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A Trigonometric Approach to Time Fractional FitzHugh-Nagumo Model on Nerve Pulse Propagation

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Abstract
The aim of this paper is to put on display the numerical solutions and dynamics of time fractional Fitzhugh-Nagumo model, which is an important nonlinear reaction-diffusion equation. For this purpose, finite element method based on trigonometric cubic B-splines are used to obtain numerical solutions of the model. In this model, the derivative which is fractional order is taken in terms of Caputo. Thus, time dicretization is made using $L1$ algorithm for Caputo derivative and space discretization is made using trigonometric cubic B- spline basis. Also, the non-linear term in the model is linearized by the Rubin Graves type linearization. The error norms are calculated for measuring the accuracy of the finite element method. The comparison of numerical and exact solutions are exhibited via tables and graphics.

Keywords: *Time fractional Fitzhugh-Nagumo model; finite element method; collocation; trigonometric B-splines*

AMS Subject Classification (2020): *Primary: 65L60 ; Secondary: 65A05; 41A15.*

1. Introduction

The roots of Fractional calculus (FC) dates back to 1965. This date is not far from the emerging date of traditional calculus. Nonetheless; though its fascinating nature and contribution of mathematicians, physicists and engineers, it has not been a huge attraction. Recent decades have seen a dramatically accelerating pace in the development of fractional calculus due to frequently appearing in various applications in fields of biomechanics, viscoelasticity, control theory, aerodynamics, physics and engineering and so on [1, 2]. The popularity of FC has attracted many researchers from all over the world, thus, this attraction produced research papers and books covering all areas of science and motivated and development of numerical methods [3, 4]. Especially, since fractional derivatives are more convenient and economical, solving fractional order differential equations has a vital role in modeling various phenomena. One can get a brief glimpse to numerical methods for solving fractional differential equations in Refs[5–14].

In this paper, Time fractional Fitzhugh–Nagumo model is going to be considered via initial and boundary conditions. In order to obtain numerical solutions a framework of combination collocation method and finite element method (FEM) based on cubic trigonometric B-spline basis will be used. The error norms will be calculated

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to know how well accurate the exact solutions and numerical ones close to each other. Comparison of exact and numerical results and dynamics of the solutions is going to be presented via tables and graphics.

2. Time fractional Fitzhugh–Nagumo model and Application of the method

The FitzHugh–Nagumo model [18]

$$\varepsilon \frac{\partial u(x,t)}{\partial t} = \Psi\left(u\left(x,t\right)\right) - v\left(x,t\right) + I,$$

$$\frac{\partial v(x,t)}{\partial t} = u\left(x,t\right) - \zeta v\left(x,t\right)$$

where ε and ζ are positive constants and $\Psi\left(u\left(x,t\right)\right) = u\left(x,t\right)\left(1-u\left(x,t\right)\right)\left(u\left(x,t\right)-r\right)$ for $r \in \left(0,\frac{1}{2}\right)$. The model takes its name from R. FitzHugh [15] and J. Nagumo [16] who proposed a model for emulating the current signal observed in a living organism's excitable cells in 1961. A lipid bilayer membrane distinct nerve cells from the extracellular region. When the cells do not run the signals, there are a a potential difference. This potential difference is known as resting potential of cells. Positively charged sodium and the potassium ions and negatively charged protein ions keep in existence of resting potential. When there is a external disturbation to the cells, depolarization and repolarization process begins. Depolarization rises up spikes toward a positive value, depolarization passes to resting potential. In a simple description, the FitzHugh-Nagumo model is a simplified model of activation and deactivation dynamics in a spiking neuron [17]. In literature, I is a constant external stimulus. $u\left(x,t\right)$ and $v\left(x,t\right)$ are unknowns and they measure the potential difference across the cell membrane transmembrane currents which influence the tendency of the cell to regain before being able to fire again, respectively [18]. It will be useful to explain that many variations of FitzHugh-Nagumo model have been derived from the original one.

