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Some Results On Silver Riemannian Structures

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Abstract. Our aim in this paper is to study of silver Riemannian structures on manifold and bundle. An integrability condition and curvature properties for silver Riemannian structure are investigated via the Tachibana operator. Twin silver Riemannian metric is defined and some properties of twin silver Riemannian metric are investigated. Examples of silver structure are given on tangent and cotangent bundles.

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1. Introduction

Artists, designers and architects throughout history have used some mathematical ratios and equations to assist them in their work. One of these mathematical ratios is the silver ratio. The silver ratio is also known as the Japanese ratio because it is used in Japanese architecture. They also used the silver ratio in anime characters.

Spinadel introduced the metallic means family [10, 11]. All of the metallic means family members are positive quadratic irrational $\sigma_{q,r} = \frac{q+\sqrt{q^2+4r}}{2}$ which are the solutions of quadratic equation $x^2 - qx - r = 0$. Inspired by the metallic means family, Hretcanu and Crasmareanu introduced the metallic structure on manifold M which is determined by a $(1,1)$ −type tensor field Θ on manifold M satisfying $\Theta^2 = q\Theta + rI$, $q, r \in \mathbb{R}$ [3]. The metallic structures are studied by many authors [1, 2]. If $q = 2$ and $r = 1$, then a $(1,1)$ −type tensor field Θ is called silver structure on M which satisfies the equation

$$\Theta^2 = 2\Theta + I,$$

where I is the $(1,1)$ −type identity tensor field. Using the notion of a silver ratio $\Theta = 1 + \sqrt{2}$ which is a positive root of the equation $x^2 - 2x - 1 = 0$, Özkan and Peltek have studied the notion of a silver structure on a differential manifold [6].

Let M be a Riemannian manifold equipped with the Riemannian metric g and the silver structure Θ such that

$$g(\Theta A, B) = g(A, \Theta B) \tag{1.1}$$

or equivalently to (1.1)

$$g(\Theta A, \Theta B) = g\left(\Theta^2 A, B\right) = g((2\Theta + I)A, B) = 2g(\Theta A, B) + g(A, B),$$

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for all vector fields A, B on M. The Riemannian metric g in equation (1.1) is called $\Theta-$compatible and (M, Θ, g) is called a Silver Riemannian manifold. Such Riemannian metrics are also referred to as pure metric [4, 12].

Let Θ be a $(1,1)$ $-$type tensor field on manifold M. A $(0,r)$ $-$type tensor field k is called a pure tensor field according to Θ if

$$\begin{aligned} k(\Theta B_1, B_2, ..., B_r) &= k(B_1, \Theta B_2, ..., B_r) \\ &\vdots \\ &= k(B_1, B_2, ..., \Theta B_r), \end{aligned}$$

for any vector fields $B_1, B_2, ..., B_r$. In 1960, Tachibana defined an operator

$$\Phi_\Theta : \Im_r^0(M) \to \Im_{r+1}^0(M)$$

which is applied to pure tensors. It is applied to $(0,r)$ $-$type pure tensor field k according to Θ by

$$\begin{aligned} (\Phi_\Theta k)(A, B_1, ..., B_r) &= (\Theta A) k(B_1, ..., B_r) - A k(\Theta B_1, ..., B_r) \\ &\quad + \sum_{\lambda=1}^{r} k(B_1, ..., (L_{B_\lambda}\Theta) A, ..., B_r), \end{aligned}$$

for any $A, B_1, ..., B_r \in \Im_0^1(M)$, where L_B denotes the Lie derivative according to B [9, 12]. This operator is referred as $\Phi-$operator (Tachibana operator) in some studies [4, 7]. Silver structure has an important status on Riemannian manifold because this structure is associated with pure Riemannian metric according to the corresponding structure. Tachibana give the definition of decomposable via $\Phi-$ operator associated with almost product structure in his study [12]. Since Riemannian silver and almost product structures are related to each other, we use the method of $\Phi-$operator in the theory of silver structure.

