right\|=0,\ \textit{for all } f\in C^2[0,1]. \tag{2.13}$$

Proof. It is obvious that (2.13) implies that (2.12). Let $f\in C^2[0,1]$ and $x\in[0,1]$ be fixed. Based on the proof of Theorem 2.1, this can be found as follows:

For every $\varepsilon>0$, there exists a $\delta>0$ such that

$$-\varepsilon-\frac{2M\beta}{\delta^2}\sigma_x''(y)\leq f''(y)-f''(x)\leq\varepsilon+\frac{2M\beta}{\delta^2}\sigma_x''(y) \tag{2.14}$$

holds for all $y\in[0,1]$ and for any $\beta\geq1$ where $M_2=\left\|f''\right\|$ and $\sigma_x(y)=-\frac{(y-x)^4}{12}+1.$

Consider the following functions on $[0,1]$:

$$u_{1,\beta}(y):=\frac{2M_2\beta}{\delta^2}\sigma_x(y)+f(y)-\frac{\varepsilon}{2}y^2-\frac{f''(y)}{2}y''\geq0,$$

$$u_{2,\beta}(y):=\frac{2M_2\beta}{\delta^2}\sigma_x(y)-f(y)-\frac{\varepsilon}{2}y^2+\frac{f''(y)}{2}y''\geq0.$$

Also then by (2.14) and for all $y\in[0,1]$,

$$u_{1,\beta}''(y)\leq0 \text{ and } u_{2,\beta}''(y)\leq0,$$

which gives $u_{1,\beta},\,u_{2,\beta}\in C_-^2$ and observe that $\sigma_x(y)\geq\frac{11}{12}$ for all $y\in[0,1]$. Then inequality

$$\frac{\left(\pm f(y)+\frac{\varepsilon}{2}\pm\frac{f''(x)}{2}y^2\right)\delta^2}{2M_2\sigma_x(y)}\leq\frac{(M_1+M_2+\varepsilon)\,\delta^2}{M_2} \tag{2.15}$$

holds for all $y \in [0,1]$, where $M_1 = \|f\|$ and $M_2 = \left\|f''\right\|$ as mentioned before. As we know, we can choose $\beta \geq 1$ such that $u_{1,\beta}(y) \geq 0$, $u_{2,\beta}(y) \geq 0$, for each $y \in [0,1]$ and hence $u_{1,\beta}$, $u_{2,\beta} \in C_{+,2} \cap C_{-}^{2}$. Then let

$$E_2 := \left\{ j \in \mathbb{N}_0 : T_j \left(C_{+,2} \cap C_{-}^{2} \right) \subset C_{-}^{2} \right\}.$$

By (2.11) it is clear that $\delta_P(E_2) = 1$ and so $\delta_P(\mathbb{N}_0 \setminus E_2) = 0$. Then we can write

$$\left[T_j(u_{i,\beta}; x)\right]'' \leq 0, \text{ for every } j \in E_2 \text{ and } i = 1,2.$$

Then we get

$$\frac{2M_2\beta}{\delta^2}\left[T_j(\sigma_x;x)\right]'' + \left[T_j(f;x)\right]'' - \frac{\varepsilon}{2}\left[T_j(f_2;x)\right]'' - \frac{f''(x)}{2}\left[T_j(f_2;x)\right]'' \leq 0,$$

$$\frac{2M_2\beta}{\delta^2}\left[T_j(\sigma_x;x)\right]'' - \left[T_j(f;x)\right]'' - \frac{\varepsilon}{2}\left[T_j(f_2;x)\right]'' + \frac{f''(x)}{2}\left[T_j(f_2;x)\right]'' \leq 0.$$

Observe that in view of $\sigma_x \in C_{+,2} \cap C_{-}^{2}$, then we can get $T_j(\sigma_x;x) \leq 0$ and using this

$$\begin{aligned}\left|\left[T_j(f;x)\right]'' - f''(x)\right| \leq & -\frac{2M_2\beta}{\delta^2}\left[T_j(\sigma_x;x)\right]'' + \frac{\varepsilon}{2}\left[T_j(f_2;x)\right]'' \\ & + \frac{\left|f''(x)\right|}{2}\left|\left[T_j(f_2;x)\right]'' - 2\right|.\end{aligned}$$