There are various approaches to FitzHugh-Nagumo model. One can read and get information in [19–22] and there in. For the reason that the main issue of this paper, it will be important to take a quick glance at fractional FitzHugh-Nagumo model; Liu et al. [23] have considered the model in two dimensional and obtained solutions via implicit numerical method. Kumar et al. [24] have focused on solutions of the model using q-homotopy analysis approach and Laplace transform approach. Injrou et al. [25] proposed a finite difference scheme to obtain numerical solutions of the model.

In this paper, a reduced problem of FitzHugh-Nagumo model in fractional form will be taken into consideration. As the starting point, time fractional nonlinear reaction–diffusion equations of the form is considered as following [25]

$$D_t^{\gamma} u_t\left(x,t\right) = \kappa u_{xx}\left(x,t\right) + \Psi\left(u\left(x,t\right)\right) \tag{2.1}$$

where x and t denote space and time derivative, respectively. κ is a physical constant which represents diffusion coefficient , Ψ accounts for all local reactions and γ is the fractional order and $0<\gamma<1$.In order to calculate wave solutions, the function $\Psi\left(u\left(x,t\right)\right)$ has to be specified; if $\kappa=1$ and $\Psi\left(u\left(x,t\right)\right) = u\left(x,t\right)\left(1-u\left(x,t\right)\right)\left(u\left(x,t\right)-r\right)$, the equation (2.1) is called as time fractional Fitzhugh–Nagumo model. Thus, time fractional initial-boundary Fitzhugh–Nagumo model [24, 26, 27] will be considered and it is formulated as

$$\begin{array}{lll} D_t^{\gamma} u\left(x,t\right) = \kappa u_{xx}\left(x,t\right) + \mu u\left(x,t\right)\left(1-u\left(x,t\right)\right)\left(u\left(x,t\right)-r\right), & & \\ & & \\ u\left(x_L,t\right) = f_1\left(t\right), & u\left(x_R,t\right) = f_2\left(t\right), & t \in [0,T] \\ u_x\left(x_L,t\right) = g_1\left(t\right), & u_x\left(x_R,t\right) = g_2\left(t\right), & x \in \left[x_L, x_R\right] \\ u\left(x,0\right) = h\left(x\right). & & \end{array} \tag{2.2}$$

As highlighted above, there are several variants of the FitzHugh-Nagumo equations. In this subsection finite element collocation method will be considered for the model given in (2.2) with its initial and boundary conditions. In order to compute the numerical solutions of the FitzHugh-Nagumo model via FEM on fixed points on a specific interval $\left[x_L, x_R\right]$, the interval should be divided N subelements with $h = \frac{x_R - x_L}{N}$ such that

$$x_L = x_0 < x_1 < x_2 < \cdots < x_{N-2} < x_{N-1} < x_N = x_R$$

where $\left\{x_m\right\}\big|_{m=0}^{N}$ are distinct grid points and number of grid points is $(N+1)$. The procedure for obtaining a numerical scheme is to compute numerical solutions define an approximate solution for the mentioned model. Let

$w(x,t)$ is an approximate solution to exact solution $u(x,t)$ is determined as follows

$$u(x,t) \approx w(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)\, TB_j^3(x)$$

where $\delta_j(t)$ are time dependent parameters which will be determined for obtaining numerical solutions at the point (x_j, t_n) and for this problem $TB_j^3(x)$ are cubic trigonometric B-spline basis given in [28] as follows

$$TB_m^3(x) = \frac{1}{\phi}\begin{cases} p^2(x_{m-2}) - p^2(x_{m-2})\,p(x_m), & x_{m-2} < x < x_{m-1} \\ -p(x_{m-2})\,p(x_{m+1})\,p(x_{m-1}) - p(x_{m+2})\,p^2(x_{m-1}), & x_{m-1} < x < x_m \\ p(x_{m-2})\,p^2(x_{m+1}) + p(x_{m+2})\,p(x_{m-1})\,p(x_{m+1}), & x_m < x < x_{m+1} \\ +p^2(x_{m+2})\,p(x_m), & \\ -p^3(x_{m+2}) & x_{m+1} < x < x_{m+2} \\ 0 & \text{otherwise} \end{cases}$$