The paper is organized as follows. In section 2, a new sufficient condition of integrability for Silver Rieamnnian structures via $\Phi-$operator is given. Some properties of twin Silver Riemannian metrics and the curvature properties of locally decomposable Silver Riemannian manifold are studied. Section 3 is devoted to some examples of locally decomposable Silver Riemannian manifold.

2. Locally Decomposable Silver Riemannian Structures

Let (M, g) be a Riemannian manifold endowed with almost product structure P, i.e. it called an almost product Riemannian manifold and represented with the triple (M, g, P). The almost product Riemannian manifold is endowed with an almost product structure P of $(1,1)$ $-$type such that

$$P^2 = I, \quad g(PA, B) = g(A, PB),$$

for all vector fields A, B and g is a Riemannian metric. The almost product structure P on manifold M is derived from the polynomial structure on manifold M. A necessary and sufficient condition for P to be integrable is that $\nabla P = 0$, where ∇ is the Riemannian connection of g. An almost product Riemannian manifold with an integrable product structure P is called locally product Riemannian manifold. If a pure tensor field k ensures the equivalence $\Phi_\varepsilon k = 0$ that ε is a $(1,1)$ $-$type tensor field and Φ is Tachibana operator, then it is called $\Phi-$tensor. If ε is a product structure, a $\Phi-$tensor is a decomposable tensor [12]. The condition $\Phi_P g = 0$ means that the pure metric g is decomposable tensor, where P is a product structure and

$$(\Phi_P g)(A, B_1, B_2) = (PA)(g(B_1, B_2)) - A(g(PB_1, B_2)) + g((L_{B_1} P) A, B_2) + g(B_1, (L_{B_2} P) A). \tag{2.1}$$

The condition $\nabla P = 0$ is equavalent to the condition $\Phi_P g = 0$ on the triple (M, g, P), where ∇ is the Riemannian connection of g [7, 9]. Then, a locally product Riemannian manifold with decomposable tensor g is called locally decomposable Riemannian manifold [13].

We can obtain integrability condition for a silver Riemannian structure via the Tachibana operator Φ.

Theorem 2.1. *Let (M, g, Θ) be a silver Riemannian manifold, where Θ is the silver structure and g is the Riemannian metric. Then,*

a) *Θ is integrable if $\Phi_\Theta g = 0$,*

b) *The condition $\Phi_\Theta g = 0$ is equivalent to $\nabla\Theta = 0$, where ∇ is the Riemannian connection of g.*

Proof. By using purity of g and the condition $\nabla g = 0$, we have

$$g\left(B_1, \left(\nabla_A \Theta\right) B_2\right) = g\left(\left(\nabla_A \Theta\right) B_1, B_2\right), \tag{2.2}$$

for all vector fields A, B_1, B_2.

By using the equation (2.2) and $[A, B] = \nabla_A B - \nabla_B A$, we can write the equation (2.1) in a different form as follows:

$$\left(\Phi_\Theta g\right)\left(A, B_1, B_2\right) = -g\left(\left(\nabla_A \Theta\right) B_1, B_2\right) + g\left(\left(\nabla_{B_1} \Theta\right) A, B_2\right) + g\left(B_1, \left(\nabla_{B_2} \Theta\right) A\right). \tag{2.3}$$

By replacing A and B_2, we have

$$\left(\Phi_\Theta g\right)\left(B_2, B_1, A\right) = -g\left(\left(\nabla_{B_2} \Theta\right) B_1, A\right) + g\left(\left(\nabla_{B_1} \Theta\right) B_2, A\right) + g\left(B_1, \left(\nabla_A \Theta\right) B_2\right). \tag{2.4}$$

If we add the equations (2.3) and (2.4), we obtain

$$\left(\Phi_\Theta g\right)\left(A, B_1, B_2\right) + \left(\Phi_\Theta g\right)\left(B_2, B_1, A\right) = 2g\left(A, \left(\nabla_{B_1} \Theta\right) B_2\right). \tag{2.5}$$

From the equation (2.5), we reach the items a) and b) of Theorem 2.1. □

Now, we are going to give the relationships between the almost product structures and silver structures.

