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Effects of Lattice Frequency on Vacancy Defect Solitons in a Medium with Quadratic Nonlinear Response

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Abstract
The impact of lattice frequency on the defect lattice solitons have been investigated in a medium with quadratic nonlinear response. Governing equation of the optical system has been formed by adding an external lattice to the nonlinear Schrödinger equation with coupling to a mean term (NLSM system), and soliton solutions of the system were calculated by the squared operator method. Moreover, stability of the fundamental solitons has been examined by the linear stability spectra and nonlinear evolution of the solitons. It has been demonstrated that although higher lattice frequency extends the existence domain of propagation constant for defective lattice solitons in a quadratic nonlinear medium, it has an adverse effect on stability dynamics of the solitons.

1. Introduction

Localized solutions of wave equations (solitons) have a significant importance in nonlinear optical systems. These nonlinear optical systems can be characterized by various equations such as the Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation or the Ginzburg-Landau equation, and soliton solutions of these equations can be obtained by analytical and numerical methods [1]. The nonlinear Schrödinger (NLS) equation is used to describe wave dynamics in centro-symmetric (or cubic Kerr) media, and it is given in the (2+1) dimension as follows

$$iu_z + \frac{1}{2}\Delta u + |u|^2 u = 0 \tag{1}$$

Here, $u(x, y, z)$ denotes the slowly-varying envelope, $\Delta u = u_{xx} + u_{yy}$ shows the wave diffraction, and the cubic term $|u|^2 u$ shows change of the refractive index (Kerr effect) of the cubic medium. However, it has been shown that quadratic effects rise in many practical optical systems [2]-[6]. Indeed, the quadratic electro-optic effect occurs in all crystal structures, irrespective of symmetry. A nonlinear medium with quadratic and cubic nonlinear response can be governed by the NLS equation with coupling to a mean term (known as NLSM system) [7]-[9]. The NLSM system is denoted by

$$\begin{aligned} iu_z + \tfrac{1}{2}\Delta u + |u|^2 u - \rho \emptyset u &= 0, \\ \emptyset_{xx} + \nu \emptyset_{yy} &= (|u|^2)_{xx} \end{aligned} \tag{2}$$

where quadratic optical effects in the medium are shown by $\emptyset(x, y)$. ρ represents the strength of the quadratic electro-optic effects and ν shows the anisotropy of the medium (optical material). These equations are emerged from the interaction between the fundamental and dc fields when second-harmonic-generation is not phase matched. Thus, the NLSM systems were procured as the nonlocal-nonlinear coupling between the first field (with the cascaded effect from the second harmonic) and a static field that arises from the zeroth harmonic (mean term) [7]-[9]. The physical equivalence of the NLSM system and its derivation were discussed in detail by Ablowitz in [1].

Adding saturable nonlinearity [10],[11] or optical lattices [12],[13] to the governing equations

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are known methods for stabilization of unstable solutions. After the existence of optical lattice solitons has been experimentally proven [14], many scientific studies have been carried out in this field. In this direction, real periodic (crystal) [5],[11],[15], quasi-periodic (quasi-crystal) [16]-[18] or complex parity-time symmetric lattices [19]-[23] were added to the model equations. Also, there were studies that investigate the nonlinear wave dynamics in defective lattices [24]-[27]. These studies generally focused on the geometric (crystal) shapes of the lattices and the point, linear or volumetric defects in the lattice structures [28]. It has been demonstrated that defects in the lattice structure have significant effects on both soliton profiles and soliton dynamics [24],[25]. In addition, it has been shown that increasing the lattice (potential) depth supports stability of solitons [5],[26],[27]. However, the impact of lattice frequency on soliton dynamics in a medium with quadratic nonlinear response has not been examined yet.

In this study, the stability dynamics of lattice solitons are investigated in a medium with quadratic and cubic nonlinear response. The external lattice is chosen as a square lattice with a vacancy defect [24],[25],[28]. The soliton solutions are obtained numerically, and the stability of solitons are tested by the linear spectra and nonlinear evolution. A vacancy defect is a point defect that occurs almost in all crystalline materials when an atom is missing from the location where it supposed to be [28]. It was shown that the defects in optical materials can be produced by irradiation of high energy particle [29], and considerable improvements has been made in the design and fabrication of lattice structures with point defects [30],[31]. In other words, vacancy defects in optical lattices can be engineered. Therefore, it is important to examine the impact of lattice frequency on stability of solitons around a vacancy defect.

