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On Delta Sets of Some Pseudo-Symmetric Numerical Semigroups with Embedding Dimension Three

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Abstract
Let S be a numerical semigroup. The catenary degree of an element s in S is a non-negative integer used to measure the distance between factorizations of s. The catenary degree of the numerical semigroup S is obtained at the maximum catenary degree of its elements. The maximum catenary degree of S is attained via Betti elements of S with complex properties. The Betti elements of S can be obtained from all minimal presentations of S. A presentation for S is a system of generators of the kernel congruence of the special factorization homomorphism. A presentation is minimal if it can not be converted to another presentation, that is, any of its proper subsets is no longer a presentation. The Delta set of S is a factorization invariant measuring the complexity of sets of the factorization lengths for the elements in S. In this study, we will mainly express the given above invariants of a special pseudo-symmetric numerical semigroup family in terms of its generators.

1. Introduction

There are many recent publications studying invariants of non-unique factorizations for finitely generated cancellative monoids. Many of these are particularly focused on numerical semigroups. Numerical semigroups provide particularly specific settings for studying these decomposition problems. One motivation for studying the factorization theory of numerical semigroups comes from its associated numerical semigroup rings. These rings usually give concrete examples of more general problems in commutative algebra [11]

The origin of factorization theory is to study the decomposition of natural numbers into their irreducible divisors. In this multiplicative monoid, the Fundamental Theorem of Arithmetic guarantees that such a decomposition is unique. By the Fundamental Theorem of Arithmetic, every positive integer greater than 1 has a prime factorization. Just as prime numbers are components that make up the natural number system using multiplication, they are generators to the components of a numerical monoid. Unlike factorization in $\mathbb{N}_0$, factorization in numerical semigroup may not be unique (where $\mathbb{N}_0$ is the set of non-negative integers)[19]. Some of the factorization invariants are length sets, delta sets and catenary degrees.

The catenary degree of the element of a numerical semigroup, which is a factorization invariant, defines the relationships between its different irreducible factorizations of the element. The catenary degree of the numerical semigroup is defined as the least upper bound of all catenary degrees of the elements in the numerical semigroup.

Calculating the Betti elements of a numerical semigroup is both complicated and difficult. It is well known that with the help of Betti elements, the maximum catenary degree of the numerical semigroup can be reached. Moreover, the numerical semigroup with embedding dimension three has at most three Betti elements [6].

Delta sets are also defined the sets of the minimum distances between any two

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factorizations with consecutive lengths. Although many research has been done on this problem, it is not an easy task to calculate the delta sets for a given numerical semigroup. Geroldinger presented the first results on Delta sets in [10]. Delta sets on numerical semigroups have been studied extensively in [3], [4], [5]. Also, the Delta set of monoids can be calculated using Euclid's greatest common divisor algorithm [8].

The structure of this paper is arranged as follows. In Section 2, we gather the background of numerical semigroups, necessary definitions, and notations that we use in the latter sections. In section 3, we obtained the formulas and the connections representing the Delta set, Betti elements, catenary degrees, graphs, and minimal presentation of the pseudo-symmetric numerical semigroup family.

2. Material and Method

Let $\mathbb{N} = \{1,2,3,\cdots\}$ be the set of positive integers and let $\mathbb{N}_0 = \{0,1,2,3,\cdots\}$ be the set of non-negative integers. If S is an additive submonoid of $\mathbb{N}_0$, S is called a numerical monoid. We say that the integers $\{m_1, m_2, \cdots, m_p\}$ generate S if $s = k_1 m_1 + k_2 m_2 + \cdots + k_p m_p = \sum_{i=1}^{p} k_i m_i$ ($k_i \in \mathbb{N}_0, i = 1,2,\cdots,p$) for $s \in S$, we denote it by $S =< m_1, m_2, \cdots, m_p >$. In terms of cardinality and set inclusion such a minimal set is the minimal generating set and it is unique. Thus, we assume that $m_1 < m_2 <\ldots< m_p$.

A numerical monoid $S = \langle m_1, m_2, \cdots, m_p \rangle$ is primitive whenever $gcd(m_1, m_2, \cdots, m_p) = 1$ (where $gcd(a,b)$ denotes the greatest common divisor of integer a and b).

The integer p is called the embedding dimension of S, denoted by $e(S)$. Also, the integer $min(S\backslash\{0\}) = m_1$ is called the multiplicity of S, denoted by $m(S)$. We know that $e(S) \leq m(S)$. If $e(S) = m(S)$, S is said to have maximal embedding dimension.

If a non-empty subset S of $\mathbb{N}_0$ satisfy the following three conditions, S is called a numerical semigroup.