Proposition 2.2. *If Θ is a silver structure on manifold M, then*

$$P = \frac{1}{\sqrt{2}}\left(\Theta - I\right) \tag{2.6}$$

is an almost product structure on M. Conversely, every almost product structure P on manifold M induces two silver structures on manifold M, given as follows:

$$\Theta_1 = \left(I + \sqrt{2}P\right), \quad \Theta_2 = \left(I - \sqrt{2}P\right).$$

Proof. Let P be an almost product structure on a Riemannian manifold (M, g), i.e. $P^2 = I$. Then each of the structures $\Theta_1 = \left(I + \sqrt{2}P\right)$ and $\Theta_2 = \left(I - \sqrt{2}P\right)$ obtained from the almost product P is a silver structure. In fact,

$$\begin{aligned} \Theta_1^2 &= I^2 + 2\sqrt{2}P + 2P^2 \\ &= I + 2\sqrt{2}P + 2I \\ &= 3I + 2\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\left(\Theta_1 - I\right)\right) \\ &= 3I + 2\Theta_1 - 2I \\ &= 2\Theta_1 + I. \end{aligned}$$

Similarly, the equation $\Theta_2^2 - 2\Theta_2 - I = 0$ is obtained for silver structure Θ_2.

Conversely, let Θ be a silver structure on a Riemannian manifold M equipped with the Riemannian metric g. Then the structure $P = \frac{1}{\sqrt{2}}\left(\Theta - I\right)$ induced by the silver structure Θ is an almost product structure. In fact,

$$P^2 = \frac{\Theta^2 - 2\Theta + I}{2} = \frac{\left(\Theta^2 - 2\Theta\right) + I}{2} = \frac{2I}{2} = I.$$

□

A Riemannian metric g is pure according to a silver structure if and only if the Riemannian metric g is pure according to corresponding almost product structure P. The relation between the Tachibana operators $\Phi_P g$ and $\Phi_\Theta g$ is given as

$$\Phi_P g = \frac{1}{\sqrt{2}}\Phi_\Theta g. \tag{2.7}$$

We know that, if the Riemannian metric g satisfies the condition $\Phi_P g = 0$, the Riemannian metric g is decomposable. Considering Theorem 2.1, we can deduce that the Silver Riemannian structure Θ is integrable if the Riemannian metric g is decomposable. If (M, Θ, g) is a locally Silver Riemannian manifold with decomposable Riemannian metric, then (M, Θ, g) is called a locally decomposable Silver Riemannian manifold. So, we get following proposition

Proposition 2.3. *Let (M, g, Θ) be a Silver Riemannian manifold, where Θ is a silver structure and g is a Riemannian metric. The manifold M is a locally decomposable Silver Riemannian manifold if and only if $\Phi_P g = 0$, where P is the corresponding almost product structure associated with Θ.*

The twin silver Riemannian metric is defined by

$$G(B_1, B_2) = g(\Theta B_1, B_2),$$

for all vector fields B_1, B_2 on M. It is easily seen that the twin silver Riemannian metric G is pure according to the silver structure Θ. If we apply the $\Phi_\Theta-$operator to the twin silver Riemannian metric G, we obtain

$$(\Phi_\Theta G)(A, B_1, B_2) = (\Phi_\Theta g)(A, \Theta B_1, B_2) + g(N_\Theta(A, B_1), B_2),$$

where N_Θ is Nijenhuis tensor consructed from Θ. So, we have

Proposition 2.4. *Let (M, Θ, g) be a locally decomposable Silver Riemannian manifold, where Θ is a silver structure and g is a silver Riemannian metric. Then, the twin silver Riemannian metric G is $\Phi-$tensor field.*

Theorem 2.5. *Let (M, Θ, g) be a locally decomposable silver Riemannian manifold, where Θ is a silver structure and g is a silver Riemannian metric. The Riemannian curvature tensor field R is a $\Phi-$ tensor field.*

Proof. The Riemannian curvature tensor field R of the silver Riemannian metric g is pure according to the silver structure Θ, i.e.