where $p(xm) = \sin\left(\frac{x-x_m}{2}\right)$ and $\phi = \sin(\frac{h}{2})\sin(h)\sin\left(\frac{3h}{2}\right)$ for $m = 0,1,2,\ldots,N$. $TB_j(x)$ basis are zero out of interval $[x_{m-2}, x_{m+2}]$. On the ground of local support property of basis $TB_m(x)$, the approximate solution can be defined over a sub interval $[x_m, x_{m+1}]$ as

$$w(x,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\, TB_j^3(x)$$

Now, when the value of $w(x,t)$ at point x_m is expressed as w_m^n, it and its required derivatives can be obtained with some calculations as follows

$$\begin{aligned} w_m^n &= \alpha_1\delta_{m-1}(t) + \alpha_2\delta_m(t) + \alpha_1\delta_{m+1}(t), \\ (w_m^n)' &= \beta_1\delta_{m-1}(t) + \beta_2\delta_{m+1}(t), \\ (w_m^n)'' &= \eta_1\delta_{m-1}(t) + \eta_2\delta_m(t) + \eta_1\delta_{m+1}(t) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \alpha_1 &= \sin^2\left(\tfrac{h}{2}\right)\csc(h)\csc\left(\tfrac{3h}{2}\right), & \alpha_2 &= \tfrac{2}{1+\cos(h)}, \\ \beta_1 &= -\tfrac{3}{4}\csc\left(\tfrac{3h}{2}\right), & \beta_2 &= \tfrac{3}{4}\csc\left(\tfrac{3h}{2}\right), \\ \eta_1 &= \tfrac{3\left((1+3\cos(h))\csc^2\left(\frac{h}{2}\right)\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}, & \eta_2 &= \tfrac{3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}. \end{aligned}$$

The solution of the problem given in (2.2) will be obtained by substituting approximate solution and its derivatives into discretization of the problem. Thus, for a first task, Fitzhugh–Nagumo model should be discretized.

3. Discretization of the Fitzhugh–Nagumo model and Application of the method

In order to compute numerical solutions of Fitzhugh–Nagumo model, it is crucial to discritize the model and obtain recursive equation system i.e numerical scheme. For this purpose, the derivatives according to space will be discretize using Crank-Nicolson method, which is a average in time, by reason of obtaining the second order accurate and the fractional order derivative according to time will be discretized using $L1$ algorithm [29] as follows

$$D_t^\gamma f(t_m) = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1} b_k^\gamma\left[f(t_{n-k}) - f(t_{n-1-k})\right]$$

where $b_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}$, $t_n = n\Delta t\ (n = 0, 1.2, ..., M)$, and final time $T = M\Delta t$

When Crank-Nicolson method and $L1$ algorithm applied to Fitzhugh–Nagumo model given in (2.2) , it yields;

$$\begin{aligned} \tfrac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\textstyle\sum_{k=0}^{n-1} b_k^\gamma\left[u^{n-k} - u^{n-1-k}\right] &= \tfrac{\kappa}{2}\left((u_{xx})^{n+1} + (u_{xx})^n\right) + \tfrac{\mu}{2}\left((u^2)^{n+1} + (u^2)^n\right) \\ &+ \tfrac{\mu r}{2}\left(u^{n+1} + u^n\right) + \tfrac{\mu}{2}\left((u^3)^{n+1} + (u^3)^n\right) - \tfrac{\mu r}{2}\left((u^2)^{n+1} + (u^2)^n\right) \end{aligned} \tag{3.1}$$

From this point of view, nonlinear terms seen in (3.1) at $(n+1)$ time level will be linearized via Rubin-Graves linearization method as following way;

$$\left(u^2\right)^{n+1} = u^{n+1}u^n + u^n u^{n+1} - u^n u^n,$$

$$\left(u^3\right)^{n+1} = u^{n+1}u^n u^n + u^n u^{n+1}u^n + u^n u^n u^{n+1} - 2u^n u^n u^n.$$