1. $0 \in S$.
2. $\forall s_1, s_2 \in S,\ s_1 + s_2 \in S$.
3. $\#(\mathbb{N}_0\backslash S) < \infty$ (where $\#(A)$ denotes the number of elements in the set A)

Namely, a numerical semigroup is a submonoid of ($\mathbb{N}_0$,+) satisfying the third condition.

A numerical semigroup S is said to be proper if $S \neq \mathbb{N}_0$. Let S be a proper numerical semigroup. We denote the complement of S in $\mathbb{N}_0$ by $G(S)$. The elements of $G(S)$ are called gaps of S. The genus of S is the number of gaps of S, denoted by $g(S)$. $F(S) = max(\mathbb{Z}\backslash S)$ is the Frobenius number of S. Note that $F(\mathbb{N}_0) = -1$. Henceforth, all numerical sets are proper.

Recall that a numerical semigroup S is symmetric if $F(S)$ is odd and $x \in \mathbb{Z} \setminus S \Rightarrow F(S) - x \in S$, and pseudo- symmetric if $F(S)$ is even and $x \in \mathbb{Z} \setminus S \Rightarrow x = F(S)/2$ or $F(S) - x \in S$.

For a numerical semigroup S and $s \in S\backslash\{0\}$, the Apéry set of S with respect to s is defined by $Ap(S,s) = \{x \in S | x - s \notin S\}$. It is well known that $Ap(S,s) = \{w_0 = 0, w_1, \cdots, w_{s-1}\}$ and $w_i = min\{x \in S: x \equiv i(mods)\}$ for $i = \{0,1,\ldots,s-1\}$. Researchers can review the definitions and results given below in more detail in [1], [17].

Let $S = \langle m_1, m_2, \cdots, m_p \rangle$. The set of factorizations of $s \in S$ is defined by

$$Z(s) = \{(\alpha_1, \alpha_2, \cdots, \alpha_p) \in \mathbb{N}_0{}^p | \alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_p m_p = s\}.$$

If a factorization has a positive entry in the p-tuple, the element can be said to be supported on the component corresponding to the generator. The length of $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_p\} \in Z(s)$ is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p$. The length set of s is expressed by $L(s) = \{ |\alpha| \mid \alpha \in Z(s)\}$.

Fix a numerical semigroup $S = \langle m_1, m_2, \cdots, m_p \rangle$, and fix $s \in S$. Writing $L(s) = \{\ell_1 < \cdots < \ell_p\}$, the delta set of s is the set $\Delta_S(s) = \{\ell_i - \ell_{i-1} | 2 \leq i \leq p\}$ of successive differences of factorization lengths of s, and $\Delta(S) = \bigcup_{s \in S} \Delta_S(s)$. For $\alpha = (x_1, x_2, \cdots, x_p) \in \mathbb{N}_0{}^p$ and $\beta = (y_1, y_2, \cdots, y_p) \in \mathbb{N}_0{}^p$, the greatest common divisor of α and β is $gcd(\alpha, \beta) = (min(x_1, y_1), \cdots, min(x_p, y_p)) \in \mathbb{N}_0{}^p$. The distance between α and β is defined as $d(\alpha,\beta) = max\{|\alpha - gcd(\alpha,\beta)|, |\beta - gcd(\alpha,\beta)|\}$.

Given $x, y \in Z(s)$ and $M \geq 1$, an M-chain from x to y is a sequence $x_1, x_2, \cdots, x_p \in Z(s)$ such that $x_1 = x, x_2, \cdots, x_p = y$, and $d(x_j, x_{j+1}) \leq M$ for every $j \in \{1,2,\cdots,i-1\}$. The catenary degree of $s \in S$, $c(s)$ is the minimal $M \in \mathbb{N}_0 \cup \{\infty\}$ such that for any two factorizations $x, y \in Z(s)$ there is an M-chain from x to y. The catenary degree of S, denoted by $C(S)$, is $C(S) = sup\{c(s) \mid s \in S\}$.

Let $S = \langle m_1, m_2, \cdots, m_p \rangle$ and $s \in S \backslash \{0\}$. Consider the graph G_s with vertex set given by its set of factorizations $Z(s)$ and an edge connecting $a, b \in Z(s)$ if a and b have disjoint support as vectors. That is, $a = (x_1, x_2, \cdots, x_p)$ and $b = (y_1, y_2, \cdots, y_p)$ are adjacent in G_s if for all i, x_i and y_i are never both non-zero. For each $s \in S\backslash\{0\}$, consider the graph $G_s$with vertex set $Z(s)$, where if $gcd(a, b) \neq 0$, then two vertices $a, b \in Z(s)$ share an edge. An element $s \in S$ is called a Betti element if G_s is disconnected. The set of Betti elements of S is denoted by $Betti(S)$. Namely, the set of Betti elements of S is

$$Betti(S) = \{s \in S | G_s \text{ is disconnected}\}$$

Calculating the Betti elements of a numerical semigroup is quite complex. The maximum catenary degree of the numerical semigroup is achieved with the help of the Betti elements of it. It is also known that numerical semigroups with an embedding dimension three have at most three Betti elements [1], [2], [7], [15].