$$R(\Theta B_1, B_2, B_3, B_4) = R(B_1, \Theta B_2, B_3, B_4) = R(B_1, B_2, \Theta B_3, B_4) = R(B_1, B_2, B_3, \Theta B_4)$$

for all vector fields B_1, B_2, B_3, B_4 on M. Applying the Tachibana operator to the Riemannian curvature R of $(0,4)-$type as in equation (2.1), we write

$$(\Phi_\Theta R)(A, B_1, B_2, B_3, B_4) = (\nabla_{\Theta A} R)(B_1, B_2, B_3, B_4) - (\nabla_A R)(\Theta B_1, B_2, B_3, B_4). \tag{2.8}$$

Considering the purity of R and applying Bianchi's 2nd identity to (2.8), we obtain

$$\begin{aligned}(\Phi_\Theta R)(A, B_1, B_2, B_3, B_4) &= g((\nabla_{\Theta A} R)(B_1, B_2, B_3) - (\nabla_A R)(\Theta B_1, B_2, B_3), B_4)\\ &= g((\nabla_{\Theta A} R)(B_1, B_2, B_3) - \Theta((\nabla_A R)(B_1, B_2, B_3)), B_4)\\ &= g(-(\nabla_{B_1} R)(B_2, \Theta A, B_3) - (\nabla_{B_2} R)(\Theta A, B_1, B_3)\\ &\quad -\Theta((\nabla_A R)(B_1, B_2, B_3)), B_4).\end{aligned} \tag{2.9}$$

And using $\nabla\Theta = 0$, we find

$$\begin{aligned}(\nabla_{B_2} R)(\Theta A, B_1, B_3) &= \nabla_{B_2}(R(\Theta A, B_1, B_3)) - R(\nabla_{B_2}(\Theta A), B_1, B_3)\\ &\quad -R(\Theta A, \nabla_{B_2} B_1, B_3) - R(\Theta A, B_1, \nabla_{B_2} B_3)\\ &= (\nabla_{B_2}\Theta)(R(A, B_1, B_3)) + \Theta(\nabla_{B_2} R(A, B_1, B_3))\\ &\quad -R((\nabla_{B_2}\Theta)A + \Theta(\nabla_{B_2} A), B_1, B_3)\\ &\quad -R(\Theta A, \nabla_{B_2} B_1, B_3) - R(\Theta A, B_1, \nabla_{B_2} B_3)\\ &= \Theta(\nabla_{B_2} R(A, B_1, B_3)) - \Theta(R(\nabla_{B_2} A, B_1, B_3))\\ &\quad -\Theta(R(A, \nabla_{B_2} B_1, B_3)) - \Theta(R(A, B_1, \nabla_{B_2} B_3))\\ &= \Theta((\nabla_{B_2} R)(A, B_1, B_3)).\end{aligned} \tag{2.10}$$

Similarly, we obtain

$$(\nabla_{B_1} R)(B_2, \Theta A, , B_3) = \Theta((\nabla_{B_1} R)(B_2, A, B_3)). \tag{2.11}$$

Subsituting (2.10) and (2.11) in (2.9) and using Bianchi's 2nd identity, we obtain

$$\begin{aligned}(\Phi_\Theta R)(A, B_1, B_2, B_3, B_4) &= g(-\Theta((\nabla_{B_1} R)(B_2, A, B_3)) - \Theta((\nabla_{B_2} R)(A, B_1, B_3))\\ &\quad -\Theta((\nabla_A R)(B_1, B_2, B_3)), B_4)\\ &= 0.\end{aligned}$$

□

By (2.6) and (2.8), we can find, in a similar way like the equation (2.7),

$$\Phi_P R = \frac{1}{\sqrt{2}}\Phi_\Theta R, \tag{2.12}$$

where Θ is the silver structure and P is its corresponding almost product structure. Based on Theorem 2.5 and the equation (2.12), we obtain following proposition

Proposition 2.6. *Let (M, Θ, g) be a locally decomposable silver Riemannian manifold, where Θ is a silver structure and g is a silver Riemannian metric. The Riemannian curvature tensor field is a decomposable tensor field.*

3. Examples

Example 3.1. Let (M, g) be a Riemannian manifold with dimension n and $T(M)$ be its tangent bundle with the bundle projection $\pi : T(M) \to M$ that the bundle projection defines the natural bundle structure of tangent bundle $T(M)$ over manifold M. Then $T(M)$ is a $2n$ dimensional smooth manifold. A system of local coordinates $\left(U, x^i\right)$ in manifold M, $U \subset M$, induces to a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i\right)$ $i = 1, ..., n$, $\bar{i} = n+1, ..., 2n$ in tangent bundle $T(M)$, where $\left(y^i\right)$ are the Cartesian coordinates in each tangent space $T_P(M)$ at $P \in M$ according to natural base.