2. Material and Method

2.1. The Model Equations

In order to describe the quadratic nonlinear medium with an external lattice $V(x,y)$, the NLSM system (2) is extended as follows.

$$\begin{aligned} iu_z + \tfrac{1}{2}\Delta u + |u|^2 u - \rho\emptyset u - V(x,y) = 0, \\ \emptyset_{xx} + \nu\emptyset_{yy} = (|u|^2)_{xx} \end{aligned} \quad (3)$$

The lattice $V(x,y)$ in the model is chosen as a square lattice with a vacancy defect and defined as follows [24]:

$$V(x,y) = \frac{V_0}{25}\left[2\cos(k_x x) + 2\cos\left(k_y y\right) + e^{i\theta(x,y)}\right]^2 \quad (4)$$

Here, the defective point is formed by the phase function $\theta(x,y)$ that is given by

$$\theta(x,y) = \tan^{-1}\left(\frac{y-y_0}{x}\right) - \tan^{-1}\left(\frac{y+y_0}{x}\right) \quad (5)$$

V_0 is the coefficient that determines the lattice depth. A perfect periodic lattice is obtained by setting $\theta(x,y) = 0$. $\left(k_x, k_y\right)$shows the wave numbers and when $K = k_x = k_y$ and $y_0 = \pi/K$ a point defect occurs in the center $(0,0)$ of the lattice. Far from the center, a lattice structure with a period of $2\pi/K$ and a frequency of K is formed [24],[25]. Therefore, the frequency of the lattice can be controlled with the parameter K. In this study, lattice solitons are examined for lattice frequencies $K = 3, K = 4$ and $K = 5$. Accordingly, top views and diagonal cross-sections of these lattices are shown in Figure 1.

As seen in Figure 1, there is a local minimum near the center of the lattice in all cases, regardless of the lattice frequency. In other words, the change in frequency does not change the structure of the lattice qualitatively, but only makes a quantitative difference.

2.2. The squared operator method (SOM) for numerical solution

A modification of the Squared Operator Method (SOM) is used to solve the system given in equation (3) [32]. In this method, an operator is defined to linearize the governing equation (3) around the solution, and the square of this operator is iterated to obtain a convergent solution. The algorithm of the SOM is explained below.

Substituting the solution suggestion $u = U(x,y)e^{i\mu z}$ in system (3), the following operator L_0 and the associated acceleration operator M_0 are defined as

$$L_0 U = -\mu U + \frac{1}{2}\Delta U + |U|^2 U - \rho\emptyset U - V(x,y)U,$$
$$M_0 = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(L_0 U)}{K^2 + c}\right). \quad (6)$$

Here $U(x,y)$ is a real valued function, μ is the propagation constant (eigenvalue), and $\mathcal{F}$

symbolizes the Fourier transform that is applied to second-order derivatives. $K = (k_x, k_y)$ denotes wave number and $K^2 = k_x{}^2 + k_y{}^2$. c is a real positive number and chosen intuitively to scale the algorithm. $L_0 U = 0$ shows the general form of equation (3) and L_1 operator is defined as linearization of L_0 around U. The operator L_1 and its associated operator M_1 are calculated as follows

