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Domain of Jordan Totient Matrix in the Space of Almost Convergent Sequences

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Abstract
In this paper, the notion of almost convergence is used to obtain a space as the domain of a regular matrix. After defining a new type of core for complex-valued sequences, certain inclusion theorems are proved.

Keywords: *Jordan totient function; regular matrix; almost convergence.*

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1. Introduction and preliminaries

The classical summability theory concerns with the generalization of the concept of convergence for series or sequences by assigning a limit for non-convergent series or sequences. For this purpose, infinite special matrices are used.

One of the fundamental subject of summability is the study of the theory of sequence spaces. By a sequence space, we mean any subspace of ω consisting all sequences with real or complex terms. We use the classical sequence spaces

$$\begin{aligned} c_0 &= \{x=(x_j)\in\omega : \lim_j x_j = 0\}, \\ c &= \{x=(x_j)\in\omega : \lim_j x_j \text{ exists }\}, \\ \ell_\infty &= \{x=(x_j)\in\omega : \sup_j |x_j| < \infty\}, \end{aligned}$$

$$cs = \left\{x=(x_j)\in\omega : \left(\sum_{i=1}^{j} x_i\right) \in c\right\}$$

and

$$bs = \left\{x=(x_j)\in\omega : \left(\sum_{i=1}^{j} x_i\right) \in \ell_\infty\right\}.$$

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In the theory of sequence spaces, the concept of Banach limit has rised as a fascinating application of the famous Hahn–Banach extension theorem. The Banach limit is known as extension of limit functional on c to the space ℓ_∞. This notion has used by Lorentz [1] to introduce a new type of convergence called almost convergence. The spaces f and f_0 of almost convergent and almost convergent to zero are given by

$$f = \left\{ x = (x_j) \in \ell_\infty : \lim_{i\to\infty} \sum_{p=0}^{i} \frac{x_{j+p}}{i+1} = \mathcal{A} \text{ uniformly in } j \right\}$$

and

$$f_0 = \left\{ x = (x_j) \in \ell_\infty : \lim_{i\to\infty} \sum_{p=0}^{i} \frac{x_{j+p}}{i+1} = 0 \text{ uniformly in } j \right\}.$$

A Banach limit $\mathcal{L}$ defined on ℓ_∞ is a non-negative linear functional such that $\mathcal{L}(\mathcal{P}x) = \mathcal{L}x$ and $\mathcal{L}(e) = 1$, where $\mathcal{P} : \omega \longrightarrow \omega$, $\mathcal{P}_j(x) = x_{j+1}$ is the shift operator. A sequence $x = (x_j)$ is said to be almost convergent to the generalized limit $\mathcal{A}$ if all Banach limits of x are coincide and are equal to $\mathcal{A}$. It is denoted by $f - \lim x_j = \mathcal{A}$. If $\mathcal{P}^p$ is the p-times composition of $\mathcal{P}$ with itself, we use the notation

$$a_{ij}(x) = \frac{1}{i+1} \sum_{p=0}^{i} (\mathcal{P}^p x)_j \text{ for all } i, j \in \mathbb{N}.$$

It is proved by Lorentz [1] that $f - \lim x_j = \mathcal{A}$ if and only if $\lim_{i\to\infty} a_{ij}(x) = \mathcal{A}$ uniformly in j. It is a known fact that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. See the papers [2–14] for more on almost convergence and Banach limit.

Given any sequence spaces $\mathcal{X}$ and $\mathcal{Y}$, an infinite matrix $S = (s_{ij})$ is considered as a matrix mapping from $\mathcal{X}$ into $\mathcal{Y}$ if the sequence $Sx = \{(Sx)_i\} = \left(\sum_j s_{ij} x_j\right) \in \mathcal{Y}$ for every $x = (x_j) \in \mathcal{X}$. By $(\mathcal{X} : \mathcal{Y})$, we denote the class of all such matrices. It is said that S regularly maps $\mathcal{X}$ into $\mathcal{Y}$ if $S \in (\mathcal{X} : \mathcal{Y})$ and $\lim_j (Sx)_j = \lim_j x_j$ for all $x \in \mathcal{X}$. This is denoted by $S \in (\mathcal{X} : \mathcal{Y})_{reg}$.

