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On the 0-Cauchy Completion of A Partial Metric Space

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Abstract. It is well known that every metric space admits a Cauchy completion which is unique up to isometry. This result was extended to partial metric spaces, which are generalization of metric spaces. It is the purpose of this paper to construct a 0-Cauchy completion of a partial metric space and we shall show that a 0-Cauchy completion is unique up to isometry. Finally, it is observed that the 0-Cauchy completion of a partial metric space is smaller than its Cauchy completion but coincides with the classical Cauchy completion when restricted to the category of metric spaces.

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1. Introduction

Partial metric spaces were introduced by S.G. Matthews [10], since then many interesting results were presented in the literature, see reference in [10] and those at the end of this article. The focus in the literature ranges from topological properties of these spaces to applications to theoretical computer science including generalizations of Banach' fixed point principle from metric spaces to partial metric spaces. In this paper we further study completeness and completion properties of partial metric spaces.

Classically, the notion of completeness for mathematical structures is done through the notion of a Cauchy sequence, metric spaces and cone metric spaces [1], [2] are such examples, this is also the case with partial metric spaces. In the literature we distinguish two types of Cauchy sequences for partial metric spaces. In particular, the notion of a 0-Cauchy sequence was introduced and used by Romaguera [11] in partial metric spaces to characterize 0-Cauchy completeness for partial metric spaces using so-called Caristi self-mappings. The characterization result obtained extends the counterpart result in metric spaces. In the paper [11] Romaguera indicates that 0-Cauchy sequences cannot necessarily be replaced by ordinary Cauchy sequences. This is very important as it shows how it is sometimes critical to study 0-Cauchy completeness instead of Cauchy completeness. The importance of 0-Cauchy sequences and 0-Cauchy completeness in partial metric spaces can also be realised by an increase in number of papers that appear in the literature that deal with 0-Cauchy completeness, see for example [13] and references therein. Furthermore, recently some papers focus on 0-Cauchy complete dislocated metric spaces [12]. We revisit the notion of a 0-Cauchy sequence and 0-Cauchy completeness in partial metric spaces and construct a 0-Cauchy completion for a partial metric space. The same construction yields the classical Cauchy completion for metric spaces, and generally, it is smaller than the Cauchy completion for partial metric spaces as constructed in the literature [9] and [5].

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2. Preliminaries

Definition 2.1 ([10]). A partial metric is a function $\sigma : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:
(i) $x = y \Leftrightarrow \sigma(x, x) = \sigma(x, y) = \sigma(y, y)$;
(ii) $\sigma(x, x) \leq \sigma(x, y)$;
(iii) $\sigma(x, y) = \sigma(y, x)$;
(iv) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z) - \sigma(y, y)$.

For a partial metric σ on X, the pair (X, σ) will be called a partial metric space. Note that for all x and y in a partial metric space $(X, \sigma), \sigma(x, y) = 0$, imply that $x = y$. The converse does not necessarily hold, also, the value $\sigma(x, x)$ usually referred to as the size or weight of x, is a feature used in applications, for instance to describe the amount of information contained in x.

Certainly a metric space is a partial metric space but not conversely. A sequence $\{x_n\}$ in a partial metric space (X, σ) is said to converge to a point $x \in X$ if $\lim_n \sigma(x_n, x) = \sigma(x, x) = \lim_n \sigma(x_n, x_n)$. The sequence $\{x_n\}$ in partial metric space is said to be Cauchy, if $\lim_{n,m} \sigma(x_n, x_m)$ exists and finite. The partial metric space (X, σ) is said to be Cauchy complete if every Cauchy sequence converges to a point $x \in X$.

The sequence $\{x_n\}$ in a partial metric space (X, σ) is said to be 0-Cauchy [11] if $\lim_{n,m} \sigma(x_n, x_m) = 0$. The partial metric space (X, σ) is said to be 0-Cauchy complete if every 0-Cauchy sequence converges to a point $x \in X$ and $\sigma(x, x) = 0$.

Note that a Cauchy complete partial metric space is referred to as a complete partial metric space and a 0-Cauchy complete partial metric space is referred to as a 0-complete partial metric space in the literature.

Remark 2.2. Every 0-Cauchy sequences is a Cauchy sequences but the converse is not necessarily true.

Definition 2.3. Let (X, σ) be a partial metric space. We shall say that a sequence $\{x_n\}$ in X, 0-converges to a point $x \in X$ if $\lim_n \sigma(x_n, x) = \sigma(x, x) = \lim_n \sigma(x_n, x_n) = 0$.

Example 2.4. Let $X = \{a, b\}$ and define $\sigma(x, y) = 1$ if $x = y$ and $\sigma(x, y) = 2$, otherwise. The sequence $\{x_n = a, n \geq 1\}$ is not 0-Cauchy but converges to a. Hence $\{x_n\}$ is a Cauchy sequence.