Putting approximate solution and its derivatives (2.3) into (3.1) , and a bit of simple arrangements gives a recursive equation system as follows

$$\begin{aligned}
&\delta_{m-1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\mu\alpha_1\rho_1 - \kappa\eta_1\right)\right] + \delta_m^{n+1}\left[\alpha_2 + \tfrac{S}{2}\left(\mu\alpha_2\rho_1 - \kappa\eta_2\right)\right] + \delta_{m+1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\mu\alpha_1\rho_1 - \kappa\eta_1\right)\right] \\
&\quad = \delta_{m-1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\alpha_1\rho_2 + \kappa\eta_1\right)\right] + \delta_m^{n+1}\left[\alpha_2 + \tfrac{S}{2}\left(\alpha_2\rho_2 + \kappa\eta_2\right)\right] + \delta_{m+1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\alpha_1\rho_2 + \kappa\eta_1\right)\right] \\
&\quad - \textstyle\sum_{k=1}^{n-1} b_k^{\gamma}\left[\left(\alpha_1\delta_{m-1}^{n-k} + \alpha_2\delta_m^{n-k} + \alpha_1\delta_{m+1}^{n-k}\right) - \left(\alpha_1\delta_{m-1}^{n-1-k} + \alpha_2\delta_m^{n-1-k} + \alpha_1\delta_{m+1}^{n-1-k}\right)\right].
\end{aligned} \tag{3.2}$$

The sub indexes seen in (3.2) are $m = 0,1,2,...,N$ and $t = 0,1,2,...,M$. Here, T is final time and Δt is time step such as $T = M.\Delta t$ coefficients of unknowns seen in recursive equation system are $S = (\Delta t)^{\gamma}\,\Gamma(2-\gamma)$, $\rho_1 = 3\left(u^n\right)^2 - 2ru^n - 2u^n + r$ and $\rho_2 = \mu u^n\left(u^n - r\right).$

3.1 Boundary condition at $x = x_L$

As in with the finite difference method, boundary conditions at boundary of the interval are so essential in applying Finite element method, the numerical scheme obtained via FEM have to satisfy predefined conditions applied on their boundary.

When one glimpse at the system in (3.2), it can be realized that systems consist of $(N+1)$ equations with $(N+3)$ unknowns. For a solvable system, there are two ways, first one is adding two equation into the system and the second one is eliminating two unknown from the system. For two choice, boundary conditions will be used to achieve to aim. In this section, eliminating two unknown choice will apply. The boundary condition $u(x_L, t) = f_1(t)$ can be written using approximate solution at grid (x_m, t_n) as

$$u(x_m, t_n) = \alpha_1\delta_{m-1}(t) + \alpha_2\delta_m(t) + \alpha_1\delta_{m+1}(t), \qquad m = 0,1,2,...,N \tag{3.3}$$

Equation (3.3) is rewritten by $m = 0$ for right boundary

$$f_1(t) = \alpha_1\delta_{-1}(t) + \alpha_2\delta_0(t) + \alpha_1\delta_1(t).$$

Thus this gives;

$$\delta_{-1}(t) = \frac{f_1(t)}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\delta_0(t) - \delta_1(t). \tag{3.4}$$

3.2 Boundary condition at $x = x_R$

When the boundary condition $u(x_R, t) = f_2(t)$ is put into equation (3.3) with $m = N$, it yields;

$$f_2(t) = \alpha_1\delta_{N-1}(t) + \alpha_2\delta_N(t) + \alpha_1\delta_{N+1}(t).$$

Thus,

$$\delta_{N+1}(t) = \frac{f_2(t)}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\delta_N(t) - \delta_{N-1}(t). \tag{3.5}$$

When substituting (3.4) and (3.5) into recursive system (3.2), there will be a system consisting of $(N+1)$ equations with $(N+1)$ unknowns. When recursive system (3.2) is used with boundary conditions, it gives following numerical scheme which will derive numerical solutions as follows

$$\begin{cases} \delta_0^{n+1}(t)\left[\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right]=\delta_0^n(t)\left[-\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right] \\ -\sum_{k=1}^{n-1}b_k^\gamma\left[u_0^{n-k}-u_0^{n-1-k}\right]+\Upsilon_1, \end{cases} \qquad m=0$$