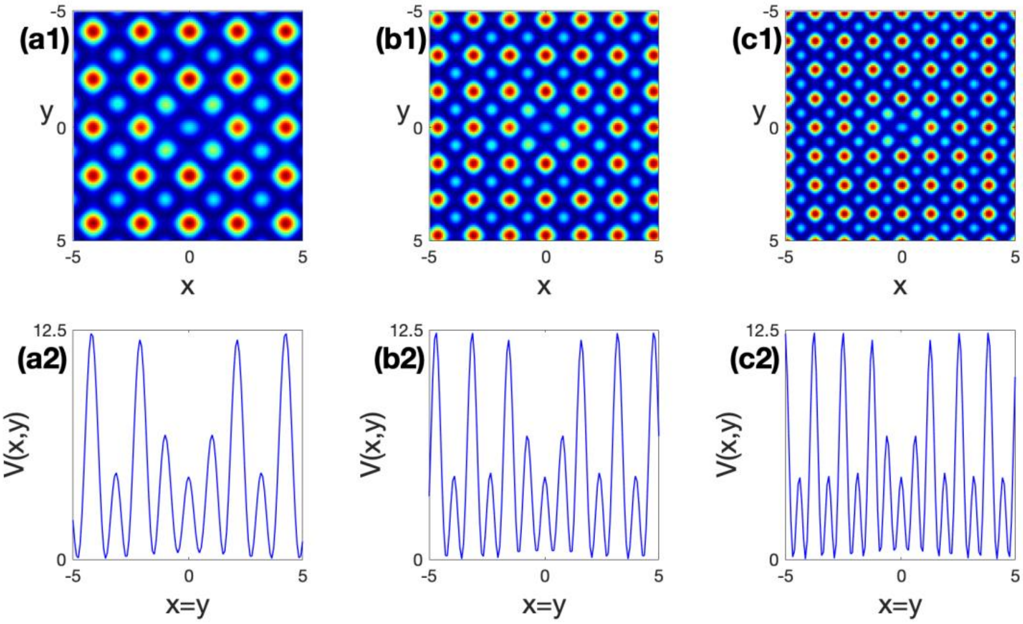

Figure 1. The top views (the first row) and diagonal cross-sections (the second row) of lattices with a vacany defect (a1)-(a2) for $K = 3$; (b1)-(b2) for $K = 4$ and (c1)-(c2) for $K = 5$. The lattice depth is $V_0 = 12.5$ for all cases considered

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$$L_1 U = -\mu M_0 + \frac{1}{2}\Delta M_0 + 3|U|^2 M_0 \\ -\rho \emptyset M_0 - V(x,y) M_0, \qquad (7) \\ M_1 = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(L_1 U)}{K^2 + c}\right)$$

After calculating the acceleration operator M_1, a convergent solution is obtained with the following iteration

$$U_{n+1} = U_n - M_1 \Delta t \qquad (8)$$

when the mean term $\emptyset(x,y)$ is iterated as follows

$$\emptyset_n = \mathcal{F}^{-1}\left(\frac{k_x{}^2}{k_x{}^2 + \nu k_y{}^2}\mathcal{F}(|U_n|^2)\right) \qquad (9)$$

Δt shows the step size of iteration, and the error E is calculated as follows at each iteration step by

$$E = \sqrt{\|U_{n+1} - U_n\|^2} \qquad (10)$$

and the iteration continues until $E < 10^{-8}$ to obtain a convergent solution.

It has been proven that, with a convenient initial condition, this algorithm produces convergent solutions for many evolution equations when Δt is less than a threshold value [32],[33]. c and Δt are fixed to 3 and 0.2 in the SOM algorithm, and the following Gaussian initial condition is used to calculate fundamental soliton solutions.

$$U_0 = e^{-[(x-x_0)+(y-y_0)]} \qquad (11)$$

The location of the initial condition on the lattice is determined by the variables x_0 and y_0. The fundamental solitons are focused on the center of the lattice ($x_0 = y_0 = 0$) near the vacancy defect, and the depth V_0 of the lattice is fixed to 12.5 for comparision with previous studies [24],[26],[27]. In addition, the quadratic term coefficient is chosen as $\rho = 0.5$ and the anisotropy coefficient as $\nu = 1.5$. These values belong to the potassium niobate ($KNbO_3$) that is an optical material used in laser systems [34].

2.3. The Linear Stability Spectra

The stability of fundamental solitons, which have been obtained by the SOM, are examined with linear eigenvalue spectra and nonlinear evolution of the peak amplitudes. The spectra are obtained by linearization of the governing equation (3) around the fundamental solitons (u_0). To do that, the solution u_0 is perturbed as follows.

$$U = e^{i\mu z}\left[u_0(x,y) + g(x,y)e^{\lambda z} + h^*(x,y)e^{\lambda^* z}\right] \quad (12)$$

where $g, h \ll 1$ are infinitesimal perturbations. Substituting U into equation (3) and neglecting second order terms, the following linear equation system is obtained

$$AV = \lambda V \quad (13)$$

where matrices are defined by

$$A = i\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad V = \begin{pmatrix} g \\ h \end{pmatrix} \quad (14)$$

and the elements of matrix A are

$$\begin{gathered} A_{11} = A_{22} = 0, \\ A_{12} = -\mu + \tfrac{1}{2}\Delta + U^2 - \rho\varphi - V, \\ A_{21} = -\mu + \tfrac{1}{2}\Delta + 3U^2 - \rho\varphi - V. \end{gathered} \quad (15)$$

The eigenvalues of the matrix A are calculated by the Fourier collocation method [33]. If there is a positive real part in the eigenvalue spectrum of the soliton, it will be considered as linearly unstable. It is also known that there is a strong relation between the power ($P = \iint_{-\infty}^{\infty} |U|^2 dxdy$) and stability of solitons. Vakhitov and Kolokolov demonstrated that solitons can be linearly stable, only if their powers increase as the propagation constant (μ) is increased [35]. In other words, a necessary condition for the linear stability is that the slope of the $P-\mu$ graph must be positive i.e.,

$$\frac{\partial P}{\partial \mu} > 0 \quad (16)$$

In order to confirm the results presented by the linear stability analysis, the nonlinear stability of the solitons are tested by investigating the evolution of peak amplitudes. For the nonlinear evolution, the derivatives in equation (3) are calculated by the finite difference method, and the evolution in the z direction is performed by the fourth-order Runge-Kutta method.

