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Research Article

Characterization of bipolar ultrametric spaces and fixed point theorems

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Abstract

Ultrametricity condition on bipolar metric spaces is considered and a geometric characterization of bipolar ultrametric spaces is given. Also embedding a bipolar ultrametric space into a pseudo-ultrametric space is discussed and, some conditions are found to be able to embed them into an ultrametric space. Finally, some fixed point theorems on bipolar ultrametric spaces are proven.

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1. Introduction

Having a considerably rich structure providing a basis to form a theory of analysis on arbitrary sets, and yet being able to represent a wide range of well-behaved topological spaces, since their introduction by Fréchet, metric spaces have not only brought about one of the major areas of study in mathematics, but also been subject to many generalizations, enrichments and modifications in a wide variety of forms [1, 2, 7, 11, 18, 21].

Among many modifications on the notion of metric space, here we are particularly interested in ultrametric spaces and bipolar metric spaces, the former being a special subclass, while the other is a generalization, of metric spaces. After presenting introductive information in this chapter, we closely investigate the notion of bipolar ultrametric space in Chapter 2, and as the main result, we give a characterization of bipolar ultrametric spaces in terms of bipolar subspaces of pseudo-ultrametric spaces, which will be stated in Corollary 2.10. This characterization, makes an important contribution to understanding the nature of bipolar ultrametric spaces. In fact, this harmonious nature of bipolar ultrametric spaces is proven to be useful in applications, by giving some fixed point theorems on this structure in Chapter 3.

Ultametric spaces date back to 1970's [21] and despite possessing many strange properties that might be seen unnatural at first glance, they have been proved to be surprisingly useful with numerous naturally occurring applications in natural sciences [9, 15, 19, 22]. They also function as useful tools on many mathematical areas such as p-adic numbers, non-Archimedean analysis, combinatorics, graph theory and especially trees [6, 8].

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An ultrametric space (X, d) is a metric space such that the inequality,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

so-called the strong triangle inequality, is satisfied for all $x, y, z \in X$. More generally, a pseudo-ultrametric space is defined to be a pseudo-metric space with the strong triangle inequality, that is an analogue of ultrametric spaces, which allows distinct points to have distance zero [21].

On the other hand, the recently introduced bipolar metric spaces [11], deal with the case where a sense of natural distance available between a pair of objects of dissimilar or somewhat opposite type, while the distances between similar objects are not well-defined, or the case that the full set of distances is not provided for economic reasons, where a relatively small subset of all distances is enough for some purpose, and they have been mainly studied from the aspect of fixed point theory [4, 5, 10–14, 16, 17, 20].

A bipolar metric space is a pair (X, Y, b), where X and Y are nonempty sets and $b : X \times Y \to \mathbb{R}^+ \cup \{0\}$ is a bipolar metric on the pair (X, Y), that is, a function having the properties

$$b(x, y) = 0 \iff x = y$$
$$b(u, v) = b(v, u)$$
$$b(x, y) \leq b(x, y') + b(x', y') + b(x', y)$$

for all $x, x' \in X$, $y, y' \in Y$ and $u, v \in X \cap Y$. More generally, if all axioms, except the implication $b(x, y) = 0 \implies x = y$, are satisfied, then (X, Y, b) is called a bipolar pseudo-metric space [11].

According to the definition of a bipolar metric space, for a given metric space (X, d), (X, X, d) is always a bipolar metric space. In fact, (X, d) and (X, X, d) are generally kept identical, which allows considering bipolar metric spaces as a generalization of metric spaces.

A bipolar subspace of a bipolar pseudo-metric space (X, Y, b) is the bipolar pseudo-metric space (A, B, b), where A and B are nonempty subsets of X and Y, respectively, and where b of (A, B, b) actually denotes the restriction of b in (X, Y, b). In particular, a subspace of (X, Y, b) is a bipolar subspace (A, B, b), such that $A = X \cap U$ and $B = Y \cap U$ for some set $U \subseteq X \cup Y$ [4].

Once a bipolar metric space (X, Y, b) is given, we have two pseudo-metric spaces, one being (X, b_X) where $b : X \times X \to \mathbb{R}^+ \cup \{0\}$ is given by

$$b_X(x_1, x_2) = \sup_{y \in Y} |b(x_1, y) - b(x_2, y)|$$

and the other (Y, b_Y) where $b : Y \times Y \to \mathbb{R}^+ \cup \{0\}$ is given by

$$b_Y(y_1, y_2) = \sup_{x \in X} |b(x, y_1) - b(x, y_2)|.$$

These are called inner pseudo-metric spaces associated with (X, Y, b). If both inner pseudo-metric spaces are in particular metric spaces, then (X, Y, b) is called to be bicharacterized [11].

Let (X, Y, b) be a bipolar metric space, $x_0 \in X$, $y_0 \in Y$ and $r > 0$. A left-centric closed ball with center x_0 and radius r is the set

$$C_X(x_0, r) = \{y \in Y : b(x_0, y) \leq r\},$$

and

$$C_Y(y_0, r) = \{x \in X : b(x, y_0) \leq r\}$$

is called a right-centric closed ball with center y_0 and radius r [4].

