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Set Invariant Means and Set Fixed Point Properties

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Abstract
In this paper, we introduce a concept of fixed point property for a semigroup S called A-fixed point property, where A is a non-empty subset of S. Also, the relationship between A-amenability and A-fixed point property is investigated.

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1. Introduction

Let S be a semitopological semigroup, i.e. S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from S into S are continuous. Let $\ell^\infty(S)$ denotes the C^*-algebra of bounded real-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let ${}_af$ and f_a denote, respectively, the left and right translations of S by a, i.e. $({}_af)(s) = (f \cdot a)(s) = f(as)$ and $(f_a)(s) = (a \cdot f)(s) = f(sa)$, $s \in S$. Let X be a closed subspace of $\ell^\infty(S)$ containing the constant functions and being invariant under translations. A linear functional $m \in X^*$ is called a mean if $\|m\| = m(1) = 1$; where 1 denotes the constant function on S with value 1. Then m is called a left invariant mean if

$$m({}_sf) = m(f),$$

for all $s \in S$ and $f \in X$. If X is a subalgebra of $\ell^\infty(S)$, then m is multiplicative if $m(fg) = m(f)m(g)$ for all $f,g \in X$.

Recently, a new version of amenability of discrete semigroups, namely set amenability is defined by authors in [1] as follows:

Definition 1.1. *Let S be a semigroup and $\emptyset \neq A \subseteq S$. We say that a mean m on $\ell^\infty(S)$ is an A-invariant mean if for all $a \in A$ and $f \in \ell^\infty(S)$ we have*

$$m({}_af) = m(f).$$

A semigroup S which admits A-invariant means is called left A-amenable. If for every pure subset A of S, S is left A-amenable, then we say that S is left set-amenable. The right A-amenability may be defined similarly. A semigroup S which is both left and right A-amenable is called A-amenable. It follows immediately that every amenable semigroup is A-amenable for all subsets A of S, but the converse is not true in general, see the examples are given in [1] for more details.

A semitopological semigroup S is said to be act on a topological space X from the left if there is a map $S \times X \to X$ denoted by $(s,x) \to s \cdot x$ for each $(s,x) \in S \times X$ such that $(st) \cdot x = s \cdot (t \cdot x)$ for all $s,t \in S$ and $x \in X$. The action is separately continuous if the mapping is continuous in each of the variables when the other is kept fixed. Moreover, the action is jointly continuous if the mapping is continuous when $S \times X$ has the product topology.

When C is a convex subset of a linear topological space X, we say that an action of S on C is affine if for each $s \in S$, the mapping from $C \to C$ defined by $x \mapsto s \cdot x \ (x \in C)$ is affine, i.e. it satisfies $s \cdot (\lambda x + (1-\lambda)y) = \lambda(s \cdot x) + (1-\lambda)(s \cdot y)$, for all $s \in S$, $x, y \in C$ and $0 \leq \lambda \leq 1$.

Let $C_b(S)$ be the Banach space of all continuous bounded real-valued functions on S with the supremum norm topology, $A(K)$ be the closed subspace of $C_b(K)$ consisting of all real valued continuous affine functions on a compact Hausdorff space K and $LUC(S)$ be the space of left uniformly continuous functions on S, that is, all $f \in C_b(S)$ such that the mappings $s \mapsto {}_s f$ from S into $C_b(S)$ are continuous when $C_b(S)$ has the supremum norm topology. Then $LUC(S)$ is a C^*-subalgebra of $C_b(S)$ invariant under translations and contains the constant functions.

A function $f \in C_b(S)$ is strongly almost periodic if $\{{}_a f : a \in S\}$ is relatively compact in the supremum norm topology of $C_b(S)$ and the set of all strongly almost periodic functions is denoted by $AP(S)$. Also, it is weakly almost periodic if $\{{}_a f : a \in S\}$ is relatively compact in the weak topology of $C_b(S)$ and the set of all weakly almost periodic functions is denoted by $WAP(S)$.

In this paper, we investigate a new version of the fixed point property for semitopological semigroups that we call A-fixed point property, where A is a non-empty subset of S. In the next section, we introduce and study the concept of set-reversibility of semitopological semigroups that is a generalization of reversibility that is defined for discrete semigroups in [1]. Section 3, introduce the notion of set-fixed point theory for semitopological semigroups and gives some relations between set-amenability and set-fixed point property for them.

Finally, in section 4, we give some examples for clarifying this new version of fixed point property for semitopological semigroups that they show that set-fixed point property is weaker than fixed point property.