Let δ be a congruence on $Free(x_1, x_2, \cdots, x_p)$ and ρ be a system of generators of δ. If the cardinality of ρ is the smallest in cardinalities of systems of generators of δ, then ρ is the minimum relation. Let S=$\langle m_1, m_2, \cdots, m_p \rangle$ and let $A = \{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ with $\alpha_i \neq \alpha_j$ for all $i \neq j$. If ρ is a minimal relation of the kernel congruence of φ: $Free(\alpha_1, \alpha_2, \cdots, \alpha_p) \to S, \varphi(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p) = x_1 m_1 + x_2 m_2 + \cdots + x_p m_p$, then ρ is called a minimal presentation.

δ is used to indicate the kernel congruence of φ. The expression set of $s \in S$ is defined as follows:

$$Z(s) = \varphi^{-1}(s) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p | x_1 m_1 + x_2 m_2 + \cdots + x_p m_p = s\}.$$

For $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p \in Free(\alpha_1, \alpha_2, \cdots, \alpha_p)$, $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p \in Free(\alpha_1, \alpha_2, \cdots, \alpha_p)$. Let the dot product of x and y be defined as $x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_p y_p$. For $x, y \in Free(\alpha_1, \alpha_2, \cdots, \alpha_p)$, xRy if $x = y = 0$ or $n_1, \cdots n_l \in Z(s)$ for some $s \in S$ such that $n_1 = x, n_l = y$ and $n_i \cdot n_{i+1}$ for all $i \in \{1, \cdots, l-1\}$. It can be easily seen that R has an equivalence relation on $Z(s)$. The element of $Z(s)/R$ are called R-classes. It is known that every finitely generated semigroup S has a finite minimal presentation and that all minimal presentations of S have equal cardinality.

A graph $G = (V, E)$ consists of a set of objects V called vertices, and another set E, whose elements are called edges, such that edge $\{u, v\}$ is identified with an unordered pair by $\overline{uv}$, where $E \subseteq \{\{u, v\} | u, v \in V, u \neq v\}$. A path of length m is a sequence of edges of the form $\overline{v_1 v_2}, \overline{v_2 v_3}, \cdots, \overline{v_m v_{m+1}}$. A graph $G = (V, E)$ is said to be connected if there is at least one path every pair vertices in $G = (V, E)$. Otherwise, $G = (V, E)$ is disconnected. A connected graph with m vertices includes in least $m - 1$ edges. Such connected graph is called a tree [14].

A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is a graph whose vertex set V' is a subset of the vertex set V, that is $V' \subseteq V$, and whose edge set E' is a subset of the edge set E that is $E' \subseteq E$. If G is a connected graph on m vertices, a generating tree for G is a subgraph of G that is a tree on m vertices.

Let $X \neq \emptyset$, γ a binary relation of X and $P = \{P_1, \ldots, P_r\}$ a partition of X. If there exists $x \in P$ and $y \in P_j$ such that $(x, y) \in \gamma \cup \gamma^{-1}$, $G\gamma = (P, E)$ is a graph associated to γ in connection with the partition P where $\overline{P_i P_j}$ with $i \neq j$.

Let $S = \langle m_1, m_2, \cdots, m_p \rangle$ and R be an equivalence binary relation on $Z(s)$ for $s \in S$. And let $s \in S$ and $P_1, \ldots, P_k$ be different equivalence classes of R contained in Z(s) for $i \in \{1, \cdots, k\}$.

$$A_i = \{m_j | x_j \leq x \text{ for some } x \in X_i\}$$

The set of vertices of the different connected components of G_s is included in these sets. To show that we first need to prove that $\{A_1, \ldots, A_k\}$ is a partition of V_s.

For $s \in S$ define the graph $G_s = (V_s, E_s)$, as $V_s = \{m_i \in \{m_1, m_2, \cdots, m_p\} | s - m_i \in S\}$ and $E_s = \{\overline{m_i m_j} | s - (m_i + m_j) \in S, i \neq j\}$.