Thus

$$\begin{aligned}\left|\left[T_j(f;x)\right]'' - f''(x)\right| \leq & \, \varepsilon + \frac{\varepsilon + \left|f''(x)\right|}{2}\left|\left[T_j(f_2;x)\right]'' - f_2''(x)\right| \\ & + \frac{2M_2\beta}{\delta^2}\left[T_j(-\sigma_x;x)\right]''. \end{aligned} \tag{2.16}$$

Now we compute the quantity $T_j(-\sigma_x;x)$ inequality (2.16). To see this observe that

$$\begin{aligned}\left[T_j(-\sigma_x;x)\right]'' = & \left[T_j\left(\frac{(y-x)^4}{12} - 1; x\right)\right]'' \\ \leq & \frac{1}{12}\left[T_j(f_4;x)\right]'' - \frac{x}{3}\left[T_j(f_3;x)\right]'' + \frac{x^2}{2}\left[T_j(f_2;x)\right]'' - \frac{x^3}{3}\left[T_j(f_1;x)\right]'' \\ & + \left(\frac{x^4}{12} - 1\right)\left[T_j(f_0;x)\right]'' \\ = & \frac{1}{12}\left\{\left[T_j(f_4;x)\right]'' - f_4''(x)\right\} - \frac{x}{3}\left\{\left[T_j(f_3;x)\right]'' - f_3''(x)\right\} \\ & + \frac{x^2}{2}\left\{\left[T_j(f_2;x)\right]'' - f_2''(x)\right\} \\ & - \frac{x^3}{3}\left\{\left[T_j(f_1;x)\right]'' - f_1''(x)\right\} + \left(\frac{x^4}{12} - 1\right)\left\{\left[T_j(f_0;x)\right]'' - f_0''(x)\right\}.\end{aligned} \tag{2.17}$$

Combining this with (2.12) and following the same steps as in the proof of Theorem 2.2 in [2]; for every $\varepsilon > 0$, we get

$$\left\|\left[T_j(f)\right]'' - f''\right\| \leq \varepsilon + A_2\sum_{i=0}^{4}\left\|\left[T_j f_i\right]'' - f_i''\right\| \tag{2.18}$$

where $A_2 = \left\{ \frac{\varepsilon + M_2}{2} + \frac{M_2 \beta}{\delta} \right\}$ and $M_2 = \left\| f'' \right\|$ as stated before. Now for a given $r > 0$, choose an $\varepsilon > 0$ such that $\varepsilon < r$, and define the following sets:

$$G := \left\{ j \in \mathbb{N}_0 : \left\| [T_j (f)]'' - f'' \right\| \geq r \right\},$$

$$G_i := \left\{ j \in \mathbb{N}_0 : \left\| [T_j (f_i)]'' - f_i'' \right\| \geq \frac{r - \varepsilon}{5A_2} \right\}, \ i = 0, 1, 2, 3, 4.$$

In this case, by (2.18),

$$G \cap E_2 \subset \bigcup_{i=0}^{4} (G_i \cap E_2),$$

which gives for every $j \in \mathbb{N}_0$, that

$$\frac{1}{p(t)} \sum_{j \in G \cap E_2} p_j t^j \leq \frac{1}{p(t)} \sum_{i=0}^{4} \left(\sum_{j \in G_i \cap E_2} p_j t^j \right) \leq \frac{1}{p(t)} \sum_{i=0}^{4} \left(\sum_{j \in G_i} p_j t^j \right). \tag{2.19}$$

Now letting $0 < t \to R^-$ in the both sides of (2.19) and using (2.8), we immediately get that

$$\lim_{0 < t \to R^-} \frac{1}{p(t)} \sum_{j \in G \cap E_2} p_j t^j = 0. \tag{2.20}$$

Furthermore since

$$\begin{aligned} \frac{1}{p(t)} \sum_{j \in G} p_j t^j &= \frac{1}{p(t)} \sum_{j \in G \cap E_2} p_j t^j + \frac{1}{p(t)} \sum_{j \in G \cap (\mathbb{N}_0 \setminus E_2)} p_j t^j \\ &\leq \frac{1}{p(t)} \sum_{j \in G \cap E_2} p_j t^j + \frac{1}{p(t)} \sum_{j \in (\mathbb{N}_0 \setminus E_2)} p_j t^j \end{aligned}$$

holds for every $j \in \mathbb{N}_0$, taking again limit $0 < t \to R^-$ in the last inequality, it follows from hypothesis and the inequailty (2.20) that

$$st_P - \lim \left\| [T_j (f)]'' - f'' \right\| = 0.$$

□

Theorem 2.3. *Let (T_j) be a sequence of linear operators from $C^1[0,1]$ into itself. Assume that*

$$\delta_P \left(\left\{ j \in \mathbb{N}_0 : T_j \left(C_{+,1} \cap C_+^1 \right) \subset C_+^1 \right\} \right) = 1. \tag{2.21}$$