By f_S, we mean the domain of an infinite matrix S in the space f; that is

$$f_S = \left\{ x = (x_j) \in \omega : Sx \in f \right\}.$$

For more on matrix domains and new sequence spaces, see [15–25]

Let $x = (x_j) \in \omega$ and C_j be the least convex closed region in complex plane containing $x_j, x_{j+1}, x_{j+2}, \ldots$ for each $j \in \mathbb{N} = \{1, 2, \ldots\}$. The Knopp Core or $\mathcal{K} - core$ of $x = (x_j)$ is defined as the intersection of all C_j ([26]). If $x \in \ell_\infty$, we have that

$$\mathcal{K} - core(x) = \bigcap_{z\in\mathbb{C}} \left\{ \tilde{z} \in \mathbb{C} : |\tilde{z} - z| \le \limsup_j |x_j - z| \right\}$$

([27]).

Knopp Core Theorem [26, p. 138] states that $\mathcal{K} - core(Sx) \subseteq \mathcal{K} - core(x)$ for all real valued sequences x and a positive matrix $S \in (c : c)_{reg}$.

Statistical convergence is another generalization of usual convergence. It is defined by the aid of natural density of a subset in $\mathbb{N}$. The natural density of a set N is

$$\delta(N) = \lim_j \frac{1}{j} |\{i \le j : i \in N\}|$$

provided that the limit exists. Here $||$ gives the cardinality of the set written inside it. It is said that a sequence $x = (x_j)$ is statistically convergent to $\mathcal{D}$ if for every $\varepsilon > 0$ the natural density of the set

$$\{j \in \mathbb{N} : |x_j - \mathcal{D}| \ge \varepsilon\}$$

equals zero. It is denoted by $st - \lim x = \mathcal{D}$ ([28]). By st_0 and st, the spaces of all statistically null and statistically convergent sequences are denoted.

The notion of the statistical core or $st - core$ of a statistically bounded sequence x is defined by Fridy and Orhan [29] as

$$st - core(x) = \bigcap_{z\in\mathbb{C}} \left\{ \tilde{z} \in \mathbb{C} : |\tilde{z} - z| \le st - \limsup_j |x_j - z| \right\}.$$

For some papers on core theorems, see [30–34].

The Jordan's function $J_r : \mathbb{N} \to \mathbb{N}$ of order r is an arithmetic function, where r is a positive integer. The value $J_r(n)$ equals to the number of r-tuples of positive integers all less than or equal to n that form a coprime $(r+1)$-tuples together with n.

In a recent paper, İlkhan et al. [35] define a new matrix $\Upsilon^r = (\upsilon^r_{nk})$ as

$$\upsilon^r_{nk} = \begin{cases} \frac{J_r(k)}{n^r} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

for each $r \in \mathbb{N}$. It is also observed that this special transformation is regular; that is a limit preserving mapping c into c.

The inverse $(\Upsilon^r)^{-1} = ((\upsilon^r_{nk})^{-1})$ is computed as

$$(\upsilon^r_{nk})^{-1} = \begin{cases} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

Here and what follows μ is the Mobius function. By using usual matrix product, the Υ^r-transform of a sequence $x = (x_j) \in \omega$ is the sequence

$$y = \Upsilon^r x = ((\Upsilon^r x)_j) = \left(\frac{1}{j^r} \sum_{d|j} J_r(d) x_d \right).$$

In this study, it is aimed to introduce and study on a new sequence space $\widehat{f}(\Upsilon^r)$ as the domain of Υ^r in the space f. Further, *Jordan Totient Core* (Υ^r-core) of a sequence is defined and characterization of matrices satisfying $\Upsilon^r - core(Sx) \subseteq \mathcal{K} - core(x)$ and $\Upsilon^r - core(Sx) \subseteq st - core(x)$ with $x \in \ell_\infty$ are given.

2. Domain of Υ^r in the space f and Jordan Totient Core

In this section, we introduce the space $\widehat{f}(\Upsilon^r)$ consisting of all sequences whose Υ^r-transforms are in f. That is,

$$\widehat{f}(\Upsilon^r) = \left\{ x = (x_j) \in \ell_\infty : \lim_{i\to\infty} \sum_{p=0}^{i} \frac{(\Upsilon^r x)_{j+p}}{i+1} = \mathcal{A} \text{ uniformly in } j \right\}.$$

One can prove that the spaces $\widehat{f}(\Upsilon^r)$ and f are linearly isomorphic.