3. Results and Discussion

Studies on Frobenius number, gaps and some properties of this numerical semigroup are included in [12], [13]. Also, in this section we study on pseudo-symmetric numerical semigroups of the form $S =< 3, 3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$. We will obtain here some invariants of such numerical semigroups. When $S = \langle m_1, m_2, m_3 \rangle$ is a numerical semigroup, for $i \in \{1, 2, 3\}$

$$c_i = min\{k \in \mathbb{N} | km_i \in \langle \{m_1, m_2, m_3\} \backslash \{m_i\} \rangle\}$$

and there exist non-negative integers r_{ij}, r_{ik} for $\{i, j, k\} = \{1,2,3\}$ such that $c_i m_i = r_{ij} m_j + r_{ik} m_k$.

Proposition 1. ([18], Proposition 2.13) Let m_1,$m_2 \in \mathbb{N}$ with $gcd(m_1, m_2) = 1$. There are the following equations:

1) $F(\langle m_1, m_2 \rangle) = m_1 m_2 - m_1 - m_2$.
2) $g(\langle m_1, m_2 \rangle) = \frac{m_1 m_2 - m_1 - m_2 + 1}{2}$.

Proposition 2. ([19], Proposition 4.1) Let $S = \langle m_1, m_2, m_3 \rangle$ be a numerical semigroup minimally generated. An element $\beta \in S$ is a Betti element if for some $i \in \{1, 2, 3\}$ $\beta_i = c_i m_i$ where $c_i = min\{k \in \mathbb{N} | km_i \in \langle \{m_1, m_2, m_3\} \backslash \{m_i\} \rangle\}$.

Lemma 3. [1], [2], [15] For any finitely generated monoid S, we have

$$c(S) = max\, C(S) = max\{c(b) : b \in Betti(S)\}$$

and

$$min\, C(S) = min\{c(b) : b \in Betti(S)\}.$$

Then

$$c(S) = max\{c(\beta) | \beta \in Betti(S)\}.$$

Lemma 4. ([18], Theorem 8.17) Let S be a numerical semigroup and let $s \in S \backslash \{0\}$. The number of connected components of G_s is equal to the number of R-classes in $Z(s)$.

Proposition 5. ([18], Proposition 31.) Let $S = \langle m_1, m_2, \dots, m_p \rangle$ be a numerical semigroup. S is a numerical semigroup with maximal embedding dimension if and only if $Ap(S, m_1) = \{0, m_2, \cdots, m_p\}$.

Proposition 6. ([18], Theorem 8.19) Let $S = \langle m_1, m_2, \cdots, m_p \rangle$ be a numerical semigroup and $s \in S \backslash \{0\}$. If G_s is disconnected, then $s = w + m_j$ with the nonzero $w \in Ap(S, m_1)$ for every $j \in \{2, \dots, e\}$.

Lemma 7. ([16], Theorem 7;[18], Lemma 4.27) The following conditions are equivalent.

1. $S =< 3,3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$.
2. S is a pseudo-symmetric numerical semigroup with maximal embedding dimension 3

Theorem 8. The set of Betti elements of the pseudo-symmetric numerical semigroup $S =< 3,3 + s, 3 + 2s >$ with $3 \nmid s$ and $s \in \mathbb{N}$ is the set $Betti(S) = \{6 + 2s, 6 + 3s, 6 + 4s\}$.

Proof. Let $S =< 3,3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$. Then

$$\begin{aligned} c_1 &= min\{k \in \mathbb{N} | 3k \in < 3 + s, 3 + 2s >\} \\ &= min\left\{k \in \mathbb{N} \middle| 3k \in \left\{0,3 + s, 6 + 2s, \boxed{3(2 + s)}, 6 + 4s, \dots\right\}\right\} \\ &= 2 + s \\ c_2 &= min\{k \in \mathbb{N} | k(3 + s) \in < 3,3 + 2s >\} \\ &= min\left\{k \in \mathbb{N} \middle| k(3 + s) \in \left\{0,3,6, \dots, 3 + 2s, \boxed{2(3 + s)}, 9 + 2s, \dots\right\}\right\} \\ &= 2 \\ c_3 &= min\{k \in \mathbb{N} | k(3 + 2s) \in < 3,3 + s >\} \\ &= min\left\{k \in \mathbb{N} \middle| k(3 + 2s) \in \left\{0,3,3 + s, 6,6 + s, 9,6 + 2s, 9 + s, \dots, \boxed{2(3 + 2s)}, \dots\right\}\right\} \\ &= 2 \end{aligned}$$