For two given bipolar metric spaces (X, Y, b) and (X', Y', b'), a *covariant mapping* (or *mapping* for short) from (X, Y, b) to (X', Y', b') is a function $f : X \cup Y \to X' \cup Y'$ such that $f(X) \subseteq X'$, $f(Y) \subseteq Y'$, and it is denoted as $f : (X, Y, b) \rightrightarrows (X', Y', b')$. Contrastly, a *contravariant mapping* from (X, Y, b) to (X', Y', b') is a function $f : X \cup Y \to X' \cup Y'$ such that $f(X) \subseteq Y'$, $f(Y) \subseteq X'$, and this is denoted as $f : (X, Y, b) \rightleftarrows (X', Y', b')$, or alternatively $f : (X, Y, b) \looparrowright (X', Y', b')$. In particular, a contravariant mapping $f : (X, Y, b) \rightleftarrows (X, Y, b)$ is said to be a contravariant self-mapping of (X, Y, b) [11].

2. Bipolar ultrametric spaces

We begin by introducing the notion of a bipolar ultrametric space.

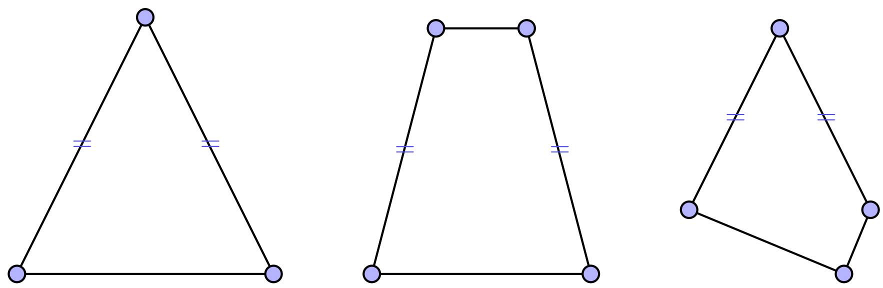
Definition 2.1. Let (X, Y, b) be a bipolar metric space. If the strong quadrilateral inequality

$$b(x, y) \le \max\{b(x, y'), b(x', y'), b(x', y)\}$$

holds for all $x, x' \in X$, and $y, y' \in Y$, then (X, Y, b) is called a bipolar ultrametric space.

Remark 2.2. If (X, Y, b) is a bipolar ultrametric space and $X = Y$, then (X, b) is an ultrametric space, and if (X, d) is an ultrametric space, then (X, X, d) is a bipolar ultrametric space.

A high isosceles triangle is an isosceles triangle whose legs are not shorter than its base. It is known that a metric d on a set X is an ultrametric if and only if every triangle generated by three different points is a high isosceles triangle [3]. In analogy with this terminology, we call a quadrilateral a high isosceles quadrilateral if it has at least two equal sides, which are not shorter than other sides.



a high isosceles triangle — a high isosceles quadrilateral with opposite long edges — a high isosceles quadrilateral with adjecent long edges

Lemma 2.3. *A bipolar metric space (X, Y, b) is an bipolar ultrametric space if and only if the quadrilateral (possibly with zero-lenght edges) formed by the distances $b(x_1, y_1)$, $b(x_2, y_1)$, $b(x_2, y_2)$ and $b(x_1, y_2)$ is a high isosceles quadrilateral for each $x_1, x_2 \in X$, $y_1, y_2 \in Y$.*

Proof. Let (X, Y, b) be a bipolar ultrametric space and $x_1, x_2 \in X$, $y_1, y_2 \in Y$. For simplicity we say $b(x_i, y_j) = a_{ij}$ for $i = 1, 2$. Then we have these inequalities:

$$\begin{aligned} a_{11} &\le \max\{a_{12}, a_{22}, a_{21}\} \\ a_{12} &\le \max\{a_{11}, a_{21}, a_{22}\} \\ a_{21} &\le \max\{a_{22}, a_{12}, a_{11}\} \\ a_{22} &\le \max\{a_{21}, a_{11}, a_{12}\} \end{aligned} \tag{2.1}$$

Note that, $a_{11} \le \max\{a_{12}, a_{22}, a_{21}\}$ implies at least one of the three cases: $a_{11} \le a_{12}$, $a_{11} \le a_{22}$ or $a_{11} \le a_{21}$. Similarly, each of the other three inequalities can be reduced to three cases. This gives 3^4 possible cases in total. Grouping and analyzing these 81 cases, we have straightforwardly seen that the quadrilateral is always a high isosceles quadrilateral.

Conversely, let (X, Y, b) be a bipolar metric space, in which distances between four different points $x_1, x_2 \in X$, $y_1, y_2 \in Y$ always form a high isosceles quadrilateral.

In the cases where two adjacent sides are equal and not shorter than the other two, say $a_{11} = a_{12} \geq a_{22}, a_{21}$, all maximums at the right sides of (2.1) are equal to $a_{11} = a_{12}$, so that the strong quadrilateral inequality will be hold in each case.

Similarly, for the cases where two opposite sides are equal and not shorter than two others, the inequalities in (2.1) are satisfied again. □

Definition 2.4. Let (X, Y, b) is a bipolar metric space. The left locus for given two points $x_1, x_2 \in X$, is the set

$$\mathrm{Loc}_X(x_1, x_2) = \{y \in Y : b(x_1, y) = b(x_2, y)\}$$

and similarly the right locus for points $y_1, y_2 \in Y$ is defined as

$$\mathrm{Loc}_Y(y_1, y_2) = \{x \in X : b(x, y_1) = b(x, y_2)\}.$$

Clearly for any point $x \in X$ and $y \in Y$, $\mathrm{Loc}_X(x, x) = Y$ and $\mathrm{Loc}_Y(y, y) = X$. Moreover, a bipolar metric space is bicharacterized if and only if $\mathrm{Loc}_X(x_1, x_2) \neq Y$ and $\mathrm{Loc}_Y(y_1, y_2) \neq X$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$. Thus, bicharacterizedness can be thought as a seperation property on bipolar metric spoaces.