$$\begin{cases} \delta_{m-1}^{n+1}\left[\alpha_1+\frac{S}{2}(\mu\alpha_1\rho_1-\kappa\eta_1)\right]+\delta_m^{n+1}\left[\alpha_2+\frac{S}{2}(\mu\alpha_2\rho_1-\kappa\eta_2)\right] \\ +\delta_{m+1}^{n+1}\left[\alpha_1+\frac{S}{2}(\mu\alpha_1\rho_1-\kappa\eta_1)\right] \\ =\delta_{m-1}^n\left[\alpha_1+\frac{S}{2}(\alpha_1\rho_2+\kappa\eta_1)\right]+\delta_m^n\left[\alpha_2+\frac{S}{2}(\alpha_2\rho_2+\kappa\eta_2)\right]+ \\ \delta_{m+1}^n\left[\alpha_1+\frac{S}{2}(\alpha_1\rho_2+\kappa\eta_1)\right] \\ -\sum_{k=1}^{n-1}b_k^\gamma\left[\left(\alpha_1\delta_{m-1}^{n-k}+\alpha_2\delta_m^{n-k}+\alpha_1\delta_{m+1}^{n-k}\right)-\left(\alpha_1\delta_{m-1}^{n-1-k}+\alpha_2\delta_m^{n-1-k}+\alpha_1\delta_{m+1}^{n-1-k}\right)\right] \end{cases} \qquad 1<m<N-1$$

$$\begin{cases} \delta_N^{n+1}(t)\left[\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right]=\delta_0^n(t)\left[-\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right] \\ -\sum_{k=1}^{n-1}b_k^\gamma\left[u_N^{n-k}-u_N^{n-1-k}\right]+\Upsilon_2. \end{cases} \qquad m=N \tag{3.6}$$

Now, the system (3.6) is solvable. Of course, (3.6) can be given in the form of a matrix in briefly such as

$$A\Lambda^{n+1}=B\Lambda^n+\Upsilon \tag{3.7}$$

where vector $\Lambda=\{\delta_0,\delta_1,...,\delta_{N-1},\delta_N\}$ and vector Υ consist of boundary terms and values which produced via sum term. Using (3.7) , the unknown vector Λ^{n+1} will be determined by values of vector Λ^n.

4. Initial State

All recursive formulae must always state on initial terms. In this section, an initial vector must be derived. Thus, subsequent terms of the sequence can be found via initial vector. The initial vector $\Lambda^{(0)}=\left\{\delta_0^{(0)},\delta_1^{(0)},...,\delta_{N-1}^{(0)},\delta_N^{(0)}\right\}$ will be computed using the initial condition $u(x,0)=h(x)$. For this process, approximate solution and its derivatives are used at point x_m as follows;

$$u(x_m,0)=w(x_m,0)=h(x_m), \qquad m=0,1,2,...,N \tag{4.1}$$

When (4.1) is written in an explicit way, one can see that the system is composed of $(N+1)$ equations with $(N+3)$ unknowns. In order to eliminate two unknowns $\delta_{-1}^{(0)}$ and $\delta_{N+1}^{(0)}$ from the system, first derivative according to space variable will be used such that

$$\begin{aligned} \left(w_m^{(0)}\right)'&=\beta_1\delta_{-1}^{(0)}+\beta_2\delta_1^{(0)}=h(x_0), && \text{for } i=0 \\ \left(w_m^{(0)}\right)'&=\beta_1\delta_{N-1}^{(0)}+\beta_2\delta_{N+1}^{(0)}=h(x_N), && \text{for } i=N \end{aligned} \tag{4.2}$$

Thus,(4.2) yields

$$\delta_{-1}^{(0)}=\frac{1}{\beta_1}\left(h(x_0)-\beta_2\delta_1^{(0)}\right),$$

$$\delta_{N+1}^{(0)}=\frac{1}{\beta_2}\left(h(x_N)-\beta_1\delta_{N-1}^{(0)}\right).$$

At the end of process, all calculations result in a system involving $(N+1)$ equations with $(N+1)$ unknowns. Solving the system results in deriving initial vector $\Lambda^{(0)}$.