2. Set-reversiblity for semigroups

For discrete semigroups, set reversibility is defined in [1, Definition 4.7], now we start off with the following definition for semitopological semigroups:

Definition 2.1. *Let S be a semigroup and $\emptyset \neq A \subseteq S$. We say that S is left A-reversible if $\overline{aS} \cap \overline{bS} \neq \emptyset$ for all $a, b \in A$. If S for every pure subset A is A-reversible, then we call it set-reversible.*

Clearly, every reversible semigroup is set-reversible, but, the converse is not true [1, Example 4.8]. For discrete semigroup S, if it is left A-amenable, then S is left A-reversible [1, Lemma 4.9]. However, a general semitopological semigroup S needs not be left A-reversible even when $C_b(S)$ has a left A-invariant mean unless S is normal (see Proposition 2.5).

Proposition 2.2. *Let S be a compact semitopological semigroup with minimal right ideal I, then S is left I-reversible.*

Proof. Since I is a right ideal, aS and bS are closed right ideals of S contained in I for each $a, b \in I$. Furthermore, since I is minimal, $aS = bS = I$. Thus

$$aS \cap bS = I \neq \emptyset.$$

□

Proposition 2.3. *Let S be a compact semitopological semigroup with subset A containig of a minimal right ideal. If S is left A-reversible, then A consists a unique minimal right ideal of S.*

Proof. Let I_1 and I_2 be two distinct minimal right ideals in A. It is easy to verify that they are closed and disjoint. Now, if we consider $a_1 \in I_1$ and $a_2 \in I_2$, then we may write

$$\overline{a_1 S} \subseteq I_1 \text{ and } \overline{a_2 S} \subseteq I_2.$$

By minimility of I_1 and I_2, we obtain that $\overline{a_1 S} = I_1$ and $\overline{a_2 S} = I_2$. Therefore

$$\overline{a_1 S} \cap \overline{a_2 S} = I_1 \cap I_2 = \emptyset.$$

This is a contradiction. □

The following Proposition is a set-reversibility version of [2, Lemma 3.1] that its proof is similar and we omit it.

Proposition 2.4. *Let S be a semitopological semigroup with non-empty subset A and X be a left translation invariant subspace of $C_b(S)$ containing constants. If X which separates closed subsets of S and has a left A-invariant mean, then S is left A-reversible.*

Above proposition implies immediately the following result:

Proposition 2.5. *Let S be a normal semitopological semigroup and $C_b(S)$ has a left A-invariant mean, then S is left A-reversible.*

In light [1, Lemma 4.9] of the above proposition we have the following result:

Proposition 2.6. *Let S be a discrete semigroup left A-amenable, then S is left A-reversible.*

If S is not a normal semitopological semigroup, then Proposition 2.5 does not hold. For example, if S is a left zero semigroup, that is a semigroup whose multiplication is defined by $st = s$ for all $s,t \in S$ and is the topological space which is regular and Hausdorff such that $C_b(S)$ consists of constant functions only. For a fixed $a \in S$ define $m(f) = f(a)$, for all $f \in C_b(S)$. Then for each subset A of S, contains more than one element, m is a left A-invariant mean on $C_b(S)$, but S is not left A-reversible.

By the following result, we show that set-reversibility can be transfered by a continuouse and onto semigroup homomorphism.

Proposition 2.7. *Let S and T be two semitopological semigroups and φ be a continuouse homomorphism of S onto T. If S is left A-reversible, then T is left $\varphi(A)$-reversible.*

Proof. For each $b_1,b_2 \in \varphi(A)$, there exist $a_1,a_2 \in A$ such that $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$. Since S is left A-reversible, $\overline{a_1S} \cap \overline{a_2S} \neq \emptyset$. Let $x_0 \in \overline{a_1S} \cap \overline{a_2S}$, then there are nets $(a_1s_\alpha) \subset a_1S$ and $(a_2t_\beta) \subset a_2S$ such that $a_1s_\alpha \longrightarrow x_0$ and $a_2t_\beta \longrightarrow x_0$. Continuity of φ implies

$$b_1\varphi(s_\alpha) = \varphi(a_1s_\alpha) \longrightarrow \varphi(x_0) \text{ and } b_2\varphi(t_\beta) = \varphi(a_2t_\beta) \longrightarrow \varphi(x_0).$$

It follows that there exists $\varphi(x_0) \in \overline{b_1T} \cap \overline{b_2T}$, hence T is left $\varphi(A)$-reversible. □

Recall that if S is a semigroup, the intersection of all the two-sided ideals of S is called the kernel of S and denoted by $K(S)$. If $K(S)$ is non-empty, it is clearly the smallest two-sided ideal of S (see [3]).