3. Results and Discussion

Using the parameter values defined above, the fundamental solitons are obtained for $K = 3, K = 4, K = 5$ and displayed in Figure 2. It is seen that the soliton amplitudes get larger as the lattice frequency K increases.

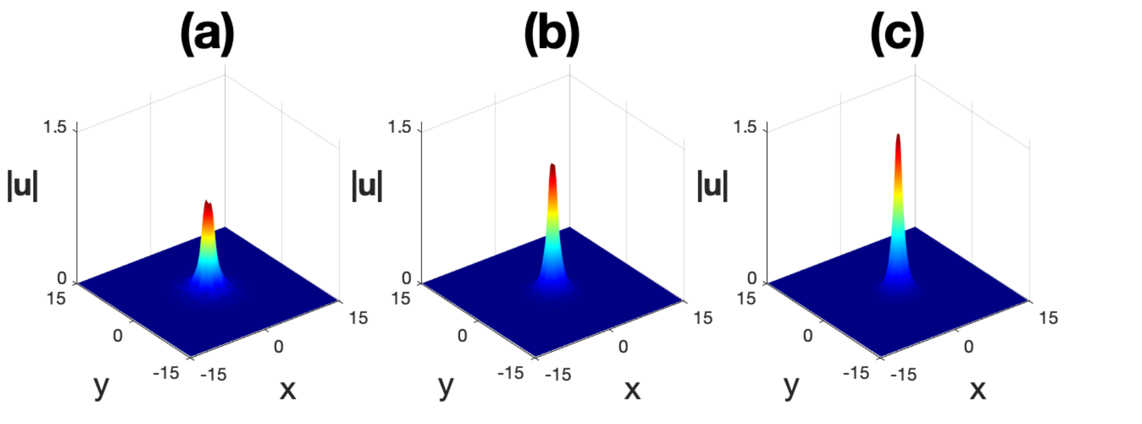

Figure 2. Fundamental solitons near the vacancy defect in a medium with quadratic nonlinear response (a) for $K = 3$; (b) for $K = 4$ and (c) for $K = 5$. $V_0 = 12.5$ and $\mu = -1.6$ in all cases.

Before the stability analysis, the power of fundamental solitons are examined in Figure 3. As seen in Figure 3 (a1), the slope of $P-\mu$ diagrams are positive in all cases ($K = 3, K = 4, K = 5$). This fact reveals that the considered solitons met the necessary condition for linear stability. In order to support this result, linear spectra of solitons are calculated at each point of the existence domain, and the linear stability (solid line) and instability (dotted line) intervals are determined for $K = 3, K = 4$ and $K = 5$ in Figure 3 (a2). As can be seen from Figure 3 (a2), if the lattice frequency $K = 3$, solitons are linearly stable when the propagation constant is less than a critical value ($\mu < -1.41$). On the other hand, solitons are linearly unstable at high values of the lattice frequency ($K = 4$ and $K = 5$) everywhere on their existence domain. Moreover, the power of solitons are investigated for varied strength of the quadratic electro-optic effects (ρ) in Figure 3 (b1), and linear stability interval of ρ is determined in Figure 3 (b2) for $K = 3, K = 4$ and $K = 5$. It can be seen that the power of solitons grow up with increased lattice frequency K (see Figure 3 (b1)), and both the domain of existence and stability interval of ρ are extended as the lattice frequency is decreased (see Figure 3 (b2)). These results reveal that higher lattice frequency has an adverse effect on dynamics of the vacancy defect solitons in a medium with quadratic nonlinear response.

It is noted that in Figure 3 (a2), the marked points 'a', 'b' and 'c' correspond to the fundamental solitons that are shown in Figure 2 (a), (b) and (c), respectively. The eigenvalue spectra of these solitons are displayed in Figure 4.