Then

$$st_P - \lim \left\| [T_j (f_i)]' - f_i' \right\| = 0, \ \textit{for } i = 0, 1, 2, 3 \tag{2.22}$$

if and only if

$$st_P - \lim \left\| [T_j (f)]' - f' \right\| = 0, \ \textit{for all } f \in C^1[0,1]. \tag{2.23}$$

Proof. It is enough to prove the implication (2.22) $\Rightarrow$ (2.23). Let $f \in C^1[0,1]$ and $x \in [0,1]$ be fixed. Then for every $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that

$$-\varepsilon - \frac{2M_3 \beta}{\delta^2} \gamma_x' (y) \leq f'(y) - f'(x) \leq \varepsilon + \frac{2M_3 \beta}{\delta^2} \gamma_x' (y) \tag{2.24}$$

holds for all $y \in [0,1]$ and for any $\beta \geq 1$ where $M_3 = \left\| f' \right\|$ and $\gamma_x (y) = \frac{(y-x)^3}{3} + 1$.

Now using the functions defined by

$$\theta_{1,\beta} (y) := \frac{2M_3 \beta}{\delta^2} \gamma_x (y) - f(y) + \varepsilon y + y f'(x),$$

$$\theta_{2,\beta} (y) := \frac{2M_3 \beta}{\delta^2} \gamma_x (y) + f(y) + \varepsilon y - y f'(x),$$

we can easily see that $\theta_{1,\beta}$ and $\theta_{2,\beta}$ belong to C_+^1 for any $\beta \geq 1$, i.e. $\theta_{1,\beta}(y) \geq 0, \theta_{2,\beta}(y) \geq 0$. Also observe that $\gamma_x(y) \geq \frac{2}{3}$ for all $y \in [0,1]$, then inequality

$$\frac{\left(\pm f(y) - \varepsilon y \pm f'(x)\, y\right)\delta^2}{2M_3\gamma_x(y)} \leq \frac{(M_1 + M_3 + \varepsilon)\,\delta^2}{M_3} \tag{2.25}$$

holds for all $y \in [0,1]$, where $M_1 = \|f\|$ as mentioned before. Now we can choose $\beta \geq 1$ such a way that $\theta_{1,\beta}(y) \geq 0, \theta_{2,\beta}(y) \geq 0$, for each $y \in [0,1]$ and hence $\theta_{1,\beta}, \theta_{2,\beta} \in C_{+,1} \cap C_+^1$. Let

$$E_3 := \left\{ j \in \mathbb{N}_0 : T_j\left(C_{+,1} \cap C_+^1\right) \subset C_+^1 \right\}.$$

By (2.21) it is clear that $\delta_P(E_3) = 1$ and so $\delta_P(\mathbb{N}_0 \setminus E_3) = 0$. Then we can write $T_j\left(\theta'_{i,\beta}; x\right) \geq 0$, for every $j \in E_3$ and $i = 1, 2$. Then we get

$$\frac{2M_3\beta}{\delta^2}\left[T_j(\gamma_x; x)\right]' - \left[T_j(f; x)\right]' + \varepsilon T_j(f_1; x) + f'(x)\left[T_j(f_1; x)\right]' \geq 0,$$

$$\frac{2M_3\beta}{\delta^2}\left[T_j(\gamma_x; x)\right]' + \left[T_j(f; x)\right]' + \varepsilon T_j(f_1; x) - f'(x)\left[T_j(f_1; x)\right]' \geq 0.$$