Similar to [4, Lemma 2.8], we have the following result:

Proposition 2.8. *Let S be a compact semitopological semigroup with unit and let A be a subset of S that consists a minimal right ideal of S. Then the following statements are equivalent:*

(a) A consists a unique minimal right ideal of S.

(b) $C_b(S)$ has a left A-invariant mean.

Proof. $(b) \Rightarrow (a)$. Let m be a left A-invariant mean on $C_b(S)$ and I_1 and I_2 be two distinct minimal right ideals in A. It is obvious that I_1 and I_2 are closed and disjoint. Now, we define $f \in C_b(S)$ by

$$f(s) = \begin{cases} 0 & s \in I_1 \\ 1 & s \in I_2. \end{cases}$$

Then for any $a \in I_1$ and $b \in I_2$, we have ${}_af = 0$ and ${}_bf = 1$. But, by the definition of m,

$$1 = m({}_af) = m(f) = m({}_bf) = 0,$$

which this is a contradiction.

$(a) \Rightarrow (b)$. By [4, Corollary 2.4], $K(S)$ is a union of compact semitopological groups that are left ideals. Normalized Haar measure on any one of these will be a left A-invariant mean for $C_b(S)$. □

By the following result, we rewrite [4, Lemma 2.10] as follows that its proof is similar to that mentioned Lemma and for clarify we state its proof.

Proposition 2.9. *Let S and T be semitopological semigroups with T be compact and $\varphi : S \to T$ be a continuous homomorphism with $\varphi(S)$ dense in T. Let $\tilde{\varphi} : C_b(T) \longrightarrow C_b(S)$ be the dual map taking f into $f \circ \varphi$, then $C_b(T)$ has a left $\varphi(A)$-invariant mean if and only if $\tilde{\varphi}(C_b(T))$ has a left A-invariant mean.*

Proof. Suppose that there is a left $\varphi(A)$-invariant mean m on $C_b(T)$. For each $f \in C_b(T)$, define n by

$$n(\tilde{\varphi} f) = m(f).$$

Furthermore, it is clear that for any f in $C_b(T)$,

$$_s(\tilde{\varphi} f) = \tilde{\varphi}(_{\varphi(s)} f) \ \text{ for all } \ s \in S.$$

Now, for all $a \in A$ and $f \in C_b(T)$, we have

$$n(_a(\tilde{\varphi} f)) = n(\tilde{\varphi}(_{\varphi(a)} f)) = m(_{\varphi(a)} f) = m(f) = n(\tilde{\varphi} f).$$

This means that n is a left A-invariant mean on $\tilde{\varphi}(C_b(T))$.

On the other hand, let n be a left A-invariant mean on $\tilde{\varphi}(C_b(T))$, we can define a mean m on $C_b(T)$ by

$$m(f) = n(\tilde{\varphi} f) \ \text{ for all } f \in C_b(T).$$

Since m satisfies

$$m(_{\varphi(a)} f) = n(\tilde{\varphi}(_{\varphi(a)} f)) = n(_a(\tilde{\varphi} f)) = n(\tilde{\varphi} f) = m(f),$$

for all $a \in A$, m is a left $\varphi(A)$-invariant mean on $C_b(T)$. □

3. Common set-fixed point

Fixed point property for semigroups is one of the interesting concepts related to the semigroups theory that investigated by many authors, see [5]-[9]. Set-fixed point property for discrete semigroups is defined in [1, Definition 4.5]. Now, we define it for semitopological semigroups as follows:

Definition 3.1. *Let X be a non-empty Hausdorff topological space and S is a semigroup acting on X from the left with $\emptyset \neq A \subseteq S$. A point $x \in X$ is called a common A-fixed point of S in X if $a \cdot x = x$ for each $a \in A$. If S for every pure subset A has a common A-fixed point, then we say that it has a common set-fixed point.*

It follows immediately that every common fixed point of S in X is a common A-fixed point. But the converse is not true in general (see the Example 4.2). In this section, we rewrite some well-known results related to fixed point properties of semitopological semigroup for the set fixed point properties.

Before stating the following result, recall that when K is convex subset of a Banach space, a mapping $T : K \longmapsto K$ is called non-expansive self-maps if $\|Tx - Ty\| \leq \|x - y\|$, for each $x, y \in K$.