(Note that the Frobenius number of $A = \langle 3,3 + s \rangle$ is $F(A) = 3 + 2s$ by Proposition 1. So $k = 2$, .)
From Proposition 2, the Betti elements of S are

$$\begin{aligned} \beta_1 &= c_1 \cdot m_1 = (2 + s) \cdot 3 = 6 + 3s, \\ \beta_2 &= c_2 \cdot m_2 = 2 \cdot (3 + s) = 6 + 2s, \\ \beta_3 &= c_3 \cdot m_3 = 2 \cdot (3 + 2s) = 6 + 4s, \end{aligned}$$

and the set of Betti elements of S is

$$Betti(S) = \{6 + 2s, 6 + 3s, 6 + 4s\}.$$

Example 9. Let us consider the set S in Theorem 8. If $s = 5$, then $S =< 3,8,13 >$. Let's find the Betti elements of the pseudo-symmetric numerical semigroup S. First, let's find the numbers c_i for $i \in \{1,2,3\}$,

$$c_1 = min\{k \in \mathbb{N} | 3k \in < 8,13 >\} = min\{k \in \mathbb{N} | 3k \in \{0,8,13,16,\boxed{21},\ldots\}\} = 7$$

$$c_2 = min\{k \in \mathbb{N} | 8k \in < 3,13 >\} = min\{k \in \mathbb{N} | 8k \in \{0,3,6,9,12,13,15,\boxed{16},\ldots\}\} = 2$$

$$c_3 = min\{k \in \mathbb{N} | 13k \in < 3,8 >\} = min\{k \in \mathbb{N} | 13k \in \{0,3,6,8,9,11,12,14,15,16,17,8,20,21,22,24,25,\boxed{26},\ldots\}\} = 2$$

From Proposition 2, the Betti elements of S

$$\left.\begin{array}{l} \beta_1 = c_1.n_1 = 7.3 = 21 \\ \beta_2 = c_2.n_2 = 2.8 = 16 \\ \beta_3 = c_3.n_3 = 2.13 = 26 \end{array}\right\} \Rightarrow Betti(S) = \{16,21,26\}$$

From Theorem 8, The set of Betti elements of S is $Betti(S) = \{6 + 2.5, 6 + 3.5, 6 + 4.5\} = \{16,21,26\}$.

Theorem 10. The catenary degree of the pseudo-symmetric numerical semigroup $S =< 3, 3 + s, 3 + 2s >$ with $3 \nmid s$ and $s \in \mathbb{N}$ is $c(S) = s + 2$.

Proof. Let $S =< 3, 3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$. Firstly, we will find the factorizations of Betti elements of S.

- We write $\beta_1 = 6 + 2s = 3x_1 + (3 + s)x_2 + (3 + 2s)x_3$ $(x_1, x_2, x_3 \in \mathbb{N}_0)$ by definition the factorizations. In this case, the solution of the linear equation is found as (0,2,0) and (1,0,1). So, $Z(\beta_1) = Z(6 + 2s) = \{(0,2,0), (1,0,1)\}$.
- We write $\beta_2 = 6 + 3s = 3x_1 + (3 + s)x_2 + (3 + 2s)x_3$ $(x_1, x_2, x_3 \in \mathbb{N}_0)$ by definition the factorizations. In this case, the solution of the linear equation is found as $(s + 2,0,0)$ and (0,1,1). So, $Z(\beta_2) = Z(6 + 3s) = \{(s + 2,0,0), (0,1,1)\}$.
- We write $\beta_3 = 6 + 4s = 3x_1 + (3 + s)x_2 + (3 + 2s)x_3$ $(x_1, x_2, x_3 \in \mathbb{N}_0)$ by definition the factorizations. In this case, the solution of the linear equation is found as $(s + 1,1,0)$ and (0,0,2). So, $Z(\beta_3) = Z(6 + 4s) = \{(s + 1,1,0), (0,0,2)\}$.

So, let's find the distance of the edge between these factorizations of Betti elements of S, then find the catenary degree of Betti elements of S.

- $gcd((0,2,0),(1,0,1)) = (min(0,1), min(2,0), min(0,1)) = (0,0,0)$

$$d((0,2,0),(1,0,1)) = max\{|(0,2,0) - (0,0,0)|, |(1,0,1) - (0,0,0)|\} = max\{|(0,2,0)|, |(1,0,1)|\} = max\{2,2\} = 2$$

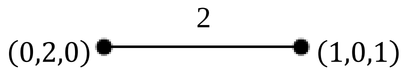

Figure 1. The catenary graph of β_1

Therefore, if we draw a graph composed of these vertices and edges in Figure 3, the catenary degree of $\beta_1 = 6 + 2s$ is 2.

- $gcd((0,1,1),(2 + s,0,0)) = (min(0,2 + s), min(1,0), min(1,0)) = (0,0,0)$

$$d(x,y) = max\{|(0,1,1) - (0,0,0)|, |(2 + s,0,0) - (0,0,0)|\} = max\{|(0,1,1)|, |(2 + s,0,0)|\} = max\{2, 2 + s\} = 2 + s$$

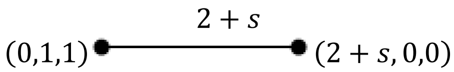

Figure 2. The catenary graph of β_2

Therefore, if we draw a graph composed of these vertices and edges in Figure 3, the catenary degree of $\beta_2 = 6 + 3s$ is $2 + s$.