The β-dual of a space $\mathcal{X}$ consists of all sequences $a = (a_j) \in \omega$ such that $xa = (x_j a_j) \in cs$ for all $x = (x_j) \in \mathcal{X}$. In order to determine the $\beta-$dual of the space $\widehat{f}(\Upsilon^r)$, we need the following result.

Lemma 2.1. *[36] $S = (s_{ij}) \in (f : c)$ if and only if*

$$\sup_{i\in\mathbb{N}} \sum_j |s_{ij}| < \infty, \tag{2.1}$$

$$\lim_{i\to\infty} s_{ij} = s_j \in \mathbb{C} \text{ for each } j \in \mathbb{N}, \tag{2.2}$$

$$\lim_{i\to\infty} \sum_j s_{ij} = s \in \mathbb{C}, \tag{2.3}$$

$$\lim_{i\to\infty} \sum_j \left|\Delta(s_{ij} - s_j)\right| = 0. \tag{2.4}$$

Theorem 2.1. *The $\beta-$dual of the sequence space $\widehat{f}(\Upsilon^r)$ is the intersection of the following sets*

$$\begin{aligned}
\mathfrak{B}_1 &= \left\{t=(t_j)\in\omega : \sup_{i\in\mathbb{N}}\sum_{j=1}^{i}\left|\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}jt_d\right|<\infty\right\},\\
\mathfrak{B}_2 &= \left\{t=(t_j)\in\omega : \lim_{i\to\infty}\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d \ \textit{exists}\right\},\\
\mathfrak{B}_3 &= \left\{t=(t_j)\in\omega : \lim_{i\to\infty}\sum_{j=1}^{i}\left[\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d\right] \ \textit{exists}\right\},\\
\mathfrak{B}_4 &= \left\{t=(t_j)\in\omega : \lim_{i\to\infty}\sum_{j}\left|\Delta\left[\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d-\alpha_j\right]\right|=0\right\}.
\end{aligned}$$

Proof. Given any $t=(t_j)\in\omega$, the equality

$$\begin{aligned}
\sum_{j=1}^{i}t_jx_j &= \sum_{j=1}^{i}t_j\Big(\sum_{d|j}\frac{\mu(\frac{j}{d})}{J_r(j)}d^r y_d\Big)\\
&= \sum_{j=1}^{i}\Big(\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d\Big)y_j\\
&= B_i(y); \qquad (i\in\mathbb{N}) \qquad (2.5)
\end{aligned}$$

holds, where the matrix $B=(b_{ji})$ is defined by

$$b_{ji}=\begin{cases}\displaystyle\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d & 1\le j\le i,\\ 0 & \text{otherwise}\end{cases} \qquad (2.6)$$

for all $j,i\in\mathbb{N}$. It follows from (2.5) that $tx=(t_jx_j)\in cs$ whenever $x=(x_j)\in c$ if and only if $By\in c$ whenever $y=(y_j)\in f$. That is, $t=(t_j)\in\{\widehat{f}(\Upsilon^r)\}^\beta$ if and only if $B\in(f:c)$. Hence the result is obtained by using Lemma 2.1. □

Now, we define Jordan totient core or Υ^r-core of a complex valued sequence.

Definition 2.1. Let C_j be the least closed convex hull containing $(\Upsilon^r x)_j,(\Upsilon^r x)_{j+1},....$ Then, Υ^r-core of x is the intersection of all C_j, i.e.,

$$\Upsilon^r-core(x)=\bigcap_{j=1}^{\infty}C_j.$$

The following result is immediate since the Υ^r-core of x is the $\mathcal{K}-core$ of the sequence $\Upsilon^r x$.

Theorem 2.2. *For any $x\in\ell_\infty$, we have*

$$\Upsilon^r-core(x)=\bigcap_{z\in\mathbb{C}}\left\{\tilde{z}\in\mathbb{C}:|\tilde{z}-z|\le\limsup_j|(\Upsilon^r x)_j-z|\right\}.$$

Recently, İlkhan et al. [37] introduced the following spaces by the aid of Jordan totient function.

$$c_0(\Upsilon^r)=\left\{x=(x_j)\in\omega:\lim_j\left(\frac{1}{j^r}\sum_{d|j}J_r(d)x_d\right)=0\right\}$$

and

$$c(\Upsilon^r)=\left\{x=(x_j)\in\omega:\lim_j\left(\frac{1}{j^r}\sum_{d|j}J_r(d)x_d\right)\ \text{exists}\right\}.$$

In order to give the necessary and sufficient conditions for an infinite matrix $S=(s_{ij})$ be in the classes $(c:c(\Upsilon^r))_{reg}$ and $(st(S)\cap\ell_\infty:c(\Upsilon^r))_{reg}$, we firstly have some auxiliary results.