Theorem 3.2. *Let S be a left I-reversible semigroup of non-expansive self-maps on a non-empty compact convex subset K of a Banach space with ideal I, then K contains a common I-fixed point.*

Proof. By using Zorn's Lemma, there is a minimal I-invariant non-empty compact convex set $X \subseteq K$. By using Zorn's Lemma again, we can find a minimal I-invariant nonempty compact set $M \subseteq X$. Since S is left I-reversible, if $\{a_1, a_2, \ldots, a_n\}$ is any finite subset of I, there is a finite subset $\{s_1, s_2, \ldots, s_n\}$ of S such that $a_1 s_1 = a_2 s_2 = \cdots = a_n s_n$. Hence

$$\bigcap_{i=1}^{n} a_i M \supseteq \bigcap_{i=1}^{n} a_i(s_i a_1 M) = a_1 s_1 a_1 M \neq \emptyset.$$

Thus the family $\{aM : a \in I\}$ has the finite intersection property. By compactness of M, $F = \bigcap_{a \in I} aM$ is non-empty. Assume that $x \in F$. For each pair $a, b \in I$, there exist $c, d \in S$ such that $ac = bd$. Since $F \subseteq caM$, $x = cay$ for some $y \in M$. Furthermore,

$$ax = a(cay) = b(day) \in bM.$$

Moreover, $aF \subseteq F$, for all $a \in I$. By minimality of M, we have $F = M$. Therefore $M = aM$, for all $a \in I$.

If we assume that M contains more than one point, there is an element u in the closed convex hull of M such that $\rho = \sup\{\|u - x\| : x \in M\} < \delta(M)$, where $\delta(M)$ is the diameter of M. Define

$$X_0 = \bigcap_{x \in M} \{y \in X : \|x - y\| \leq \rho\},$$

then X_0 is a proper non-empty compact convex subset of X such that it is I-invariant, which contradicts the minimality of X. Hence M is a singleton set, which proves the theorem. □

Note that in the above Theorem if we replace I by S we have the following result:

Corollary 3.3. *Let S be a left reversible semigroup of non-expansive self-maps on a non-empty compact convex subset K of a Banach space, then K contains a common fixed point of S.*

Definition 3.4. *An action S on a convex subset K of a linear topological space X is A-affine if $a\cdot(\lambda x+(1-\lambda)y)=\lambda(a\cdot x)+(1-\lambda)(a\cdot y)$, for all $a\in A$, $x,y\in K$ and $0\le\lambda\le 1$.*

Clearly every affine action is an A-affine, but the converse is not true in general. For example, let S be a semigroup of real-valued functions on $\mathbb{R}$ with function composition operation and $A=\{f:\mathbb{R}\to\mathbb{R}\mid f(x)=kx\}$. Define the action S on $\mathbb{R}$ by $f\cdot x=f(x)$. It is easy to see that this action is A-affine, but it does not act affinely on $\mathbb{R}$.

There is a strong connection between left A-amenability and A-fixed point properties. By a similar method for discrete semigroups in [1, Theorem 4.6] and [9, Theorem 2.1], we have the following result:

Theorem 3.5 (Day's Fixed Point Theorem)**.** *Let S be a semigroup. Then the following statements are equivalent:*

(a) S is left A-amenable.

(b) Whenever S acts A-affinely on a non-empty compact convex subset K of a locally convex space, there is a common A-fixed point of S in K.

Proof. According to [1, Theorem 4.6], it suffices that we prove $(b)\Rightarrow(a)$.

$(b)\Rightarrow(a)$. As well-known that $M(S)$ the set of all means on $\ell^\infty(S)$ is a w^*-compact convex subset of $\ell^\infty(S)^*$. For $s\in S$ and $m\in M(S)$, we can define the action S on the left $\ell^\infty(S)^*$ by $(s\cdot m)(f)=m(f\cdot s)$ for each $f\in\ell^\infty(S)$. It is easy to verify that the map $m\mapsto s\cdot m$ is w^*-w^*-continuous and S acts A-affinely on $(M(S),w^*)$. Hence, by the assumption there is a common A-fixed point of S in $M(S)$ which is a left A-invariant mean on $\ell^\infty(S)$. □

Theorem 3.6. *Let S be a semitopological semigroup. Then the following statements are equivalent:*

(a) $LUC(S)$ has a multiplicative left A-invariant mean.

(b) Whenever action of S on a compact Hausdorff space X is jointly continuous, then X contains a common A-fixed point of S.