5. Numerical Approximation

In this section, the cubic trigonometric B-splines will be used to calculate the numerical solutions of FitzHugh–Nagumo model. Hence, accuracy and efficiency of the method will be tested of proposed numerical scheme. So as to calculate error norms, following formulas will be used;

$$L_2 = \sqrt{h \sum_{m=0}^{N} \left| u_m - w_m \right|^2}, \qquad L_\infty = \max_m \left| u_m - w_m \right|.$$

5.1 Case 1:

In this section, we apply the proposed technique on two cases. Consider the following time fractional FitzHugh-Nagumo model given in (2.2) under the following conditions $\kappa = \mu = 1, r = 4/10$, Thus, the problem is translated into following form;

$$D_t^\gamma u(x,t) = u_{xx}(x,t) + u(x,t)(1 - u(x,t))(u(x,t) - 4/10),$$

$$\begin{cases} u(x_L,t) = f_1(t), & u(x_R,t) = f_2(t), \\ u_x(x_L,t) = g_1(t), & u_x(x_R,t) = g_2(t), \quad (x,t) = [0,1] \times [0,1] \\ u(x,0) = h(x) \end{cases} \tag{5.1}$$

where $h(x) = 1/e^{-\frac{x}{\sqrt{2}}}$ and exact solution of the model is

$$u(x,t) = \frac{1}{e^{-\frac{x}{\sqrt{2}}}} - \frac{t^\gamma (2r-1) e^{\frac{x}{\sqrt{2}}}}{2\Gamma(\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2} - \frac{t^{2\gamma}(2r-1)^2 \left(e^{\frac{x}{\sqrt{2}}} - 1\right)}{4\Gamma(2\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}$$
$$- \frac{t^{3\gamma}(2r-1)^3 e^{\frac{x}{\sqrt{2}}}\left(e^{\sqrt{2}x} + 4e^{\frac{x}{\sqrt{2}}} + 1\right)}{16\Gamma(2\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4} + \frac{t^{4\gamma}(2r-1)^4 e^{\frac{x}{\sqrt{2}}}\left(-11e^{\sqrt{2}x} + e^{\frac{3x}{\sqrt{2}}} + 11e^{\frac{x}{\sqrt{2}}} - 1\right)}{96\Gamma(2\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}.$$

Boundary conditions can be seen taking $x = x_L$ and $x = x_R$ at exact solution.

As the first example, the interval which problem discussed on is chosen as $\Omega = [0,1] \times [0,1]$. The numerical solutions of (5.1) for different values of space step partition N and time step Δt and presented in Table 5.1-2 for $\gamma = 0.25$. Then different values for fractional order derivative are presented in Table 3 for $N = 800$ and $\Delta t = 0.001$. In last table i.e Table 4, comparisons are presented between [25] and present method for $\gamma = 0.9$ and different values of time step Δt.

It can be seen clearly from the table 5.1-2 that, the error norms L_2 and L_∞ are getting smaller with increasing number of space and time discretization. So that, for $N = 50$ and $\Delta t = 1/10$, L_2 =2.06214883$\times 10^{-5}$, L_∞ = 3.53959227$\times 10^{-5}$and for $N = 800$ and $\Delta t = 1/1000$, $L_2 = 1.26949713 \times 10^{-5}$, $L_\infty = 1.70531113 \times 10^{-5}$. We can conclude that the finite element method using trigonometric cubic splines derives quite accurate solutions to exact ones. Additionally, the results given in Table 4 are calculated using "mean absolute error" given in [25]. It can be seen from the table 4, the error norms which are obtained using trigonometric cubic splines are in agreement with the error norms given in [25] and obtained more accurate solutions for many values of Δt.

Table 1. Case 1: Error norms L_2 and L_∞ of Time fractional FitzHugh-Nagumo model for different values of Δt and N.