Lemma 2.2. *$S=(s_{ij})\in(\ell_\infty:c(\Upsilon^r))$ if and only if*

$$\sup_i\sum_j\left|\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}\right|<\infty, \tag{2.7}$$

$$\lim_i\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}=\gamma_j \quad \textit{for each } j, \tag{2.8}$$

$$\lim_i\sum_j\left|\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}-\gamma_j\right|=0. \tag{2.9}$$

Lemma 2.3. *$S=(s_{ij})\in(c:c(\Upsilon^r))_{reg}$ if and only if (2.7) and (2.8) hold with $\gamma_j=0$ for each j and*

$$\lim_i\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}=1. \tag{2.10}$$

Lemma 2.4. *$S=(s_{ij})\in(st\cap\ell_\infty:c(\Upsilon^r))_{reg}$ if and only if $S\in(c:c(\Upsilon^r))_{reg}$ and*

$$\lim_i\sum_{j\in N,\delta(N)=0}\left|\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}\right|=0. \tag{2.11}$$

Proof. It is a known fact that $c\subset st\cap\ell_\infty$ holds. So we have $S\in(c:c(\Upsilon^r))_{reg}$. Now let $\delta(N)=0$ and $x\in\ell_\infty$. Define a sequence $\tilde{x}=(\tilde{x}_j)$ as $\tilde{x}_j=x_j$ if $j\in N$ and $\tilde{x}_j=0$ otherwise. Clearly $\tilde{x}\in st_0$. Hence we have $S\tilde{x}\in c_0(\Upsilon^r)$. Further the equality

$$\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}\tilde{x}_j=\sum_{j\in N}\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}x_j$$

yields that $\hat{S}=(\hat{s}_{ij})\in(\ell_\infty:c(\Upsilon^r))$, where

$$\hat{s}_{ij}=\begin{cases}\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij} & , \text{ if } j\in N\\ 0 & , \text{ if } j\notin N.\end{cases}$$

Thus we deduce (2.11) from Lemma 2.2.

Conversely, choose a sequence $x\in st\cap\ell_\infty$ with $st-\lim x=\mathcal{D}$. Given any $\varepsilon>0$, we have $\delta(N)=\delta(\{j:|x_j-\mathcal{D}|\geq\varepsilon\})=0$. By letting $i\to\infty$ in the following equality

$$\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}x_j=\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}(x_j-\mathcal{D})+\mathcal{D}\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}, \tag{2.12}$$

the inequality

$$\left|\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}(x_j-\mathcal{D})\right|\leq\|x\|\sum_{j\in N}\left|\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}\right|+\varepsilon\sum_j\left|\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}\right|,$$

and (2.10) with (2.11) yield that

$$\lim_i\sum_j\frac{1}{i^r}\sum_{j|i}J_r(j)s_{ij}x_j=\mathcal{D}.$$

This means that $S\in(st\cap\ell_\infty:c(\Upsilon^r))_{reg}$. □

Lemma 2.5. *[30] Let $S = (s_{ij})$ be a matrix satisfying the conditions $\sum_j |s_{ij}| < \infty$ and $\lim_i s_{ij} = 0$. Then we have*

$$\limsup_i \sum_j s_{ij} x_j = \limsup_i \sum_j |s_{ij}|$$

for some $x \in \ell_\infty$ with $\|x\| \leq 1$.

Now, we are ready to give our main theorems.