It is known that every bipolar metric space (X, Y, b) is embeddable into a pseudo-metric space $(X \cup Y, d)$. In particular, it is possible to embed a bicharacterized bipolar metric space into a metric space [11]. In the following, we obtain some similar results for the case of bipolar ultrametric spaces.

Theorem 2.5. *A bipolar ultrametric $b : X \times Y \longrightarrow \mathbb{R}^+ \cup \{0\}$ can be extended to a pseudo-ultrametric $d : (X \cup Y) \times (X \cup Y) \longrightarrow \mathbb{R}^+ \cup \{0\}$.*

Proof. We need to construct a function $d : (X \cup Y) \times (X \cup Y) \longrightarrow \mathbb{R}^+ \cup \{0\}$, which satisfies ultrametricity conditions on $X \cup Y$, and accepts the function $b : X \times Y \longrightarrow \mathbb{R}^+ \cup \{0\}$ as a restriction. For these purpose, we need to consider the infimal distance between the parts X and Y of the bipolar ultrametric space (X, Y, b), which clearly equals to 0 if X and Y have a nonempty intersection, but may be equal or greater than 0, if $X \cap Y = \varnothing$. Denote this infimal distance by m, that is, let $m = \inf\{b(x, y) : x \in X, y \in Y\} \geq 0$. We define the function $d : (X \cup Y)^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ as follows:

For $x \in X, y \in Y$, define $d(x, y) := b(x, y)$ and $d(y, x) := b(x, y)$.

For $x_1, x_2 \in X$, as $x_1 = x_2$ implies $\mathrm{Loc}_X(x_1, x_2) = Y$, we have three distinct cases: $x_1 = x_2$, $x_1 \neq x_2$ where $\mathrm{Loc}_X(x_1, x_2) = Y$ and $x_1 \neq x_2$ where $\mathrm{Loc}_X(x_1, x_2) \neq Y$. For the case $x_1 = x_2$, we define $d(x_1, x_2) := 0$. If $x_1 \neq x_2$ and $\mathrm{Loc}_X(x_1, x_2) = Y$ we define $d(x_1, x_2) := m$, where m is the invariant of the bipolar metric space (X, Y, b), defined above. On the other hand, if $x_1 \neq x_2$ but $\mathrm{Loc}_X(x_1, x_2) \neq Y$; then we pick an arbitrary $y \in Y$ such that $y \notin \mathrm{Loc}_X(x_1, x_2)$, and define $d(x_1, x_2) := \max\{b(x_1, y), b(x_2, y)\}$. Below, we will show that this value is independent of the selection of y.

Similarly, for $y_1, y_2 \in Y$ we define $d(y_1, y_2) := 0$ if $y_1 = y_2$, $d(y_1, y_2) := m$ if $\mathrm{Loc}_Y(y_1, y_2) = X$ where $y_1 \neq y_2$ and $d(y_1, y_2) := \max\{b(y_1, x), b(y_2, x)\}$ if $x \notin \mathrm{Loc}_Y(y_1, y_2)$ where $x \in X$.

First, we shortly deal with well-definedness of d.

For the case where $y, y' \notin \mathrm{Loc}_X(x_1, x_2)$ and $y, y' \in Y$, we have two definitions of $d(x_1, x_2)$: $d(x_1, x_2) = \max\{b(x_1, y), b(x_2, y)\}$ and $d(x_1, x_2) = \max\{b(x_1, y'), b(x_2, y')\}$. Hence, one have to verify that

$$\max\{b(x_1, y), b(x_2, y)\} = \max\{b(x_1, y'), b(x_2, y')\}.$$

Since $y, y' \notin \mathrm{Loc}_X(x_1, x_2)$, we have $b(x_1, y) \neq b(x_2, y)$ and $b(x_1, y') \neq b(x_2, y')$. Thus the high isosceles quadrilateral of these four distances has one of its longest sides from the set $\{b(x_1, y), b(x_2, y)\}$ and the other longest side from the set $\{b(x_1, y'), b(x_2, y')\}$. This shows that

$$\max\{b(x_1, y), b(x_2, y)\} = \max\{b(x_1, y'), b(x_2, y')\},$$

that is $d(x_1, x_2)$ is independent of the choice of $y \in Y \setminus \mathrm{Loc}_X(x_1, x_2)$.

Another obstacle in the definition of the function d, arises in the case where $X \cap Y \neq \varnothing$. For instance, let $x \in X$, and $u \in X \cup Y$. Then, since $x \in X$ and $u \in Y$, $d(x,u)$ is defined to be equal to $b(x,u)$. But, since $x \in X$ and $u \in X$ at the same time, the definition of $d(x,u)$ must also agree the formula given for two elements of X. So, we must also verify for $x \in X$ and $u \in X \cap Y$ that two different definitions for $d(x,u)$, one is given by considering $u \in X$, and the other is given by considering $u \in Y$, coincide. Under the consideration that $x \in X$ and $u \in Y$, we have $d(x,y) = b(x,y)$. On the other hand, if we consider u as an element of X, then there are three cases. In the first case, where $x = u$, the two definitions for $d(x,u)$ coincide on the value zero. The second case, where $\mathrm{Loc}_X(x,u) = Y$ and $x \neq u$ is impossible since $u \in Y = \mathrm{Loc}_X(x,u)$ contradicts with $x \neq u$ by $b(x,u) = b(u,u) = 0$. In the third case, where there is some $y \in Y$ such that $y \notin \mathrm{Loc}_X(x,u)$, we have $b(x,y) \neq b(u,y)$. Since $u \in \mathrm{Loc}_X(x,u)$ implies $x = y$ contradictorily with $b(x,y) \neq b(u,y)$, we have $u \notin \mathrm{Loc}_X(x,u)$, which means that

$$d(x,u) = \max\{b(x,y), b(u,y)\} = \max\{b(x,u), b(u,u)\} = b(x,u).$$

Hence, both definitions for $d(x,u)$ coincide.