Let $A = A^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field A. Then the vertical lift ${}^V A$ and horizontal lift ${}^H A$ of A are given according to induced coordinates in $T(M)$

$$
\begin{aligned}
{}^V A &= \begin{pmatrix} {}^V A^i \\ {}^V A^{\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^i \end{pmatrix}, \\
{}^H A &= \begin{pmatrix} {}^H A^i \\ {}^H A^{\bar{i}} \end{pmatrix} = \begin{pmatrix} A^i \\ -A^s y^t \Gamma^i_{ts} \end{pmatrix},
\end{aligned}
$$

where Γ^i_{ts} are the coefficients of the Riemannian connection ∇ of g.

The Sasaki metric ${}^S g$ on the tangent bundle $T(M)$ is defined by

$$
{}^S g\left({}^V A, {}^V B\right) = {}^V (g(A, B)) , \tag{3.1}
$$

$$
{}^S g\left({}^V A, {}^H B\right) = {}^S g\left({}^H A, {}^V B\right) = 0 , \tag{3.2}
$$

$$
{}^S g\left({}^H A, {}^H B\right) = {}^V (g(A, B)) , \tag{3.3}
$$

for any vector fields A, B on manifold M [14].

We define a silver structure J_Θ on $T(M)$

$$
\begin{cases} J_\Theta\left({}^H A\right) = {}^H A + \sqrt{2}\, {}^V A \\ J_\Theta\left({}^V A\right) = {}^V A + \sqrt{2}\, {}^H A \end{cases} \tag{3.4}
$$

which implies $J_\Theta^2 - 2J_\Theta - I = 0$.

We write

$$
K\left(\widehat{A}, \widehat{B}\right) = {}^S g\left(J_\Theta \widehat{A}, \widehat{B}\right) - {}^S g\left(\widehat{A}, J_\Theta \widehat{B}\right), \tag{3.5}
$$

for any vector fields $\widehat{A}, \widehat{B}$. Replacing $\widehat{A}, \widehat{B}$ with ${}^V A$, ${}^V B$ or ${}^H A$, ${}^H B$, respectively, from (3.1)-(3.3) and (3.5) we have $K\left(\widehat{A}, \widehat{B}\right) = 0$. It is mean that ${}^S g$ is pure according to the silver structure J_Θ. So we obtain:

Theorem 3.2. *Let (M, g) be a Riemannian manifold and $T(M)$ be its tangent bundle endowed with the Sasaki metric ${}^S g$ and silver structure J_Θ defined by (3.4). The triple $\left(T(M), J_\Theta, {}^S g\right)$ is a silver Riemannian manifold.*

We determined the Sasaki metric ${}^S g$ and the silver structure in equations (3.1) – (3.3) and (3.4), respectively. Using the several properties of lifts on tangent bundle $T(M)$ that ${}^V A\, {}^V (g(B_1, B_2)) = 0$ and ${}^H A\, {}^V (g(B_1, B_2)) = {}^V (Ag(B_1, B_2))$, we calculate

$$
\begin{aligned}
\left(\Phi_{J_\Theta}\, {}^S g\right)\left(\widehat{A}, \widehat{B}_1, \widehat{B}_2\right) &= \left(J_\Theta \widehat{A}\right)\left({}^S g\left(\widehat{B}_1, \widehat{B}_2\right)\right) - \widehat{A}\left({}^S g\left(J_\Theta \widehat{B}_1, \widehat{B}_2\right)\right) \\
&\quad + {}^S g\left(\left(L_{\widehat{B}_1} J_\Theta\right)\widehat{A}, \widehat{B}_2\right) + {}^S g\left(\widehat{B}_1, \left(L_{\widehat{B}_2} J_\Theta\right)\widehat{A}\right),
\end{aligned}
$$