Since the function $\gamma_x \in C_{+,1} \cap C_+^1$, we have $T_j(\gamma_x) \in C_+^1$ and

$$\begin{aligned}\left|\left[T_j(f; x)\right]' - f'(x)\right| &\leq \varepsilon + \left(\varepsilon + \left|f'(x)\right|\right)\left|\left[T_j(f_1; x)\right]' - f_1'(x)\right| \\ &\quad + \frac{2M_3\beta}{\delta^2}\left[T_j(\gamma_x; x)\right]',\end{aligned} \tag{2.26}$$

holds. Since

$$\begin{aligned}\left[T_j(\gamma_x; x)\right]' &= \left[T_j\left(\frac{(y-x)^3}{3} + 1; x\right)\right]' \\ &\leq \frac{1}{3}\left\{\left[T_j(f_3; x)\right]' - f_3'(x)\right\} - x\left\{\left[T_j(f_2; x)\right]' - f_2'(x)\right\} \\ &\quad + x^2\left\{\left[T_j(f_1; x)\right]' - f_1'(x)\right\} + \left(1 - \frac{x^3}{3}\right)\left\{\left[T_j(f_0; x)\right]' - f_0'(x)\right\},\end{aligned} \tag{2.27}$$

combining this with (2.26), by using similar lines as in the proof of Theorem 2.3 in [2]; for every $\varepsilon > 0$, we get

$$\left\|\left[T_j(f)\right]' - f'\right\| \leq \varepsilon + A_3 \sum_{i=0}^{3}\left\|\left[T_j(f_i)\right]' - f_i'\right\| \tag{2.28}$$

where $A_3 = \left\{\varepsilon + M_3 + \frac{2M_3\beta}{\delta}\right\}$. Now for a given $r > 0$, choose an $\varepsilon > 0$ such that $\varepsilon < r$, and define the following sets:

$$R := \left\{ j \in \mathbb{N}_0 : \left\|\left[T_j(f)\right]' - f'\right\| \geq r \right\},$$

$$R_i := \left\{ j \in \mathbb{N}_0 : \left\|\left[T_j(f_i)\right]' - f_i'\right\| \geq \frac{r - \varepsilon}{4A_3} \right\},\ i = 0, 1, 2, 3.$$

In this case, by (2.28),

$$R \cap E_3 \subset \bigcup_{i=0}^{3}(R_i \cap E_3),$$

which gives for every $j \in \mathbb{N}_0$, that

$$\frac{1}{p(t)} \sum_{j \in R \cap E_3} p_j t^j \leq \frac{1}{p(t)} \sum_{i=0}^{3} \left(\sum_{j \in R_i \cap E_3} p_j t^j \right) \leq \frac{1}{p(t)} \sum_{i=0}^{3} \left(\sum_{j \in R_i} p_j t^j \right). \tag{2.29}$$

Now letting $0 < t \to R^-$ in the both sides of (2.29) and using (2.26), we immediately get that

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j \in R \cap E_3} p_j t^j = 0. \tag{2.30}$$

Furthermore since

$$\begin{aligned} \frac{1}{p(t)} \sum_{j \in R} p_j t^j &= \frac{1}{p(t)} \sum_{j \in R \cap E_3} p_j t^j + \frac{1}{p(t)} \sum_{j \in R \cap (\mathbb{N}_0 \backslash E_3)} p_j t^j \\ &\leq \frac{1}{p(t)} \sum_{j \in R \cap E_3} p_j t^j + \frac{1}{p(t)} \sum_{j \in (\mathbb{N}_0 \backslash E_3)} p_j t^j \end{aligned}$$

holds for every $j \in \mathbb{N}_0$.

Thus it follows from hypothesis and the last inequailty that

$$st_P - \lim \left\| [T_j(f)]' - f' \right\| = 0.$$

□

3. An application

In this section, we give an application showing that in general, our results are stronger than classical ones and we provide some graphs in order to illustrate the efficiency of our result when it is compared with other results in the literature.