Similar discussions apply for the symmetrical cases in which the roles of the sets X and Y are interchanged. Hence, d is well-defined and clearly extends b from $X \times Y$ to $(X \cup Y)^2$. So, it only remains to show that d is a pseudo-ultrametric on the set $X \cup Y$.

If any two of three given points z_1, z_2 and z_3 from $X \cup Y$ are equal, then d trivially satisfies the strong triangle inequality for z_1, z_2 and z_3. Herewith we suppose in all the following cases that the three points given are different from each other.

Case 1. In this case we show that

$$d(x_1, x_3) \leq \max\{d(x_1, x_2), d(x_2, x_3)\} \tag{2.2}$$

for $x_1, x_2, x_3 \in X$.

Case 1.a. If $d(x_1, x_3) = m = \inf\{b(x,y) : x \in X, y \in Y\}$, then the inequality (2.2) is satisfied since all three points have been assumed to be different. To see this, note that under the definition of d, the minimum possible value of $d(x_1, x_2)$ for $x_1 \neq x_2$, is m.

Case 1.b. If $d(x_1, x_3) > m$, then by the definition of d, there exists a $y \in \mathrm{Loc}_X(x_1, x_3)$ such that $d(x_1, x_3) = \max\{b(x_1, y), b(x_3, y)\}$. Then also $b(x_1, y) \neq b(x_3, y)$. This means that also either $b(x_1, y) \neq b(x_2, y)$ or $b(x_2, y) \neq b(x_3, y)$.

Case 1.b.i. If $b(x_1, y) = b(x_2, y)$, then $b(x_2, y) \neq b(x_3, y)$ so that $y \in Y \setminus \mathrm{Loc}_X(x_1, x_3)$ and

$$d(x_2, x_3) = \max\{b(x_2, y), b(x_3, y)\} = \max\{b(x_1, y), b(x_3, y)\} = d(x_1, x_3)$$

which makes the inequality (2.2) satisfied.

Case 1.b.ii. If $b(x_2, y) = b(x_3, y)$, then this leads to inequality (2.2) similarly to the previous case.

Case 1.b.iii. If $b(x_1, y) \neq b(x_2, y) \neq b(x_3, y)$, then

$$\begin{aligned} d(x_1, x_3) &\leq \max\{b(x_1, y), b(x_3, y)\} \\ &\leq \max\{b(x_1, y), b(x_2, y), b(x_3, y)\} \\ &= \max\{\max\{b(x_1, y), b(x_2, y)\}, \max\{b(x_2, y), b(x_3, y)\}\} \\ &= \max\{d(x_1, x_2), d(x_2, x_3)\}. \end{aligned}$$

Case 2. In this case we show that

$$d(x_1, x_2) \leq \max\{d(x_1, y_1), d(y_1, x_2)\} \tag{2.3}$$

for $x_1, x_2 \in X, y_1 \in Y$.

Case 2.a. If $y_1 \in \mathrm{Loc}_X(x_1, x_2)$, then $b(x_1, y_1) = b(x_2, y_1)$.

Case 2.a.i. If $d(x_1, x_2) = m$, then (2.3) is easily seen, since we assume that x_1, x_2 and y_1 are different points.

Case 2.a.ii. If $d(x_1,x_2) > m$ then there exists a $y \in \mathrm{Loc}_X(x_1,x_2)$, that is $b(x_1,y) \neq b(x_2,y)$ and $d(x_1,x_2) = \max\{b(x_1,y), b(x_2,y)\}$. $b(x_1,y) \neq b(x_2,y)$ and $b(x_1,y_1) = b(x_2,y_1)$ imply that $b(x_1,y_1)$ is one of the longest sides of the high isosceles quadrilateral $b(x_1,y), b(x_2,y), b(x_2,y_1), b(x_1,y_1)$, that is

$$\begin{aligned} d(x_1,x_2) &\leq \max\{b(x_1,y), b(x_2,y)\} \\ &\leq \max\{b(x_1,y), b(x_2,y), b(x_1,y_1), b(x_2,y_1)\} \\ &= b(x_1,y_1) \\ &= d(x_1,y_1) \\ &= \max\{d(x_1,y_1), d(x_2,y_1)\}. \end{aligned}$$

Case 2.b. If $y_1 \notin \mathrm{Loc}_X(x_1,x_2)$, then this leads to (2.3) since the equalities

$$d(x_1,x_2) = \max\{b(x_1,y_1), b(x_2,y_1)\} = \max\{d(x_1,y_1), d(x_2,y_1)\}$$

are satisfied.