Theorem 2.3. *Let $S \in (c, c(\Upsilon^r))_{reg}$ and $x \in \ell_\infty$. The inclusion $\Upsilon^r - core(Sx) \subseteq \mathcal{K} - core(x)$ holds if and only if*

$$\lim_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| = 1. \tag{2.13}$$

Proof. By combining Lemma 2.3 and Lemma 2.5 we obtain the equality

$$\left\{ \tilde{w} \in \mathbb{C} : |\tilde{w}| \leq \limsup_i \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} x_j \right\} = \left\{ \tilde{w} \in \mathbb{C} : |\tilde{w}| \leq \limsup_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| \right\}$$

for some $x = (x_j) \in \ell_\infty$ with $\|x\| \leq 1$. Since the inclusions

$$\Upsilon^r - core(Sx) \subseteq \mathcal{K} - core(x) \subseteq \{\tilde{w} \in \mathbb{C} : |\tilde{w}| \leq 1\}$$

hold, (2.13) follows from the inclusion

$$\left\{ \tilde{w} \in \mathbb{C} : |\tilde{w}| \leq \limsup_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| \right\} \subseteq \{\tilde{w} \in \mathbb{C} : |\tilde{w}| \leq 1\}.$$

Now, let $\tilde{w} \in \Upsilon^r - core(Sx)$. We have

$$\begin{aligned} |\tilde{w} - w| &\leq \limsup_i |(\Upsilon^r(Sx))_i - w| \qquad (2.14) \\ &= \limsup_i \left| w - \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} x_j \right| \\ &\leq \limsup_i \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| + \limsup_i |w| \left| 1 - \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| \\ &= \limsup_i \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| \end{aligned}$$

for any $w \in \mathbb{C}$. Put $\limsup_j |x_j - w| = l$. Given any $\varepsilon > 0$ there exists j_0 such that $|x_j - w| \leq l + \varepsilon$ for $j \geq j_0$. Hence, it follows that

$$\begin{aligned} \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| &= \left| \sum_{j<j_0} \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) + \sum_{j \geq j_0} \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| \qquad (2.15) \\ &\leq \sup_j |w - x_j| \sum_{j<j_0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| + (l + \varepsilon) \sum_{j \geq j_0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| \\ &\leq \sup_j |w - x_j| \sum_{j<j_0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| + (l + \varepsilon) \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right|. \end{aligned}$$

Hence (2.14) and (2.15) yield that

$$|\tilde{w}-w| \leq \limsup_i \left|\sum_j \frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}(w-x_j)\right| \leq l+\varepsilon.$$

This implies that $\tilde{w} \in \mathcal{K}-core(x)$. Hence the desired inclusion holds. □

Theorem 2.4. *Let $S \in (st \cap \ell_\infty : c(\Upsilon^r))_{reg}$ and $x \in \ell_\infty$. The inclusion $\Upsilon^r - core(Sx) \subseteq st - core(x)$ holds if and only if (2.13) holds.*

Proof. Since $st-core(x) \subseteq \mathcal{K}-core(x)$ holds, the inclusion $\Upsilon^r - core(Sx) \subseteq st-core(x)$ implies (2.13) by Theorem 2.3.

Now, let $\tilde{w} \in \Upsilon^r - core(Sx)$. Similarly we have inequality (2.14). Put $st - \limsup|x_j - w| = \hat{l}$. Given any $\varepsilon > 0$, we have $\delta(\tilde{N}) = \delta(\{j : |x_j - w| > \hat{l}+\varepsilon\}) = 0$ (see [38]). Hence it follows that

$$\begin{aligned}
\left|\sum_j \frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}(w-x_j)\right| &= \left|\sum_{j\in\tilde{N}} \frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}(w-x_j) + \sum_{j\notin\tilde{N}} \frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}(w-x_j)\right| \\
&\leq \sup_j |w-x_j| \sum_{j\in\tilde{N}} \left|\frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}\right| + (\hat{l}+\varepsilon)\sum_{j\notin\tilde{N}} \left|\frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}\right| \\
&\leq \sup_j |w-x_j| \sum_{j\in\tilde{N}} \left|\frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}\right| + (\hat{l}+\varepsilon)\sum_{j} \left|\frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}\right|.
\end{aligned}$$

Consequently, by (2.11) and (2.13), we have

$$\limsup_i \left|\sum_j \frac{1}{i^r}\sum_{j|i} J_r(j)s_{ij}(w-x_j)\right| \leq \hat{l}+\varepsilon. \tag{2.16}$$

If we combine (2.14) with (2.16), we deduce that

$$|\tilde{w}-w| \leq st - \limsup_j |x_j - w|.$$

This implies that $\tilde{w} \in st-core(x)$. Hence the desired inclusion holds. □

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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