- $gcd((0,0,2),(s + 1,1,0)) = (min(0,s + 1), min(0,1), min(2,0)) = (0,0,0)$

$$d(x,y) = max\{|(0,0,2) - (0,0,0)|, |(s + 1,1,0) - (0,0,0)|\} = max\{|(0,0,2)|, |(s + 1,1,0)|\} = max\{2, 2 + s\} = 2 + s$$

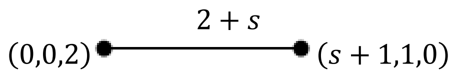

Figure 3. The catenary graph of β_3

Therefore, if we draw a graph composed of these vertices and edges in Figure 3, the catenary degree of $\beta_3 = 6 + 4s$ is $2 + s$.

According to Lemma 3,

$c(S) = max\{c(\beta)|\beta \in Betti(S)\} = max\{2, s + 2\} = s + 2.$

Example 11. If $s = 7$, then $S = \langle 3,10,17\rangle$. Let's find the catenary degree of the pseudo-symmetric numerical semigroup S.

Using the GAP package numericalsgps [8], the following results are obtained

```
S: =NumericalSemigroup([3,10,17]);
<Numerical semigroup with 3 generators>
gap> BettiElementsOfNumericalSemigroup(se);
[20, 27, 34]

gap> Factorizations(20,S);
[[0, 2, 0], [1, 0, 1]]
gap> Factorizations(27,S);
[[9, 0, 0], [0, 1, 1]]
gap> Factorizations(34,S);
[[8, 1, 0], [0, 0, 2]]
```

The catenary degree of Betti elements of S.

- $gcd\big((0,2,0),(1,0,1)\big) = (min(0,1), min(2,0), min(0,1)) = (0,0,0)$

$$d((0,2,0),(1,0,1)) = max\{|(0,2,0) - (0,0,0)|, |(1,0,1) - (0,0,0)|\} = max\{|(0,2,0)|, |(1,0,1)|\} = max\{2,2\} = 2$$

$C(\beta_1) = 2.$

- $gcd\big((0,1,1),(9,0,0)\big) = (min(0,0), min(1,0), min(1,0)) = (0,0,0)$

$$d(x,y) = max\{|(0,1,1) - (0,0,0)|, |(9,0,0) - (0,0,0)|\} = max\{|(0,1,1)|, |(9,0,0)|\} = max\{2,9\} = 9$$

$C(\beta_2) = 9.$

- $gcd\big((0,0,2),(8,1,0)\big) = (min(0,8), min(0,1), min(2,0)) = (0,0,0)$

$$d(x,y) = max\{|(0,0,2) - (0,0,0)|, |(8,1,0) - (0,0,0)|\} = max\{|(0,0,2)|, |(8,1,0)|\} = max\{2,9\} = 9$$

$C(\beta_3) = 9.$

According to Lemma 3,

$c(S) = max\{c(\beta)|\beta \in Betti(S)\} = max\{2,9\} = 9.$

From Theorem 10, the catenary degree of the pseudo-symmetric numerical semigroup S $c(S) = s + 2 = 7 + 2 = 9$.

Theorem 12. The graph of the pseudo-symmetric numerical semigroup $S =< 3, 3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$ is as following:

Table 1. The graph of $S =< 3, 3 + s, 3 + 2s >$

Graph	Connected components	Relations	Factorizations
G_{6+2s}	$\{3, 3 + 2s\}, \{3 + s\}$	$(x_1 + x_3, 2x_2)$	$(1,0,1), (0,2,0)$
G_{6+3s}	$\{3\}, \{3 + s, 3 + 2s\}$	$\big((2 + s)x_1\big), (x_2 + x_3)$	$(s + 2,0,0), (0,1,1)$
G_{6+4s}	$\{3, 3 + s\}, \{3 + 2s\}$	$\big((s + 1)x_1 + x_2\big), (2x_3)$	$(s + 1,1,0), (0,0,2)$

Proof. Let $S =< 3, 3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$. Then, $Ap(S,3) = \{w_0 = 0, w_1, w_2\} = \{0, 3 + s, 3 + 2s\}$ from Proposition 5. According to Proposition 6, $w \in Ap(S,3)\backslash\{0\} = \{3 + s, 3 + 2s\}$ and $s = w + m_j$ for $j \in \{1,2,3\}$. Thus, $s \in (Ap(S,3)\backslash\{0\}) + \{3 + s, 3 + 2s\} = \{3 + s, 3 + 2s\} + \{3 + s, 3 + 2s\} = \{6 + 2s, 6 + 3s, 6 + 4s\} = Betti(S)$. Table 1 is expressed with those obtained.