Case 3. In this case, we want to show that

$$d(x_1,y_1) \leq \max\{d(x_1,x_2), d(x_2,y_1)\} \tag{2.4}$$

for $x_1, x_2 \in X, y_1 \in Y$. We assume the contrary that

$$d(x_1,y_1) > \max\{d(x_1,x_2), d(x_2,y_1)\}.$$

Then $y_1 \in \mathrm{Loc}_X(x_1,x_2)$ by $b(x_1,y_1) = d(x_1,y_1) \neq d(x_2,y_1) = b(x_2,y_1)$. Here

$$d(x_1,x_2) = \max\{d(x_1,y_1), d(x_2,y_1)\} \geq d(x_1,y_1)$$

contradics with

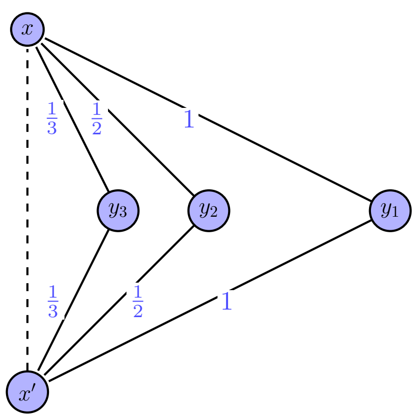
$$d(x_1,y_1) > \max\{d(x_1,x_2), d(x_2,y_1)\},$$

so that we have (2.4).

Other cases regarding the strong triangle inequalities with three points from Y and with one point from X, two points from Y are taken, are essentially similar to the cases taken into account above. Hence, d is a pseudo-metric extension of b, in other words the bipolar ultrametric space (X,Y,b) is embeddable into the pseudo-ultrametric space $(X \cup Y, d)$. $\square$

It is not always possible to embed a bipolar ultrametric space into an ultrametric space as illustrated in the following example.

Example 2.6. Let $X = \{x, x'\}$ and $Y = \{y_n : n \in \mathbb{N}\}$. Let the distances on $X \times Y$ be defined as in the following diagram:



Clearly since all quadrilaterals are high isosceles quadrilateral, this gives a bipolar ultrametric space and it can be embedded into a pseudo-ultrametric space by Theorem 2.5, however it cannot be embedded into an ultrametric space since the high isosceles triangle characterization for ultrametric spaces, forces $d(x,x') \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, that is $d(x,x') = 0$.

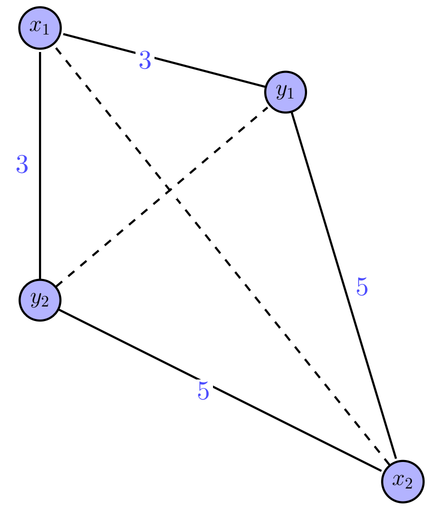
The following corollary, which is easily seen from the construction of d in the proof of Theorem 2.5, gives a sufficient condition for a bipolar ultrametric space to be embeddable into an ultrametric space.

Corollary 2.7. *Any bipolar ultrametric space (X,Y,b) with the property that*

$$\inf\{b(x,y) : x \in X \backslash Y,\, y \in Y \backslash X\} \neq 0,$$

is embeddable into an ultrametric space. In particular every finite bipolar ultrametric space is embeddable into an ultrametric space.

Example 2.8. Given two disjoint sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and a bipolar ultrametric b on (X,Y) with the following diagram:



To extend b to an ultrametric d on $X \cup Y$, in addition to the zero distances on the diagonal of $(X \cup Y)^2$, one must also define the non-trivial distances $d(x_1,x_2) = d(x_2,x_1)$ and $d(y_1,y_2) = d(y_2,y_1)$.

To make the triangle x_1, y_1, x_2 an high isosceles, the only option for the value of $d(x_1,x_2) = d(x_2,x_1)$ is 5, however $d(y_1,y_2)$ can get any value from the interval $(0,3]$.

As seen in the above example, an embedding of a bipolar ultrametric space into an ultrametric space, doesn't necessarily have to be unique. The following corollary express a condition for the uniquiness of such an embedding.

Theorem 2.9. *Every bicharacterized bipolar ultrametric space (X,Y,b) is embeddable into an ultrametric space $(X \cup Y, d)$ in a unique way.*

Proof. Since (X,Y,b) is bicharacterized, the inner pseudo-metric b_X on the set X is a metric, so that for $x_1, x_2 \in X$, $x_1 \neq x_2$,

$$b_X(x_1,x_2) = \sup_{y \in Y} |b(x_1,y) - b(x_2,y)| = c > 0.$$

This means that $b(x_1,y) \neq b(x_2,y)$ for some $y \in Y$, that is $y \notin \mathrm{Loc}_X(x_1,x_2)$. Following the construction in the proof of Theorem 2.5, we see that

$$d(x_1,x_2) = \max\{b(x_1,y), b(x_2,y)\} > 0,$$

since $b(x_1,y) \neq b(x_2,y)$. Similar result is also true for pairs of different elements of Y. Hence d is a metric on $X \cup Y$.

To see the uniqueness, we observe from the above discussion that for $x_1, x_2 \in X$, $x_1 \neq x_2$, there is some $y \in Y$ such that

$$d(x_1, y) = b(x_1, y) \neq b(x_2, y) = d(x_2, y) = d(y, x_2).$$

Then $d(x_1, y)$ and $d(y, x_2)$ are different lengths on the triangle $x_1 y x_2$, and the high isosceles triangle characterization for ultrametric spaces, leaves only one option for the value of $d(x_1, x_2)$. □

By Theorem 2.5, we know that every bipolar ultrametric space (X, Y, b) can be embedded into a pseudo-ultrametric space $(X \cup Y, d)$, so that (X, Y, b) will be equal to the bipolar subspace (X, Y, d) of $(X \cup Y, d) = (X \cup Y, X \cup Y, d)$. On the other side, given a pseudo-ultrametric space (Z, d) and two nonempty subsets of Z, say X and Y. It is clear that if (X, Y, d) forms a bipolar ultrametric space, then it is embeddable in (Z, d) and $d(x, y) \neq 0$ for $x \neq y$, $x \in X$, $y \in Y$. Conversely, if X and Y has the property that $d(x, y) \neq 0$ for all $x \in X$, $y \in Y$ such that $x \neq y$, then (X, Y, d) satisfies all the requirements in Definition 2.1. Accordingly, we have a full characterization of bipolar ultrametric spaces in terms of bipolar subspaces of pseudo-ultrametric spaces, as phrased in the following.