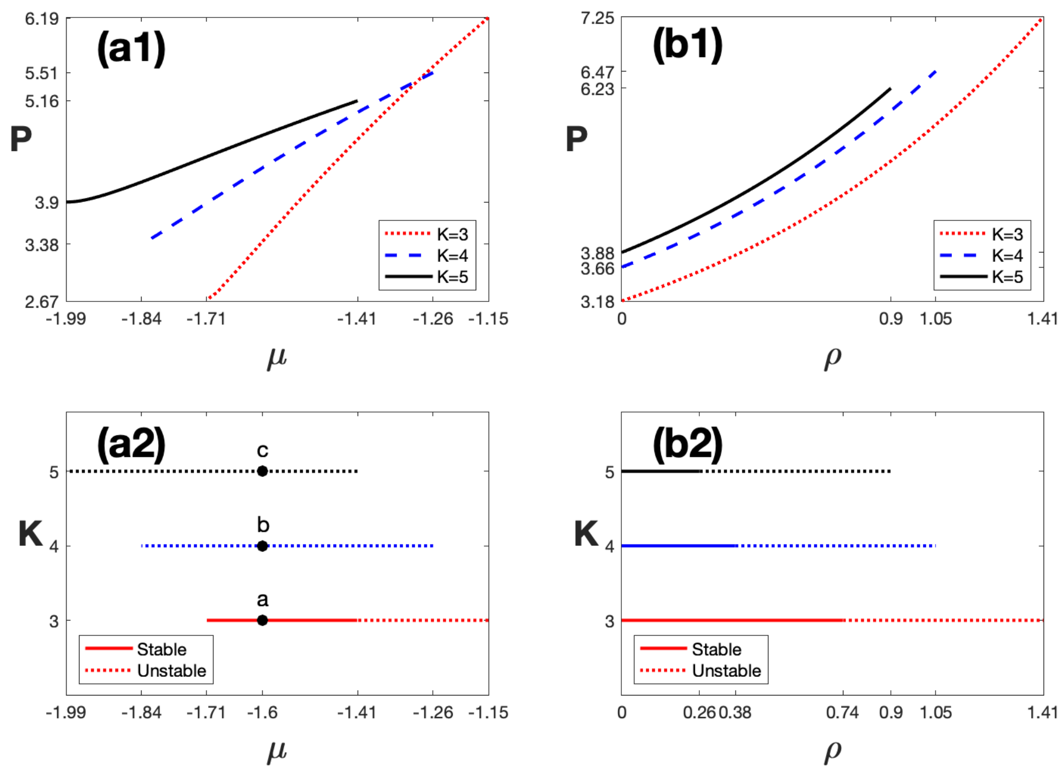

Figure 3. Power of solitons (a1) for varied eigenvalue (μ) and (b1) for varied strength of the quadratic electro-optic effects ρ when $K = 3, K = 4$ and $K = 5$. Stability intervals of solitons (a2) for μ and (b2) for ρ when $K = 3, K = 4$ and $K = 5$. The points 'a', 'b', 'c' correspond to the fundamental solitons shown in Figure 2 (a), (b), (c), respectively.

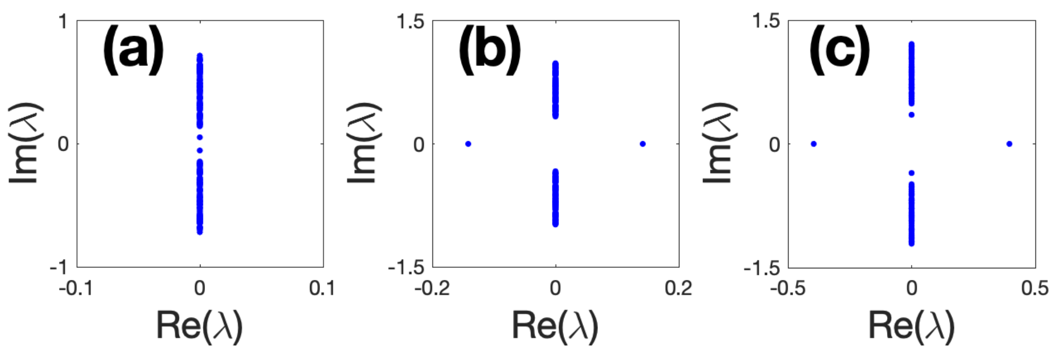

Figure 4. Linear spectra of fundamental solitons that are obtained at points 'a', 'b' and 'c' in Figure 3 (a2).