$\gamma = 0.25$	$\Delta t = 0.01$		$\Delta t = 0.01$	
N	$L_2 \times 10^5$	$L_\infty \times 10^5$	$L_2 \times 10^5$	$L_\infty \times 10^5$
50	2.06214883	3.53959227	1.34963953	1.89004865
100	2.01958305	3.44947107	1.28629959	1.80653553
200	2.00910969	3.42689595	1.27040051	1.78548711
400	2.00650198	3.42151241	1.26641599	1.78020364
800	2.00585031	3.42009868	1.26541792	1.77888708

Table 2. Case 1: Error norms L_2 and L_∞ of Time fractional FitzHugh-Nagumo model for different values of Δt and N.

$\gamma = 0.25$	$\Delta t = 0.05$		$\Delta t = 0.001$	
N	$L_2 \times 10^5$	$L_\infty \times 10^5$	$L_2 \times 10^{-5}$	$L_\infty \times 10^5$
50	1.31240768	1.84955706	1.28811292	1.81625015
100	1.24911235	1.76581920	1.22486871	1.73269800
200	1.23322360	1.74499717	1.20899298	1.71185235
400	1.22924146	1.73972543	1.20501403	1.70662887
800	1.22824392	1.73841335	1.20401729	1.70531113

Table 3. Case 1: Error norms L_2 and L_∞ of Time fractional FitzHugh-Nagumo model for different values of Δt and $N = 800$.

$N = 800$	$\Delta t = 0.01$		$\Delta t = 0.001$	
γ	$L_2 \times 10^5$	$L_\infty \times 10^5$	$L_2 \times 10^5$	$L_\infty \times 10^5$
0.1	1.10005315	1.58375582	1.04760205	1.49860510
0.3	1.30542358	1.83538958	1.24669134	1.75903888
0.5	1.41379340	1.96531926	1.36063388	1.89242299
0.8	1.40460709	1.92744306	1.33323670	1.82888194
0.9	1.33077714	1.82130939	1.26949713	1.73682731

Table 4. Case 1: A comparison between error norms L_2 and L_∞ for Time fractional FitzHugh-Nagumo model

$\gamma = 0.9$	Present method	[25]
Δt	MAE	MAE
$1/10$	$1.86895787 \times 10^{-5}$	5.35×10^{-5}
$1/20$	$1.51876756 \times 10^{-5}$	2.81×10^{-5}
$1/40$	$1.33900185 \times 10^{-5}$	1.48×10^{-5}
$1/80$	$1.25240772 \times 10^{-5}$	7.74×10^{-6}
$1/120$	$1.22481462 \times 10^{-5}$	4.04×10^{-6}

5.2 Case 2:

For this case, the solution interval is chosen as $x = [-10, 10]$ and $T = 1$. Also, the parameters seen in the model are chosen as $\kappa = \mu = 1, r = 0.8$ [30]. Similar to case 1, Table 5 is prepared to check effect of time partition on numerical solution. Thus, Table is prepared for various values of Δt . In table 6, numerical solutions of FitzHugh-Nagumo model for several values of fractional derivative are demonstrated.

Figures show numerical behaviour of Time fractional FitzHugh-Nagumo model for $r = 0.8, \gamma = 0.9, \Delta t = 0.01, N = 400$ and different final times in . Figure clears behaviour of model at final time $T = 5$, and is the 3-dimensional representation of the model.

Table 5. Case 2: Error norms L_2 and L_∞ for different values of Δt and γ for $N = 400$.

$N = 400$	$\gamma = 0.5$		$\gamma = 0.9$	
Δt	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.1	11.9763931	7.29847571	5.10996928	3.09912528
0.01	8.30005134	5.34968587	3.10841816	2.02770504
0.005	8.10092924	5.24171639	2.96691736	1.94862517
0.0025	8.00168118	5.18768771	2.89126522	1.90592140
0.001	7.94222728	5.15525135	2.84271156	1.87833803

Table 6. Case 2: A comparison between values of solutions of Time fractional FitzHugh-Nagumo model and absolute errors for $\gamma = 0.8$ and $x = 0.01$