Corollary 2.10. *Every bipolar ultrametric space is identical to a bipolar subspace (X, Y, b) of a pseudo-metric space (Z, d), such that $x \neq y$ implies $d(x, y) \neq 0$ for all $x \in X$, $y \in Y$.*

3. Fixed point and coincidence point theorems

Definition 3.1. A bipolar ultrametric space is said to be spherically complete, if every chain consisting of only left-centric or only right-centric balls has nonempty intersection.

Theorem 3.2. *Let (X, Y, b) be a spherically complete bipolar ultrametric space and $T : (X, Y, b) \rightleftarrows (X, Y, b)$ be a contravariant self-mapping of (X, Y, b) such that*

$$b(Ty, Tx) < \max\{b(x, y), b(x, Tx), b(Ty, y)\} \tag{3.1}$$

for all $(x, y) \in X \times Y$. Then T has a unique fixed point.

Proof. Let $L_x = C_X(x, b(x, Tx))$ for each $x \in X$, and $\mathfrak{L} = \{L_x : x \in X\}$. Then $L_{x_1} \lesssim L_{x_2} \iff L_{x_2} \subseteq L_{x_1}$ defines a partial order on $\mathfrak{L}$. Similarly, for $y \in Y$, we define $R_y = C_Y(y, b(Ty, y))$.

Let $\mathfrak{C}$ be a chain in $(\mathfrak{L}, \lesssim)$. Then $L := \bigcap\limits_{L_x \in \mathfrak{C}} L_x \neq \varnothing$ since (X, Y, b) is spherically complete.

Given an $x \in X$ such that $L_x \in \mathfrak{C}$ and let $y \in L$, $z \in R_y$. Then

$$\begin{aligned} b(z, y) &\leq b(Ty, y) \\ &\leq \max\{b(Ty, Tx), b(x, Tx), b(x, y)\} \\ &= \max\{b(Ty, Tx), b(x, Tx)\} \end{aligned}$$

since $y \in L \subseteq L_x$ implies $b(x, y) \leq b(x, Tx)$.

If $\max\{b(Ty, Tx), b(x, Tx)\} = b(Ty, Tx)$ then either $b(x, Tx) \leq b(Ty, Tx) = 0$ gives that x is a fixed point of T, or $b(Ty, Tx) > 0$ yields

$$\begin{aligned} b(z, y) &\leq b(Ty, y) \\ &\leq b(Ty, Tx) \\ &< \max\{b(Ty, Tx), b(x, Tx), b(x, y)\} \\ &= \max\{b(Ty, Tx), b(x, Tx)\} \end{aligned}$$

by (1). Since $\max\{b(Ty, y), b(x, Tx)\} = b(Ty, y)$ leads to the contradiction that $b(Ty, y) < b(Ty, y)$, we have

$$\max\{b(Ty, y), b(x, Tx)\} = b(x, Tx),$$

thus $b(z, y) \leq b(x, Tx)$.

On the other hand, also in the case that $\max\{b(Ty,Tx), b(x,Tx)\} = b(x,Tx)$, we have $b(z,y) \le b(x,Tx)$.

Now, let $w \in L_z$, that is $b(z,w) \le b(z,Tz)$. Since $z \in R_y$ implies $b(z,y) \le b(Ty,y)$, we get

$$\begin{aligned} b(z,w) &\le b(z,Tz) \\ &\le \max\{b(z,y), b(Ty,y), b(Ty,Tz)\} \\ &= \max\{b(Ty,y), b(Ty,Tz)\}. \end{aligned}$$

Similarly, either y is a fixed point of T, or both $\max\{b(Ty,y), b(Ty,Tz)\} = b(Ty,y)$ and $\max\{b(Ty,y), b(Ty,Tz)\} = b(Ty,Tz)$ lead to $b(z,w) \le b(Ty,y)$.

Now, we want to see that $b(Ty,y) \le b(x,Tx)$. By assuming the contrary, the inequality

$$\begin{aligned} b(Ty,y) &\le \max\{b(Ty,Tx), b(x,Tx), b(x,y)\} \\ &= \max\{b(x,Tx), b(Ty,Tx)\} \end{aligned}$$

gives $\max\{b(Ty,Tx), b(x,Tx)\} \ne b(x,Tx)$, thus $b(Ty,y) \le b(Ty,Tx) > 0$ and by (3.1) we get the contradiction

$$\begin{aligned} b(Ty,y) &\le b(Ty,Tx) \\ &< \max\{b(x,y), b(Ty,y), b(x,Tx)\} \\ &= \max\{b(Ty,y), b(x,Tx)\} \\ &= b(Ty,y). \end{aligned}$$

Hence we have $b(Ty,y) \le b(x,Tx)$.