For the point 'a', there is no real part of the eigenvalue in the spectrum of the fundamental soliton that is obtained with a low lattice frequency ($K = 3$). On the contrary, at the points 'b' and 'c', there are positive real parts in the eigenvalue spectra of the solitons that are obtained for $K = 4$ and $K = 5$, respectively. This analysis indicates the linear instability of the solitons that are generated with higher frequencies at points 'b' and 'c'.

In order to cofirm the linear stability results, the nonlinear evolution of fundamental solitons, that are obtained at points 'a', 'b' and 'c' in Figure 3 (a2), are examined in Figure 5. As seen in Figure 5, value of the peak amplitude oscillates with relatively small amplitudes for the soliton obtained with the low lattice frequency ($K = 3$) at the point 'a'. This result shows the nonlinear stability of the soliton considered. The amplitudes of solitons obtained at 'b' and 'c' points show a significant increase during the evolution. Therefore solitons obtained with high lattice frequencies ($K = 4$ and $K = 5$) are nonlinearly unstable. Furthermore, it is observed that results of the linear and nonlinear stability analysis are consistent.

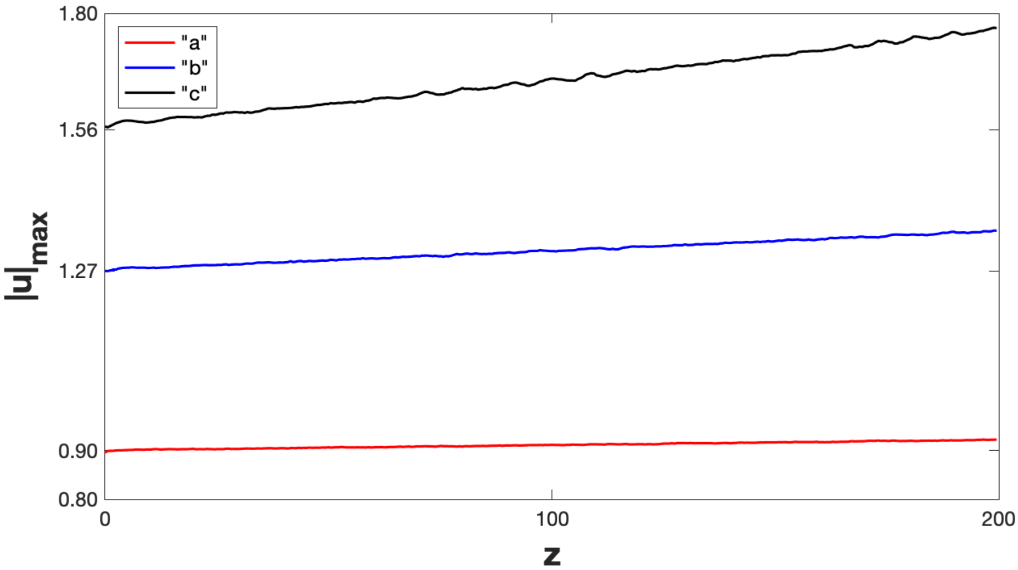

Figure 5. The nonlinear evolution of the peak amplitudes for the fundamental solitons that are obtained at points 'a', 'b' and 'c' in Figure 3 (a2). Here, the step size in the z direction is $dz = 0.01$.

4. Conclusion and Suggestions

The effects of lattice frequency on defect lattice solitons have been investigated in a quadratic medium. An external lattice was added to the NLSM system to describe the optical system, and convergent soliton solutions were obtained by the squared operator method. Stability analyses of the calculated solitons have been examined by the linear spectra and nonlinear evolutions. After performing power analysis in the domain of existence for the solitons with varied frequencies, the linear stability intervals were determined by eigenvalue spectra.

The linear spectra analysis showed that solitons obtained with low lattice frequency can be stable when the propagation constant is less than a certain threshold value ($\mu < -1.41$), whereas solitons obtained with high lattice frequency are unstable everywhere on their existence domain. These results were confirmed by examining the nonlinear evolutions of the solitons considered. Furthermore, the stability of solitons has been investigated for varied strength of the quadratic electro-optic effects (ρ) by variation of lattice frequency and, it has been observed that higher lattice frequency has an adverse effect on dynamics of the vacancy defect solitons in a medium with quadratic nonlinear response.

As a result, it has been observed that although higher lattice frequency extends the existence domain of propagation constant μ for vacancy defect solitons in a quadratic nonlinear medium, it negatively affects the stability properties of the solitons.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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