Proof. $(a)\Rightarrow(b)$. From [8, Theorem 1], for each $x\in X$ and each $f\in C_b(X)$, we have $f_x\in LUC(S)$, where f_x is defined by $f_x(s)=f(s\cdot x)$. Now, for $x\in X$ we consider $T_x:C(X)\to LUC(S)$ by $T_x(f)=f_x$ for each $f\in C(X)$. Let $T_x^*:LUC(S)^*\to C(X)^*$ be the adjoint map of T_x. Thus, if m is a multiplicative left A-invariant mean on $LUC(S)$, then there exists a point $x_0\in X$ such that $f(x_0)=(T_x^*m)(f)=m(T_xf)$ for all $f\in C(X)$.

For each $s\in S$, define $\theta_s:C(X)\to C(X)$ by $(\theta_sf)(x)=f(s\cdot x)$ for all $f\in C(X)$, $x\in X$. Hence, for each $t\in S$, we have

$$(T_x(\theta_sf))(t)=(\theta_s)(t\cdot x)=f(s\cdot(t\cdot x))=f(st\cdot x)=f_x(st)=(T_xf)(st)={}_s(T_xf)(t).$$

Therefore, $T_x(\theta_sf)={}_s(T_xf)$. Thus it follows that for all $f\in C(X)$ and $a\in A$,

$$\begin{aligned} f(a\cdot x_0) &= (\theta_af)(x_0)\\ &= m(T_x(\theta_af))\\ &= m({}_a(T_xf))\\ &= m(T_xf)\\ &= f(x_0). \end{aligned}$$

But $C(X)$ separates points of X and this implies that x_0 is the A-fixed point.

$(b)\Rightarrow(a)$. Assume that X is the compact Hausdorff space of the set of all multiplicative means on $LUC(S)$, where X is given by the w^*-topology of $LUC(S)^*$. By a simillar method in [8, Theorem 1] and use the notations there in, one can show the action of S on X its jointly continuous. Thus, (b) implies that there exists $m_0\in LUC(S)^*$ such that $m_0({}_af)=m_0(f)$, for all $f\in LUC(S)$ and $a\in A$. □

Theorem 3.7. *Let S be a semitopological semigroup. Then the following properties are equivalent:*

(a) $LUC(S)$ has a left A-invariant mean.

(b) *Whenever action of S on a nonempty compact convex subset X of a locally convex linear topological space is A-affine, then X contains a common A-fixed point of S.*

Proof. $(a) \Rightarrow (b)$. For each $x \in X$ and each $f \in C_b(X)$, the proof of Theorem 3.6 yields $f_x \in LUC(S)$, where f_x is defined by $f_x(s) = f(s \cdot x)$. For a specific $x \in X$ we consider $T_x : A(X) \to LUC(S)$ by $T_x(f) = f_x$ for each $f \in A(X)$. Now, by simmilar method in Theorem 3.6 we can obtain (b).

$(b) \Rightarrow (a)$. Assume that X is the compact convex set of the space of all means on $LUC(S)$, with the w^*-topology of $LUC(S)^*$. Define the A-affine action of S on X by $s \cdot m = l_s^* m$, for each $s \in S$ and $m \in X$. Now, the argument used in the proof of Theorem 3.6 can be used to show that the action is jointly continuous, hence by (b), there exists an A-fixed point of S on X. □

An action of S on a compact convex subset K of locally convex space X is equicontinuous if for each neighborhood U of 0, there exists a neighborhood V of 0 in X such that $x, y \in K$ and $x - y \in V$ imply $s \cdot x - s \cdot y \in U$ for each $s \in S$.

In the following Theorem, we state a relation between the existence of left A-invariant mean on $AP(S)$, the space of continuous almost periodic functions on S and A-fixed point properties of S acting on certain subsets of a locally convex space. In light of [10, Theorem 3.2], we have the following result which its proof is similar to the mentioned result and for clarifying we write its proof completely.

Theorem 3.8. *Let S be a semitopological semigroup. Then the following statements are equivalent:*

(a) *$AP(S)$ is left A-amenable.*

(b) *Whenever action of S on a compact convex subset K of a separated locally convex space is separately continuous, equicontinuous and A-affine, then there exists a common A-fixed point of S in K.*