Finally we observe from a similar reasoning that

$$\begin{aligned} b(x,w) &\le \max\{b(x,y), b(z,y), b(z,w)\} \\ &= \max\{b(x,Tx), b(Ty,y)\} \\ &= b(x,Tx), \end{aligned}$$

so that $w \in L_x$. Therefore, $L_z \subseteq L_x$, that is $L_x \lesssim L_z$. Since $L_x \in \mathfrak{C}$ is arbitrary and independent of the choice of y and z, L_z is an upper bound of the chain $\mathfrak{C}$ in $(\mathfrak{C}, \lesssim)$ and by Zorn's Lemma $(\mathfrak{C}, \lesssim)$ has a maximal element.

Let L_u be maximal in $(\mathfrak{C}, \lesssim)$. If $u = Tu$ or $Tu = T^2u$, then T has a fixed point, namely u or Tu. Assume that $Tu \ne T^2u$. Then $u \ne Tu$. So by (3.1)

$$b(T^2u, Tu) < \max\{b(u,Tu), b(u,Tu), b(T^2u,Tu)\}$$

gives $b(T^2u,Tu) < b(u,Tu)$, and

$$b(T^2u, T^3u) < \max\{b(T^2u,Tu), b(T^2u,T^3u), b(T^2u,Tu)\}$$

gives $b(T^2u,T^3u) < b(T^2u,Tu)$.

If $v \in L_{T^2u}$, then

$$b(T^2u, v) \le b(T^2u, T^3u) < b(T^2u, Tu) < b(u,Tu),$$

and

$$b(u,v) < \max\{b(u,Tu), b(T^2u,Tu), b(T^2u,v)\} = b(u,Tu),$$

which gives $v \in L_u$. Thus $L_{T^2u} \subseteq L_u$ and $L_u \lesssim L_{T^2u}$. However, we also have $Tu \in L_u$ since $b(u,Tu) \le b(u,Tu)$, and $Tu \notin L_{T^2u}$ since $b(T^2u,Tu) \le b(T^2u,T^3u)$. So $L_u \ne L_{T^2u}$ and this contradicts with the maximality of L_u. Consequently, our assumption that $Tu \ne T^2u$ is false, which proves that Tu is a fixed point of T.

Now we show the uniqueness of the fixed point of T. Assume that u_1 and u_2 are two fixed points of T, such that $u_1 \ne u_2$. Since $u_1 = Tu_1$ and $u_2 = Tu_2$, we have $u_1, u_2 \in X \cap Y$.

Also $u_1 \neq u_2$ gives $Tu_1 \neq Tu_2$ so that $b(Tu_1, Tu_2) > 0$. Then by (3.1),

$$\begin{aligned} b(u_1, u_2) &= b(Tu_1, Tu_2) \\ &< \max\{b(u_2, u_1), b(u_2, Tu_2), b(Tu_1, u_1)\} \\ &= \max\{b(u_2, u_1), b(u_2, u_2), b(u_1, u_1)\} \\ &= b(u_2, u_1) \\ &= b(u_1, u_2) \end{aligned}$$

gives a contradiction. Hence the fixed point is unique. □

In the folllowing, $\mathfrak{F}$ stands for the set of all functions $f : (0, \infty) \longrightarrow \mathbb{R}$ such that

(i) $x < y \implies f(x) < f(y)$, for all $x, y \in (0, \infty)$,

(ii) $(a_n) \to 0 \iff (f(a_n)) \to -\infty$, for all sequences (a_n) on $(0, \infty)$,

(iii) $\exists k \in (0, 1),\ \lim\limits_{x \to 0^+} x^k f(x) = 0$.

Theorem 3.3. *Let (X, Y, b) be a spherically complete bipolar ultrametric space and $T : (X, Y, b) \rightleftarrows (X, Y, b)$ be a contravariant self-mapping of (X, Y, b) such that*

$$b(Ty, Tx) > 0 \implies f(b(Ty, Tx)) + c \leq f(\max\{b(x, y), b(x, Tx), b(Ty, y)\}) \tag{3.2}$$

for all $(x, y) \in X \times Y$, where $f \in \mathfrak{F}$ and $c > 0$. Then T has a unique fixed point.

Proof. Let $L_x = C_X(x, b(x, Tx))$ for each $x \in X$, and $\mathfrak{L} = \{L_x : x \in X\}$. Then $L_{x_1} \lesssim L_{x_2} \iff L_{x_2} \subseteq L_{x_1}$ defines a partial order on $\mathfrak{L}$. Similarly, for $y \in Y$, we define $R_y = C_Y(y, b(Ty, y))$. Let $\mathfrak{C}$ be a chain in $(\mathfrak{L}, \lesssim)$. Then $L := \bigcap\limits_{L_x \in \mathfrak{C}} L_x \neq \varnothing$ since (X, Y, b) is spherically complete.

Given an $x \in X$ such that $L_x \in \mathfrak{C}$ and let $y \in L$, $z \in R_y$. Then

$$\begin{aligned} b(z, y) &\leq b(Ty, y) \\ &\leq \max\{b(Ty, Tx), b(x, Tx), b(x, y)\} \\ &= \max\{b(Ty, Tx), b(x, Tx)\} \end{aligned}$$

since $y \in L \subseteq L_x$ implies $b(x, y) \leq b(x, Tx)$.