Example 3.1. We consider the following linear operator on $C^2[0,1]$

$$T_j(f;x) = \begin{cases} L_j(f;x), & if\ j = 2l, \\ M_j(f;x), & if\ j = 2l+1, \end{cases} \tag{3.1}$$

where M_j and L_j, $j \in \mathbb{N}_0$ are defined as follows

$$M_j(f;x) = \int_0^1 (1+j)\, t^j f(tx)\, dt, \ \ x \in [0,1]$$

and

$$L_j(f;x) = \int_0^1 (1-j)\, t^j f(tx)\, dt, \ \ x \in [0,1].$$

Also, assume that the power series method is given by

$$p_j = \begin{cases} 0, & \text{if } j = 2l, \\ 1, & \text{if } j = 2l+1. \end{cases}$$

It is easy to see that $M_j(f_0;x) = f_0(x)$, $M_j(f_1;x) = \frac{1+j}{2+j} f_1(x)$, $M_j(f_2;x) = \frac{1+j}{3+j} f_2(x)$. Now, we have

$$T_j(f_0;x) - f_0(x) = \begin{cases} \frac{-2j}{j+1}, & if\ j = 2l, \\ 0, & if\ j = 2l+1. \end{cases}$$

Hence, we get

$$st_P - \lim \|T_j(f_0) - f_0\| = 0.$$

Also, it is clear that

$$T_j(f_1;x) - f_1(x) = \begin{cases} \frac{-x(1+2j)}{j+2}, & if\ j = 2l, \\ \frac{-x}{j+2}, & if\ j = 2l+1, \end{cases}$$

and we get

$$st_P - \lim \|T_j(f_1) - f_1\| = 0.$$

Finally,

$$T_j(f_2;x) - f_2(x) = \begin{cases} \frac{-2x^2(1+j)}{j+3}, & if\ j = 2l, \\ \frac{-2x^2}{j+3}, & if\ j = 2l+1, \end{cases}$$

and we have

$$st_P - \lim \|T_j(f_2) - f_2\| = 0.$$

Hence we conclude that our operator satisfies all assumptions of Theorem 2.1. Therefore we obtain

$$st_P - \lim \|T_j(f) - f\| = 0.$$

However it can be easily seen that $(T_j(f_0))$ is not convergent and statistically convergent to f_0. Hence, these show that Proposition 1 of [9] and statistical Korovkin theorem ([3]) do not work for our operators T_j defined by (3.1) (It is illustrated for the function $f(x) = x^3+1$ in Figure 1).

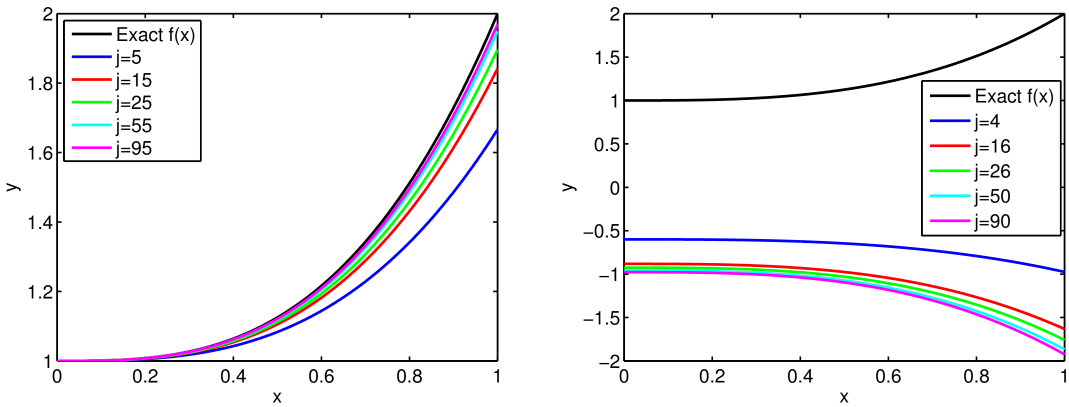

Figure 1. (Left) The function f and operators $T_j(f;x)$ for $j = 5, 15, 25, 55, 95$; (Right) the operators $T_j(f;x)$ for $j = 4, 16, 26, 50, 90$ where $f(x) = x^3 + 1$.

4. Rate of convergence

In this section, we prove some results which give the degree of approximation by means of linear operators.

The modulus of continuity, denoted by $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{|y-x|\leq\delta} |f(y) - f(x)|$$

where δ is a positive constant, $f \in C[a,b]$. It is easy to see that, for any $c > 0$ and all

$$\omega(f;\delta) \leq (1 + [c])\omega(f;\delta)$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

Now we present some estimates of rates of power series method for Korovkin-type theorems.