Proof. $(a) \Rightarrow (b)$. Suppose that m is a left A-invariant mean on $AP(S)$. Since the finite means are w^*-dense in the set of means, we can find a net of finite means $\varphi_\alpha = \sum_{i_\alpha=1}^n \lambda_{i_\alpha} \delta_{s_{i_\alpha}}$, $\lambda_{i_\alpha} > 0$ and $\sum_{i_\alpha=1}^n \lambda_{i_\alpha} = 1$ such that w^*-converges to m in $AP(S)^*$. Let $x \in K$ be fixed and x_0 be a cluster point of the net $(\sum_{i_\alpha=1}^n \lambda_{i_\alpha} s_{i_\alpha} \cdot x)_\alpha$ in K. Now by [10, Lemma 3.1], for each $f \in A(K)$, we have $f_x \in AP(S)$ and hence

$$\begin{aligned}
f(a \cdot x_0) &= f(a \cdot \lim_\alpha \sum_{i_\alpha=1}^n \lambda_{i_\alpha} s_{i_\alpha} \cdot x) = f(\lim_\alpha \sum_{i_\alpha=1}^n \lambda_{i_\alpha} a s_{i_\alpha} \cdot x) \\
&= \lim_\alpha f(\sum_{i_\alpha=1}^n \lambda_{i_\alpha} a s_{i_\alpha} \cdot x) = \lim_\alpha (\sum_{i_\alpha=1}^n \lambda_{i_\alpha} f(a s_{i_\alpha} \cdot x)) \\
&= \lim_\alpha \sum_{i_\alpha=1}^n \lambda_{i_\alpha} \delta_{s_{i_\alpha}}({}_a(f_x)) = \lim_\alpha \varphi_\alpha({}_a(f_x)) \\
&= m({}_a(f_x)) = m(f_x) \\
&= f(x_0),
\end{aligned}$$

for all $a \in A$. Since $A(K)$ separates points, this shows that x_0 is an A-fixed point for S.

$(b) \Rightarrow (a)$. Let the compact convex set K be the space of all means on $AP(S)$, where K has the w^*-topology of $AP(S)^*$. Let the A-affine action of S on K be given by $s \cdot m = l_s^* m$, for each $s \in S$ and $m \in K$. By the similar method in [10, Theorem 3.2], the action of S on (K, w^*) is both separately continuous and equicontinuous. Consequently, any A-fixed point in K under this action is a left A-invariant mean on $AP(S)$. □

Recall that the right translation operators r_a on the Banach space $AP(S)$, clearly, form an almost periodic semigroup of operators. In fact, the strong operator closure of this semigroup is a compact semitopological semigroup, having jointly continuous multiplication, in the strong (or equivalently weak) operator topology. It will be denoted by S^a and called the almost periodic compactification of S.

Corollary 3.9. *Let S be a semitopological semigroup with subset A containig of minimal right ideal. If S is left A-reversible, then $AP(S)$ is left A-amenable.*

Proof. Assume that S is left A-reversible. In light of [4, Theorem 6.1], the homomorphism $r : S \mapsto \overline{S}^a$ defined by $r(a) = r_a$ is continuous. This implies that $\overline{S}^a$ is also $r(A)$-reversible. By Proposition 2.3, $\overline{S}^a$ has a unique minimal right ideal in $r(A)$. Hence, by Theorem 2.8, $C_b(\overline{S}^a)$ has a left $r(A)$-invariant mean. Consequently, again it follows from [4, Theorem 6.1] and Proposition 2.9, $AP(S)$ has a left A-invariant mean. □

Form the above Corollary and Proposition 2.5, we immediately have the following result:

Corollary 3.10. *Let S be a normal semitopological semigroup and $C_b(S)$ has a left A-invariant mean, then $AP(S)$ has a left A-invariant mean.*

Note that the converse of Corollary 3.9 is false in general, since there exist the exampels of topological semigroups such as S such that they are not left A-reversible, but $AP(S)$ (or even $C_b(S)$) has a left A-invariant mean (see Exampel 4.3).

Definition 3.11. *Let Q be a (fixed) family of continuous semi-norms on a separated locally convex space X which determines the topology of X. Then an action of S on a subset $K \subseteq X$ is Q_A-nonexpansive if $\rho(a\cdot x - a\cdot y) \leq \rho(x-y)$ for all $a \in A$, $x,y \in K$ and $\rho \in Q$.*

Theorem 3.12. *Let S be a semitopological semigroup. Then the following statements are equivalent:*

(a) $AP(S)$ is left A-amenable.

(b) Whenever S is a separately continuous and Q_A-non-expansive action on a compact convex subset K of a separately locally convex space, there is a common A-fixed point of S in K.

Proof. $(a) \Rightarrow (b)$. Assume that m is a left A-invariant mean on $AP(S)$. An application of Zorn's Lemma shows that there exists a minimal non-empty compact convex $X \subseteq K$, that is invariant under A. In particular, If X is not a singleton, apply Zorn's Lemma for the second time to get a minimal non-empty compact $F \subseteq X$, that is invariant under A.