If $\max\{b(Ty, Tx), b(x, Tx)\} = b(Ty, Tx)$ then either $b(x, Tx) \leq b(Ty, Tx) = 0$ gives that x is a fixed point of T, or $b(Ty, Tx) > 0$ yields

$$\begin{aligned} b(z, y) &\leq b(Ty, y) \\ &\leq b(Ty, Tx) \\ &< \max\{b(Ty, Tx), b(x, Tx), b(x, y)\} \\ &= \max\{b(Ty, Tx), b(x, Tx)\} \end{aligned}$$

by (1). Since $\max\{b(Ty, y), b(x, Tx)\} = b(Ty, y)$ leads to the contradiction that $b(Ty, y) < b(Ty, y)$, we have

$$\max\{b(Ty, y), b(x, Tx)\} = b(x, Tx),$$

thus $b(z, y) \leq b(x, Tx)$.

On the other hand, also in the case that $\max\{b(Ty, Tx), b(x, Tx)\} = b(x, Tx)$, we have $b(z, y) \leq b(x, Tx)$.

Now, let $w \in L_z$, that is $b(z, w) \leq b(z, Tz)$. Since $z \in R_y$ implies $b(z, y) \leq b(Ty, y)$, we get

$$\begin{aligned} b(z, w) &\leq b(z, Tz) \\ &\leq \max\{b(z, y), b(Ty, y), b(Ty, Tz)\} \\ &= \max\{b(Ty, y), b(Ty, Tz)\}. \end{aligned}$$

Similarly, either y is a fixed point of T, or both $\max\{b(Ty, y), b(Ty, Tz)\} = b(Ty, y)$ and $\max\{b(Ty, y), b(Ty, Tz)\} = b(Ty, Tz)$ lead to $b(z, w) \leq b(Ty, y)$.

Now, we want to see that $b(Ty, y) \le b(x, Tx)$. By assuming the contrary, the inequality

$$\begin{aligned} b(Ty, y) &\le \max\{b(Ty, Tx), b(x, Tx), b(x, y)\} \\ &= \max\{b(x, Tx), b(Ty, Tx)\} \end{aligned}$$

gives $\max\{b(Ty, Tx), b(x, Tx)\} \neq b(x, Tx)$, thus $b(Ty, y) \le b(Ty, Tx) > 0$ and by (3.1) we get the contradiction

$$\begin{aligned} b(Ty, y) &\le b(Ty, Tx) \\ &< \max\{b(x, y), b(Ty, y), b(x, Tx)\} \\ &= \max\{b(Ty, y), b(x, Tx)\} \\ &= b(Ty, y). \end{aligned}$$

Hence we have $b(Ty, y) \le b(x, Tx)$.

Finally we observe that

$$\begin{aligned} b(x, w) &\le \max\{b(x, y), b(z, y), b(z, w)\} \\ &= \max\{b(x, Tx), b(Ty, y)\} \\ &= b(x, Tx), \end{aligned}$$

so that $w \in L_x$. Therefore, $L_z \subseteq L_x$, that is $L_x \lesssim L_z$. Since $L_x \in \mathfrak{C}$ is arbitrary and independent of the choice of y and z, L_z is an upper bound of the chain $\mathfrak{C}$ in $(\mathfrak{C}, \lesssim)$ and by Zorn's Lemma $(\mathfrak{C}, \lesssim)$ has a maximal element.

Let L_u be maximal in $(\mathfrak{C}, \lesssim)$. If $u = Tu$ or $Tu = T^2u$, then T has a fixed point, namely u or Tu. Assume that $Tu \neq T^2u$. Then $u \neq Tu$. So by (3.2)

$$f(b(T^2u, Tu)) + c \le f(\max\{b(u, Tu), b(u, Tu), b(T^2u, Tu)\})$$

gives $f(b(T^2u, Tu)) < f(b(u, Tu))$, and since f is strictly increasing we also have $b(T^2u, Tu) < b(u, Tu)$. Similarly

$$f(b(T^2u, T^3u)) + c \le f(\max\{b(T^2u, Tu), b(T^2u, T^3u), b(T^2u, Tu)\})$$

gives $b(T^2u, T^3u) < b(T^2u, Tu)$. If $v \in L_{T^2u}$, then

$$b(T^2u, v) \le b(T^2u, T^3u) < b(T^2u, Tu) < b(u, Tu),$$

and

$$b(u, v) < \max\{b(u, Tu), b(T^2u, Tu), b(T^2u, v)\} = b(u, Tu),$$

which gives $v \in L_u$. Thus $L_{T^2u} \subseteq L_u$ and $L_u \lesssim L_{T^2u}$. However, we also have $Tu \in L_u$ since $b(u, Tu) \le b(u, Tu)$, and $Tu \notin L_{T^2u}$ since $b(T^2u, Tu) \le b(T^2u, T^3u)$. So $L_u \neq L_{T^2u}$ and this contradicts with the maximality of L_u. Consequently, our assumption that $Tu \neq T^2u$ is false, which proves that Tu is a fixed point of T.

Now we show the uniqueness of the fixed point of T. Assume that u_1 and u_2 are two fixed points of T, such that $u_1 \neq u_2$. Since $u_1 = Tu_1$ and $u_2 = Tu_2$, we have $u_1, u_2 \in X \cap Y$. Also $u_1 \neq u_2$ gives $Tu_1 \neq Tu_2$ so that $b(Tu_1, Tu_2) > 0$. Then by (3.2),

$$\begin{aligned} f(b(u_1, u_2)) + c &= f(b(Tu_1, Tu_2)) + c \\ &\le f(\max\{b(u_2, u_1), b(u_2, Tu_2), b(Tu_1, u_1)\}) \\ &= f(\max\{b(u_2, u_1), b(u_2, u_2), b(u_1, u_1)\}) \\ &= f(b(u_2, u_1)) \\ &= f(b(u_1, u_2)) \end{aligned}$$

gives a contradiction. Hence the fixed point is unique. □

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