Remark 13. Assume now that $S = \langle m_1, m_2, m_3\rangle$. For $\{i,j,k\} = \{1,2,3\}$, let the non-negative integers c_i, r_{ij} and r_{ik} be defined as below.

- If $c_1m_1 = c_2m_2 = c_3m_3$, then

$$\{(c_1x_1, c_2x_2), (c_1x_1, c_3x_3)\}$$

is a minimal presentation of S.

- If $c_1m_1 \neq c_2m_2 = c_3m_3$, then $c_1m_1 = am_2 + bm_3$ with $a, b \in \mathbb{N}$. Thus,

$$\{(c_1x_1, ax_2 + bx_3), (c_2x_2, c_3x_3)\}$$

is a minimal presentation of S.

- If $\#(\{c_1x_1, c_2x_2, c_3x_3\}) = 3$, then $c_im_i = r_{ij}m_j + r_{ik}m_k$ for some non-negative integers r_{ij} and r_{ik}. Then,

$$\{(c_1x_1, r_{12}x_2 + r_{13}x_3), (c_2x_2, r_{21}x_1 + r_{23}x_3), (c_3x_3, r_{31}x_1 + r_{32}x_2)\}$$

is a minimal presentation of S.

Theorem 14. The minimal presentation of the pseudo-symmetric numerical semigroup $S =< 3,3 + s, 3 + 2s >$ with $3 \nmid s$ and $s \in \mathbb{N}$ is as following:

$$\{((s + 2)x_1, x_2 + x_3), (2x_2, x_1 + x_3), (2x_3, (s + 1)x_1 + x_2)\}$$

or

$$\{((s + 2,0,0), (0,1,1)), ((0,2,0), (1,0,1)), ((0,0,2), (s + 1,1,0))\}.$$

Proof. Let $S =< 3,3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$. For $i \in \{1,2,3\}$, the integers c_i were obtained in proof of Theorem 8. For $\{i, j, k\} = \{1,2,3\}$, the non-negative integers are obtained according to the definitions of r_{ij} and r_{ik} as follows.

$r_{21} = r_{23} = r_{32} = r_{12} = r_{13} = 1$ and $r_{31} = s + 1$.

By Remark 13, The minimal presentation of S is

$$\{((s + 2)x_1, x_2 + x_3), (2x_2, x_1 + x_3), (2x_3, (s + 1)x_1 + x_2)\}$$

or

$$\Big\{\big(((s + 2),0,0), (0,1,1)\big), \big((0,2,0), (1,0,1)\big), \Big((0,0,2), \big((s + 1), 1,0\big)\Big)\Big\}.$$

Proposition 15. ([9], Proposition 2**)** If the numerical semigroup $S = \langle m_1, m_2 m_3\rangle$ is not symmetric, then the r_{ij}, $r_{ik} \in \mathbb{N}$ are unique for $\{i, j, k\} = \{1,2,3\}$. In addition,

$$c_i = r_{ji} + r_{ki}.$$

From $m_1 < m_2 < m_3$ we will get the following result.

Lemma 16. ([9], Lemma 3) Let the numbers c_i, r_{ji} and r_{ik} be defined as we previously determined, $c_1 > r_{12} + r_{13}$ and $c_3 > r_{31} + r_{32}$. Set

$$\delta_i = |c_i - r_{ji} - r_{ik}|$$

for every $\{i, j, k\} = \{1, 2, 3\}$.

From Lemma 16**,** $\delta_1 = c_1 - r_{12} - r_{13}$ and $\delta_3 = r_{31} - r_{32} - c_3$. Also, from Proposition 15, $\delta_2 = |\delta_1 - \delta_3|$.

Lemma 17. ([9], Lemma 4) Under the standing hypothesis, $min\, \Delta\, (S) = obeb(\delta_1, \delta_3)$ and $max\, \Delta\, (S) = max\{\delta_1, \delta_3\}$.

Remark 18. Given Lemma 17, we can think $\delta_1 \neq \delta_3$ because in other case we will write $min\, \Delta\, (S) = max\, \Delta\, (S) = \delta_1 = \delta_3$. And then $\Delta(S) = \{\delta_1\}$.

Theorem 19. The Delta set of the pseudo-symmetric numerical semigroup $S =< 3,3 + s, 3 + 2s >$ with $3 \nmid s$ and $s \in \mathbb{N}$ is $\Delta(S) = \{s\}$.