Let $x \in X$ be a fixed. By using [10, Lemma 3.1], we may define a mean μ on $C(F)$ by $\mu(f) = m(f_x)$ for all $f \in C(F)$. Since $\mu(f) \geq 0$ whenever $f \geq 0$, and $\mu(1) = 1$ and,

$$\mu({}_a f) = m(({}_a f)_x) = m({}_a(f_x)) = m(f_x) = \mu(f).$$

It is easy to see that μ is a left A-invariant mean on $C(F)$. From Riesz representation Theorem, μ can be viewed as a regular probability measure on F and it satisfies $\mu(B) = \mu(a^{-1}B)$ for each Borel set $B \subseteq F$ and $a \in A$, where as usual, $a^{-1}B = \{x \in F : a\cdot x \in B\}$. Let $\Gamma = \{B \subseteq F : \text{B is closed subset, } \mu(B) = 1\}$. Set $F_0 = \bigcap_{B\in\Gamma} B$. Then by finite intersection property F_0 is a non-empty compact subset of F. Since for each $B \in \Gamma$ and $a \in A$, we have $a^{-1}B \in \Gamma$ then $a^{-1}F_0 \supseteq F_0$ or $F_0 \supseteq aF_0$. Hence $F = F_0$ by the minimality of F. Since

$$\mu(aF) = \mu(a^{-1}(aF)) = \mu(F) = 1,$$

$aF \in \Gamma$ for all $a \in A$. Consequently, $F \supseteq aF \supseteq F_0 = F$. This means that $aF = F$ for all $a \in A$.

Now, if F is a singleton we are done, otherwise, there exists a continuous seminorm ρ in Q such that $r = \sup\{\rho(x-y) : \ x,y \in F\} > 0$. Then, by De Marr's Lemma [6], there exists an element u in the closed convex hull of F such that $r_0 = \sup\{\rho(u-x) : \ x \in F\} < r$. Consider

$$X_0 = \bigcap_{x\in F}\{y \in X : \rho(y-x) \leq r_0 \}.$$

Then $u \in X_0$ and X_0 is a nonempty closed convex proper subset of X. From $aF = F$ for each $a \in A$ and Q_A-nonexpansiveness of S on X, we can write

$$\rho(a\cdot x_0 - a\cdot y) \leq \rho(x_0 - y) \leq r_0,$$

for each $x_0 \in X_0$ and $y \in F$. This leads to $aX_0 \subseteq X_0$ for all $a \in A$, contradicting the minimality of X. Consequently, F contains only one point, which, in fact, is a common A-fixed point for S.

$(b) \Rightarrow (a)$. We can prove it by the same argument in Theorem 3.8.

□

Finally, in this section, by the similar method in [7, Theorem 3.4], we have the following result:

Theorem 3.13. *Let S be a semitopological semigroup with separable ideal I. Then the following statements are equivalent:*

(a) $WAP(S)$ is left I-amenable.

(b) Whenever S acts on a weakly compact convex subset K of a separated locally convex space and the action is weakly separately continuous, weakly quasi-equicontinuous and Q_I-non-expansive, there is a common I-fixed point of S in K.

4. Examples

Example 4.1. *Consider the semigroup $S = \{a,b,c,d\}$ defined as follows:*

	a	b	c	d
a	a	a	c	c
b	b	b	d	d
c	a	a	c	c
d	b	b	d	d

(i) Let $A = \{a,b\}$. The sets $aS = \{a,c\}$ and $bS = \{b,d\}$ are disjoint minimal right ideals. Thus S is not left A-reversible. Letting $f = \chi_{\{a,c\}}$ be the characteristic function. It is easy to see that ${}_af = 1$ and ${}_bf = 0$. Now, if we take m a left A-invariant mean on $AP(S) = WP(S) = C_b(S)$, then

$$1 = m({}_af) = m(f) = m({}_bf) = 0,$$

which is impossible.

(ii) Let $A = \{a,c\}$. Since $aS \cap cS = \{a,c\}$, S is left A-reversible. Now, if we for a fix $a \in S$ define $m(f) = f(a)$ for all $f \in AP(S)$. It is evident that ${}_af = {}_cf$. Also, we have

$$m({}_af) = {}_af(a) = f(aa) = f(a) = m(f).$$

This means that m is a left A-invariant mean on $AP(S)$.

In the following, we denote the cardinal number of a set A by $|A|$.