		[30]				
t	u_{RPSM}	u_{HAM}	u_{FVIM}	u_{NIM}	u_{FEMT}	$\lvert u(x,t) - w(x,t)\rvert_{FEM}$
0.01	0.499745	0.499765	0.499774	0.497779	0.500038	0.0002929366
0.05	0.494437	0.494699	0.494541	0.487317	0.495499	0.0010628858
0.1	0.489004	0.489798	0.489186	0.476613	0.490853	0.0018553202
0.15	0.484113	0.485631	0.484366	0.46698	0.486670	0.0025749224
0.2	0.479543	0.481948	0.479864	0.457985	0.482762	0.0032545373

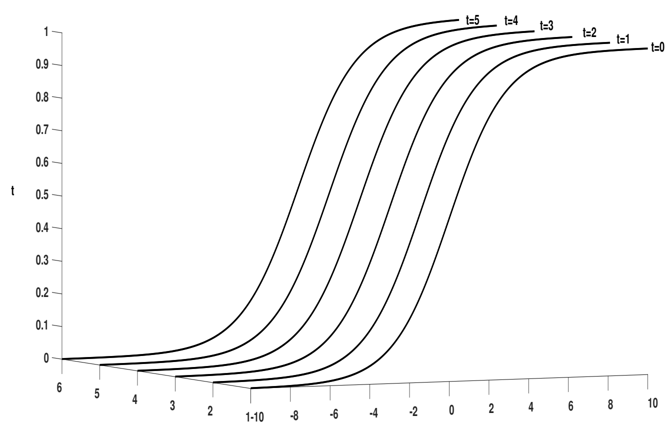

Figure 1. Numerical simulation of FitzHugh-Nagumo model: $r = 0.8, \gamma = 0.9, \Delta t = 0.01, N = 400$

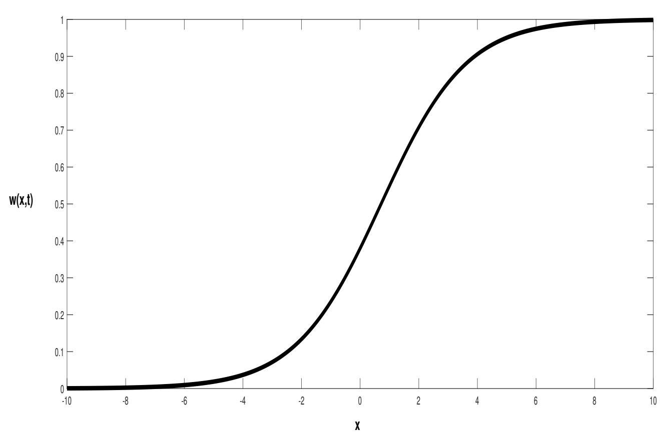

Figure 2. Numerical simulation of FitzHugh-Nagumo model: $r = 0.8, \gamma = 0.25, \Delta t = 0.01, N = 400$

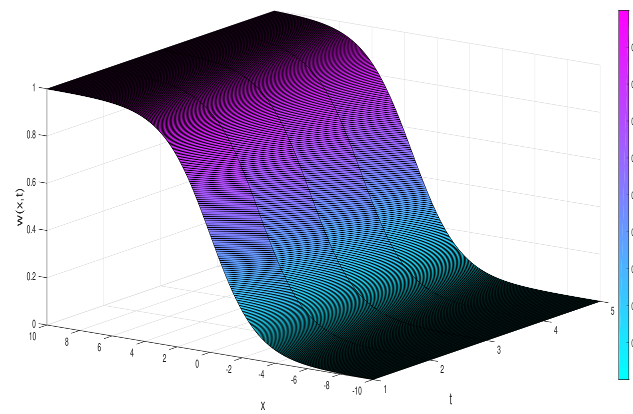

Figure 3. Numerical simulation of FitzHugh-Nagumo model: $r = 0.8$, $\gamma = 0.9$, $\Delta t = 0.01$,$N = 400$

6. Conclusion

In the present paper, finite element method based on trigonometric cubic B-spline is applied to time fractional FitzHugh-Nagumo model. Because the fractional derivative is taken in terms of Caputo sense, time discretization is made by $L1$ algorithm. Numerical solutions with comparison tables and error norms are presented. As it is seen from the numerical results, the results demonstrate that finite element method with trigonometric cubic basis is accurate and effective among existing methods.

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