Theorem 4.1. *Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself and $T_j\left(C_{+,2}\cap C_+^2\right)\subset C_{+,2}$, for all $j\in\mathbb{N}_0$. Assume that the following conditions hold:*

$$st_P-\lim\|T_j(f_0)-f_0\|=0 \tag{4.1}$$

and

$$st_P-\lim\omega(f,\delta_j)=0 \tag{4.2}$$

where $\delta_j:=\sqrt{\|T_j\varphi_x\|}$ and $\varphi_x(y)=(y-x)^2$, then we have, for all $f\in C^2[0,1]$

$$st_P-\lim\|T_j(f)-f\|=0.$$

Proof. Let $x\in[0,1]$ be fixed and let $f\in C^2[0,1]$. We can write that

$$-\left(1+\frac{\beta}{\delta^2}\varphi_x(y)\right)\omega(f,\delta)\le f(y)-f(x)\le\left(1+\frac{\beta}{\delta^2}\varphi_x(y)\right)\omega(f,\delta) \tag{4.3}$$

for all $y\in[0,1]$ and for any $\beta\ge1$ where $\varphi_x(y)=(y-x)^2$. Then by (4.3) we get that

$$g_{1,\beta}(y):=\left(1+\frac{\beta}{\delta^2}\varphi_x(y)\right)\omega(f,\delta)+f(y)-f(x)\ge0, \tag{4.4}$$

$$g_{2,\beta}(y):=\left(1+\frac{\beta}{\delta^2}\varphi_x(y)\right)\omega(f,\delta)-f(y)+f(x)\ge0. \tag{4.5}$$

Also for all $y\in[0,1]$,

$$g''_{1,\beta}(y):=\frac{2\beta}{\delta^2}\omega(f,\delta)+f''(y)\text{ and }g''_{2,\beta}(y):=\frac{2\beta}{\delta^2}\omega(f,\delta)-f''(y).$$

Because of f'' is bounded on $[0,1]$ we can choose $\beta\ge1$ such a way that $g''_{1,\beta}(y)\ge0$, $g''_{2,\beta}(y)\ge0$, for each $y\in[0,1]$. Hence $g_{1,\beta},\ g_{2,\beta}\in C_{+,2}\cap C_+^2$ and then by the hypothesis

$$T_j(g_{i,\beta};x)\ge0,\text{ for all } j\in\mathbb{N}_0,\ x\in[0,1]\text{ and } i=1,2 \tag{4.6}$$

and hence

$$T_j(g_{i,\beta};x)\ge0,\text{ for } t\in(0,R),\ x\in[0,1]\text{ and } i=1,2.$$

From (4.4) – (4.6) and the linearity of (T_j) we get

$$T_j(f_0;x)\omega(f,\delta)+\frac{\beta\omega(f,\delta)}{\delta^2}T_j(\varphi_x;x)+T_j(f;x)-f(x)T_j(f_0;x)\ge0,$$

$$T_j(f_0;x)\omega(f,\delta)+\frac{\beta\omega(f,\delta)}{\delta^2}T_j(\varphi_x;x)-T_j(f;x)+f(x)T_j(f_0;x)\ge0,$$

thus

$$\begin{aligned}-T_j(f_0;x)\omega(f,\delta)-\frac{\beta\omega(f,\delta)}{\delta^2}T_j(\varphi_x;x)&\le f(x)T_j(f_0;x)-T_j(f;x)\\&\le T_j(f_0;x)\omega(f,\delta)+\frac{\beta\omega(f,\delta)}{\delta^2}T_j(\varphi_x;x).\end{aligned}$$

Then we obtain

$$|T_j(f;x)-f(x)|\le\omega(f,\delta)+(\omega(f,\delta)+|f(x)|)|T_j(f_0;x)-f_0(x)|+\frac{\beta\omega(f,\delta)}{\delta^2}T_j(\varphi_x;x).$$

If we take $\delta:=\delta_j:=\sqrt{\|T_j(\varphi_x;x)\|}$, $M_1=\|f(x)\|$ and taking supremum $x,y\in[0,1]$, then we get

$$\|T_j(f)-f\|\le(1+\beta)\omega(f,\delta_j)+(\omega(f,\delta)+M_1)\|T_j(f_0)-f_0\|. \tag{4.7}$$