Example 4.2. *Let $K = [0,1]$ and consider a semigroup $S = \{h_s : s \in K\}$ with functional composition operation. Define the action S on K by $h_s(x) = s$ for each $x \in K$. Then for any subset A of S we have:*

(i) If $|A| = 1$, then there is an A-fixed point of S in K.

(ii) If $|A| \geq 2$, then there is no common A-fixed point of S in K.

Example 4.3. *Consider the partially bicyclic semigroups $S_2 = \langle e,a,b,c|ab = e, ac = e\rangle$ and $S_{1,1} = \langle e,a,b,c,d|ac = e, bd = e\rangle$. For $A_1 = \{b,c\}$ and $A_2 = \{b,d\}$, since*

$$bS_2 \cap cS_2 = \emptyset \text{ and } cS_{1,1} \cap dS_{1,1} = \emptyset$$

S_2 and $S_{1,1}$ are not left A_i-reversible $(i = 1,2)$, respectively. Of course, it is worth to mention that both $AP(S_2)$ and $AP(S_{1,1})$ have an invariant mean [7, Proposition 4.6].

We consider for a semitopological semigroup S the following A-fixed point property:

(F_A): Every jointly continuous action of S on a non-empty compact convex set K of a separated locally convex topological vector space has a common A-fixed point.

Proposition 4.4. *If a semitopological semigroup S has the common A-fixed point property (F_A), then $LUC(S)$ has a left A-invariant mean.*

Proof. Suppose that $M(S)$ is the set of all means on $LUC(S)$, where $M(S)$ is given the w^*-topology of $LUC(S)^*$. Then, $M(S)$ is w^*-compact convex subset of $LUC(S)^*$. Define an action of S on X by $s \cdot m = l_s^* m$ for each $s \in S$ and each $m \in M(S)$. This action is jointly continuous on $M(S)$. Therefore, the common A-fixed point of this action gives a left A-invariant mean on $LUC(S)$. □

Corollary 4.5. *If S is a discrete semigroup with the common A-fixed point property (F_A), then S is left A-amenable.*

In the following, by A_i's we mean the sets in Example 4.3.

Example 4.6. *We know that the partially bicyclic semigroups S_2 and $S_{1,1}$ are not left A_i-amenable $(i = 1,2)$, respectively. Hence, by Corollary 4.5, they do not have the common A_i-fixed point property (F_A).*

Proposition 4.7. *Let S and T be two semigroups and φ be a homomorphism of S onto T. If S has the common A-fixed point property (F_A), then T has the common $\varphi(A)$-fixed point property (F_A).*

Proposition 4.8. *Let S and T be semigroups such that $S \times T$ has the common $(A \times B)$-fixed point property (F_A).Then both semigroups S and T have the common A-fixed point and common B-fixed point property (F_A), respectively.*

Proof. Consider the projection homomorphisms $\pi_S : S \times T \longrightarrow S$ and $\pi_T : S \times T \longrightarrow T$ defined by $\pi_S(s,t) = s$ and $\pi_T(s,t) = t$, respectively. Let $S \times T$ has the common $(A \times B)$-fixed point property (F_A). Then by $\pi_S(A \times B) = A$ and Proposition 4.7, we obtain that S has the common A-fixed point property (F_A). Similarly, we conclude that T has the common B-fixed point property (F_A). □

In the following, we show that the converse of the Proposition 4.8, is not true in general.

Example 4.9. *Commutative free semigroup on two generators does not have (F_A).*

First recall from the well-known Schauder's Fixed Point Theorem that every free commutative discrete semigroup on one generator has the fixed point property (F_A). Let N_0 denote the additive semigroup of non-negative integers, which is the free commutative semigroup on one generator. Hence, it has the common A-fixed point property (F_A).

We know from [11] that there are two continuous functions f and g mapping the unit interval $[0,1]$ into itself which commute under the function composition but do not have any common fixed point in $[0,1]$.

Consider set $A = \{(0,0),(1,0),(0,1)\}$ and define the action of $N_0 \times N_0$ on $[0,1]$ by

$$(0,0) \cdot x = x, \ (1,0) \cdot x = f(x) \ \text{ and } \ (0,1) \cdot x = g(x).$$

Then, $N_0 \times N_0$ has no common A-fixed point on $[0,1]$. Therefore, this semigroup does not has (F_A), since it is isomorphic to $N_0 \times N_0$.

5. Conclusion

In this paper, we investigate a new version of the fixed point property for semitopological semigroups and also we introduce and study the concept of set-reversibility of semitopological semigroups that is a generalization of reversibility that is defined for discrete semigroups. Finally, some examples are given to illustrate the theoretical results.

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