for all vector fields A, B_1, B_2. Then we obtain

$$\begin{aligned}
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^V A,\,{}^V B_1,\,{}^V B_2\right) &= 0\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^V A,\,{}^V B_1,\,{}^H B_2\right) &= -\sqrt{2}\,{}^S g\left({}^V B_1,\,{}^V\left(R\left(B_2,A\right)y\right)\right)\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^V A,\,{}^H B_1,\,{}^V B_2\right) &= -\sqrt{2}\,{}^S g\left({}^V\left(R\left(B_1,A\right)y\right),\,{}^V B_2\right)\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^V A,\,{}^H B_1,\,{}^H B_2\right) &= 0\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^H A,\,{}^V B_1,\,{}^V B_2\right) &= 0\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^H A,\,{}^V B_1,\,{}^H B_2\right) &= 0\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^H A,\,{}^H B_1,\,{}^V B_2\right) &= 0\\
\left(\Phi_{J_\Theta}\,{}^S g\right)\left({}^H A,\,{}^H B_1,\,{}^H B_2\right) &= \sqrt{2}\,{}^S g\left({}^H\left(R\left(B_1,A\right)y\right),\,{}^H B_2\right)\\
&\quad +\sqrt{2}\,{}^S g\left({}^H B_1,\,{}^H\left(R\left(B_2,A\right)y\right)\right).
\end{aligned}$$

So, from Proposition 2.3 and equation (2.7), we obtain following theorem.

Theorem 3.3. *Let (M,g) be a Riemannian manifold and let $T(M)$ be its tangent bundle endowed with the Sasaki metirc ${}^S g$ and the silver structure J_Θ defined by (3.4). The triple $\left(T(M), J_\Theta, {}^S g\right)$ is locally decomposable silver Riemannian manifold if and only if the Riemannian manifold is locally flat.*

Example 3.4. Let (M,g) be a Riemannian manifold with dimension n and $T^*(M)$ be its cotangent bundle with the bundle projection $\pi : T^*(M) \to M$ that the bundle projection defines the natural bundle structure of cotangent bundle $T^*(M)$ over manifold M. Then $T^*(M)$ is a $2n$ dimensional smooth manifold. A system of local coordinates $\left(U, x^i\right)$ in manifold M, $U \subset M$, induces to a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i\right)$ $i = 1,...,n$, $\bar{i} = n+1,...,2n$ in cotangent bundle $T^*(M)$, where (p_i) are the Cartesian coordinates in each cotangent space ${T^*}_P(M)$ at $P \in M$ according to natural base.

Let $\alpha = \alpha_j dx^j$ be the local expression in U of a $1-$form α. Then the vertical lift ${}^V\alpha$ of $1-$form α and horizontal lift ${}^H A$ of vector field A are given according to induced coordinates in $T^*(M)$

$${}^V\alpha = \begin{pmatrix} {}^V\alpha_j \\ {}^V\alpha_{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_j \end{pmatrix}$$

$${}^H A = \begin{pmatrix} {}^H A^j \\ {}^H A^{\bar{j}} \end{pmatrix} = \begin{pmatrix} A^j \\ p_s \Gamma^s_{ji} A^i \end{pmatrix}.$$

The Sasaki metric ${}^S g$ on the cotangent bundle $T^*(M)$ is defined by

$${}^S g\left({}^V\alpha,\,{}^V\beta\right) = {}^V\left(g^{-1}(\alpha,\beta)\right) = g^{-1}(\alpha,\beta)\circ\pi, \tag{3.6}$$

$${}^S g\left({}^V\alpha,\,{}^H B\right) = 0, \tag{3.7}$$

$${}^S g\left({}^H A,\,{}^H B\right) = {}^V\left(g(A,B)\right) = g(A,B)\circ\pi, \tag{3.8}$$

for any vector fields A, B and $1-$forms α,β. The Sasaki metric ${}^S g$ is assigned by equations (3.6)$-$(3.8). The Riemannian connection ${}^S\nabla$ of ${}^S g$ compensates the following relations:

$$\begin{aligned}
&(i)\ {}^S\nabla_{{}^V\alpha}\,{}^V\beta = 0\\
&(ii)\ {}^S\nabla_{{}^V\alpha}\,{}^H B = \tfrac{1}{2}\,{}^H\left(p\left(g^{-1}\circ R(\,,B)\widetilde{\alpha}\right)\right)\\
&(iii)\ {}^S\nabla_{{}^H A}\,{}^V\beta = {}^V\left(\nabla_A\beta\right) + \tfrac{1}{2}{}^H\left(p\left(g^{-1}\circ R(\,,A)\widetilde{\beta}\right)\right)\\
&(iv)\ {}^S\nabla_{{}^H A}\,{}^H B = {}^H\left(\nabla_A B\right) + \tfrac{1}{2}{}^V\left(pR(A,B)\right),
\end{aligned} \tag{3.9}$$

for all vector fields A, B and $1-$forms α,β, where $\widetilde{\alpha} = g^{-1}\circ\alpha$ is a vector field, $R(\,,A)\widetilde{\beta}$ is a $(1,1)-$type tensor field and $g^{-1}\circ R(\,,A)\widetilde{\beta}$ is a $(2,0)-$type tensor field [5, 8].

We define a silver structure J_{Θ^*} on $T^*(M)$ by

$$\begin{cases} J_{\Theta^*}\left({}^H A\right) = {}^H A + \sqrt{2}\,{}^V\widetilde{A} \\ J_{\Theta^*}\left({}^V\alpha\right) = {}^V\alpha + \sqrt{2}\,{}^H\widetilde{\alpha} \end{cases} \tag{3.10}$$

for any vector field A and any $1-$form α, where $\widetilde{A} = g \circ A$ is a $1-$form, $\widetilde{\alpha} = g^{-1} \circ \alpha$ is a vector field. And the Sasaki metric is pure according to silver structure J_{Θ^*}. Then we obtain following theorem:

Theorem 3.5. *Let (M, g) be a Riemannian manifold and let $T^*(M)$ be its cotangent bundle endowed with the Sasaki metric ${}^S g$ and silver structure J_{Θ^*} defined by (3.10). The triple $\left(T^*(M), J_{\Theta^*}, {}^S g\right)$ is a silver Riemannian manifold.*

We investigate the covariant derivative of J_{Θ^*}. Considering the equations $(i) - (ii)$ of (3.9) and (3.10), we have

$$\begin{array}{lcl} \left({}^S\nabla_{{}^H A} J_{\Theta^*}\right)\left({}^H B\right) & = & \frac{\sqrt{2}}{2}\,{}^H\left(p\left(g^{-1}\circ\left(R(\,,A)\,B - R(A,B)\right)\right)\right) \\ \left({}^S\nabla_{{}^V \alpha} J_{\Theta^*}\right)\left({}^H B\right) & = & -\frac{\sqrt{2}}{2}\,{}^V\left(pR(\,,B)\,\widetilde{\alpha}\right) \\ \left({}^S\nabla_{{}^H A} J_{\Theta^*}\right)\left({}^V \beta\right) & = & \frac{\sqrt{2}}{2}\,{}^V\left(pR\left(A,\widetilde{\beta}\right) - pR(\,,A)\,\widetilde{\beta}\right) \\ \left({}^S\nabla_{{}^V \alpha} J_{\Theta^*}\right)\left({}^V \beta\right) & = & \frac{\sqrt{2}}{2}\,{}^H\left(p\left(g^{-1}\circ R\left(\,,\widetilde{\beta}\right)\widetilde{\alpha}\right)\right), \end{array} \tag{3.11}$$

for all vector fields A, B and $1-$forms α, β. From equations (3.11) we obtain:

Theorem 3.6. *Let (M, g) be a Riemannian manifold and $T^*(M)$ be its cotangent bundle endowed with the Sasaki metric ${}^S g$ and silver structure J_{Θ^*} defined by (3.10). The triple $\left(T^*(M), J_{\Theta^*}, {}^S g\right)$ is locally decomposable silver Riemannian manifold if and only if the Riemannian manifold is flat.*

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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