Given $\varepsilon > 0$ define the following set

$$\begin{aligned} S &: = \{j : \|T_j(f) - f\| \geq \varepsilon\}, \\ S_1 &: = \left\{j : \omega(f, \delta_j) \geq \frac{\varepsilon}{3M_1}\right\}, \\ S_2 &: = \left\{j : \|\omega(f, \delta_j) T_j(f_0) - f_0\| \geq \frac{\varepsilon}{3M_1}\right\}, \\ S_3 &: = \left\{j : \|T_j(f_0) - f_0\| \geq \frac{\varepsilon}{3M_1}\right\}. \end{aligned}$$

Then we easily see that $S \subset S_1 \cup S_2 \cup S_3$ and also defining

$$\begin{aligned} S_2' &: = \left\{j : \|\omega(f, \delta_j)\| \geq \sqrt{\frac{\varepsilon}{3M_1}}\right\}, \\ S_2'' &: = \left\{j : \|T_j(f_0) - f_0\| \geq \sqrt{\frac{\varepsilon}{3M_1}}\right\}, \end{aligned}$$

one can deduce that $S_2 \subset S_2' \cup S_2''$. Hence we get $S \subset S_1 \cup S_2' \cup S_2'' \cup S_3$, So we get that

$$\begin{aligned} \frac{1}{p(t)} \sum_{j \in S} p_j t^j &\leq \frac{1}{p(t)} \sum_{j \in S_1} p_j t^j + \frac{1}{p(t)} \sum_{j \in S_2'} p_j t^j \\ &\quad + \frac{1}{p(t)} \sum_{j \in S_2''} p_j t^j + \frac{1}{p(t)} \sum_{j \in S_3} p_j t^j \end{aligned}$$

from the hypothesis and the last inequailty we obtain that

$$st_P - \lim \|T_j(f) - f\| = 0,$$

that is, the assertion. □

Note that the following theorems may be proved as in Theorem 4.1. So we omit their proofs.

Theorem 4.2. *Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself and $T_j(C_{+,2} \cap C_-^2) \subset C_-^2$, for all $j \in \mathbb{N}_0$. Assume that the following conditions hold:*

$$st_P - \lim \left\| [T_j(f_0)]'' - f_0'' \right\| = 0 \tag{4.8}$$

and

$$st_P - \lim \omega\left(f'', \delta_j\right) = 0 \tag{4.9}$$

where $\delta_j := \sqrt{\|T_j(-\sigma_x)\|}$ and $\sigma_x(y) = -\frac{(y-x)^4}{12} + 1$, then we have, for all $f \in C^2[0,1]$

$$st_P - \lim \left\| [T_j(f)]'' - f'' \right\| = 0.$$

Theorem 4.3. *Let (T_j) be a sequence of linear operators from $C^1[0,1]$ into itself and $T_j(C_{+,1} \cap C_+^1) \subset C_+^1$, for all $j \in \mathbb{N}_0$. Assume that the following conditions hold:*

$$st_P - \lim \left\| [T_j(f_0)]' - f_0' \right\| = 0 \tag{4.10}$$

and

$$st_P - \lim \omega\left(f', \delta_j\right) = 0 \tag{4.11}$$

where $\delta_j := \sqrt{\|T_j \gamma_x\|}$ and $\gamma_x(y) = \frac{(y-x)^3}{3} + 1$, then we have, for all $f \in C^1[0,1]$

$$st_P - \lim \left\| [T_j(f)]' - f' \right\| = 0.$$

5. Conclusions

Finally we give the following concluding remarks.

◇ Let (T_j) be a sequence of linear operators from $C[0,1]$ into itself and $T_j(C_+) \subset C_+$, for all $j \in \mathbb{N}_0$. Then for all $f \in C[0,1]$,

$$st_P - \lim \|T_j(f) - f\| = 0 \tag{5.1}$$

if and only if

$$st_P - \lim \|T_j(f_i) - f_i\| = 0, \; i = 0,1,2 \text{ (see [30])}. \tag{5.2}$$

◇ We remark that all our theorems also work on any compact subset of $\mathbb{R}$ instead of the unit interval $[0,1]$.

◇ Theorem 2.3 works if we replace the condition $T_j(C_{+,1} \cap C_+^1) \subset C_+^1$ by $T_j(C_{+,1} \cap C_-^1) \subset C_-^1$. To prove this, it is enough to consider the function $\mu_x(y) = -\frac{(y-x)^3}{3} + 1$ instead of $\gamma_x(y)$ defined in the proof of Theorem 2.3.

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