Proof. Let $S =< 3,3 + s, 3 + 2s >$ for $3 \nmid s$ and $s \in \mathbb{N}$. For $\{i, j, k\} = \{1,2,3\}$, the non-negative integers c_i, r_{ij} and r_{ik} were obtained in proof of Theorem 8 and Theorem 14 as follows.

$c_1 = 2 + s, c_2 = c_3 = 2,$ $r_{21} = r_{23} = r_{32} = r_{12} = r_{13} = 1$ and $r_{31} = s + 1$.

When we write the obtained values into the equations in Lemma 16, the following equations are obtained.

$$\delta_1 = |c_1 - r_{21} - r_{13}| = |2 + s - 1 - 1| = |s| = s$$
$$\delta_2 = |c_2 - r_{12} - r_{23}| = |2 - 1 - 1| = |0| = 0$$
$$\delta_3 = |c_3 - r_{31} - r_{32}| = |2 - s - 1 - 1| = |-s| = s$$

Namely, $\delta_1 = \delta_3 = s$. Taking into account Remark 18,

$\Delta(S) = \{\delta_1\} = \{s\}$.

Example 20. If $s = 4$, then $S =< 3, 7, 11 >$.

Using the GAP package numericalsgps [8], the following results are obtained. We show how we can compute Betti elements, the catenary degree, the minimal presentation, the delta set of S, factorizations of elements of S and the graph associated to in S using GAP package numericalsgps.

```
gap> se:=NumericalSemigroup([3,7,11]);
<Numerical semigroup with 3 generators>
gap> BettiElements(se);
[14, 18, 22]
gap> CatenaryDegree(se);
6
gap> MinimalPresentation(se);
[[[0, 2, 0], [1, 0, 1]], [[5, 1, 0], [0, 0, 2]], [[6, 0, 0], [0, 1, 1]]]
gap> DeltaSet(se);
[4]
gap> Factorizations(14,se);
[[0, 2, 0], [1, 0, 1]]
gap> Factorizations(18,se);
[[6, 0, 0], [0, 1, 1]]
gap> Factorizations(22,se);
[[5, 1, 0], [0, 0, 2]]
gap>
GraphAssociatedToElementInNumericalSemigroup(14,se);
[[3, 7, 11], [[3, 11]]]
gap>
GraphAssociatedToElementInNumericalSemigroup(18,se);
[[3, 7, 11], [[7, 11]]]
gap>
GraphAssociatedToElementInNumericalSemigroup(22,se);
[[3, 7, 11], [[3, 7]]]
```

Namely, we obtain the set of Betti elements, the catenary degree, the minimal presentation and the Delta set of S from Theorem 8, Theorem 10, Theorem 14, Theorem 19.

$Betti(S) = \{6 + 2s, 6 + 3s, 6 + 4s\} = \{6 + 2 \cdot 4, 6 + 3 \cdot 4, 6 + 4 \cdot 4\} = \{14,18,22\}$,
$c(S) = s + 2 = 4 + 2 = 6$,

The minimal presentation of the pseudo-symmetric numerical semigroup S is

$$\{((s + 2,0,0), (0,1,1)), ((0,2,0), (1,0,1)), ((0,0,2), (s + 1,1,0))\} = \{((4 + 2,0,0), (0,1,1)), ((0,2,0), (1,0,1)), ((0,0,2), (4 + 1,1,0))\} = \{((6,0,0), (0,1,1)), ((0,2,0), (1,0,1)), ((0,0,2), (5,1,0))\}$$

$\Delta(S) = \{s\} = \{4\}$.

And from Theorem 12, we can construct Table 2 as follows.

Table 2. The graph of $S = \langle 3,7,11 \rangle$

Graph	Connected components	Relations	Factorizations
G_{14}	$\{3,11\}, \{7\}$	$(x_1 + x_3, 2x_2)$	$(1,0,1), (0,2,0)$
G_{18}	$\{3\}, \{7,11\}$	$(6x_1), (x_2 + x_3)$	$(6,0,0), (0,1,1)$
G_{22}	$\{3,7\}, \{11\}$	$(5x_1 + x_2), (2x_3)$	$(5,1,0), (0,0,2)$

4. Conclusion and Suggestions

This study aims to present the relationship between the Delta Set, Betti elements, catenary degree, graphs, and minimal representation of the family of pseudo-symmetric numerical semigroups and their generators. Such numerical semigroups are also of great interest in ring theory because numerical semigroups have many applications in ring theory and algebraic geometry via the valuations of one-dimensional local Noetherian domains whose value groups are numerical semigroups. Therefore, the results in this manuscript can be extended and studied by ring theory researchers.

Contributions of the authors

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Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics

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