However, the following holds:

Theorem 2.5. *Let (X, σ) be a partial metric space and $\{x_n\}$ be a sequence in X. If a sequence $\{x_n\}$ 0-converges to a point x in X, then $\{x_n\}$ is a 0-Cauchy sequence.*

Note:
(i) A subsequence of a 0-Cauchy sequence in a partial metric space is a 0-Cauchy sequence.
(ii) If a 0-Cauchy sequence in a partial metric space converges to a point $x \in X$, then $\sigma(x, x) = 0$.
(iii) In the category of metric spaces, 0-Cauchy sequences and Cauchy sequences coincide. This is however not true in partial metric spaces. This remarkable feature will made explicit in the construction of the completion of a partial metric space using 0-Cauchy sequences.

Theorem 2.6. *Let (X, σ) be a partial metric space. Then (X, σ) is a metric space if and only if every Cauchy sequence is a 0-Cauchy sequences.*

Proof. If a partial metric space (X, σ) is a metric space, then every Cauchy sequences is a 0-Cauchy sequence. Suppose that every Cauchy sequence is a 0-Cauchy sequence. A constant sequence in a partial metric space (X, σ) is a Cauchy sequence. That is for all $a \in X$, we consider, $\{x_n = a, n \geq 1\}$. Then we have $\lim_{m,n} \sigma(x_m, x_n) = \sigma(a, a) = 0$. Hence $\sigma(x, x) = 0$, for all $x \in X$. So $\sigma(x, y) = 0$, if and only if $x = y$, and $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$ for all $x, y, z \in X$. The symmetric property is clear. This shows that (X, σ) is a metric space. □

Proposition 2.7. *Let (X, σ) be a partial metric space and $\{x_n\}$ be a 0-Cauchy sequence in X. Suppose that $\{x_n\}$ converges to x and y in X. Then $x = y$.*

Theorem 2.8. *Let (X, σ) be a partial metric space and $\{x_n\}$ be a 0-Cauchy sequence in X. If the subsequence $\{x_{n_k}\}$ converges to $x \in X$, then $\{x_n\}$ converges to x.*

Proof. See Lemma 1 (2), in [5], put $r = 0$. □

Theorem 2.9 ([5]). *Let (X, σ) be a partial metric space and $\{x_n\}, \{y_n\}$ be sequences in X that converges to $x, y \in X$, respectively. Then*

$$|\lim_n \sigma(x_n, y_n) - \sigma(x, y)| = 0.$$

Theorem 2.10 ([5]). *Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a partial metric space (X, σ). Then $\{\sigma(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$.*

3. Main Results

We are now ready to discuss a 0-Cauchy completion of a partial metric space. We have already indicated that Cauchy sequences are not the same as 0-Cauchy sequences in partial metric spaces. It is also worth considering that given a partial metric space (X, σ) the function $d_\sigma : X \times X \to [0, \infty)$ defined by $d_\sigma(x, y) = \sigma(x, y) - \sigma(x, x)$ is a quasi-metric on X, for all $x, y \in X$. Hence (X, d_σ) is a quasi-metric space. All Cauchy sequences are d^s_σ-Cauchy sequences, where $d^s_\sigma(x, y) = d_\sigma(x, y) \vee d_\sigma(y, x)$, for $x, y \in X$. Hence a Cauchy sequence in (X, σ) is a 0-Cauchy sequence in (X, d^s_σ). The reader should note that (X, d^s_σ) is a metric space. The Cauchy completion of a partial metric space (X, σ) can be obtained via the bicompletion of a quasi-metric space (X, d_σ) see [9]. Furthermore, given a partial metric space (X, σ) define a function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} \sigma(x, y) & \text{if} \quad x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

The space (X, d) is a metric space. Note that (X, σ) is 0-Cauchy complete if and only if (X, d) is Cauchy complete [6]. In this paper we will construct a 0-Cauchy completion of a partial metric space using 0-Cauchy sequences without using neither the associated quasi-metric space (X, d_σ) nor the associated metric space (X, d).

Definition 3.1 ([5]). Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a map between partial metric spaces. We say that f is an isometry if

$$\sigma_Y(f(x), f(y)) = \sigma_X(x, y),$$

for all $x, y \in X$. We shall say that f is continuous if

$$\lim_n \sigma_X(x_n, x) = \sigma_X(x, x) = \lim_n \sigma_X(x_n, x_n),$$

implies that

$$\lim_n \sigma_Y(f(x_n), f(x)) = \sigma_Y(f(x), f(x)) = \lim_n \sigma_Y(f(x_n), f(x_n)).$$

Lemma 3.2 ([5]). *Let $f : (X, \sigma_X) \to (Y, \sigma_Y)$ be a map between partial metric spaces. If f is an isometry, then it is continuous.*

Definition 3.3 ([5]). Let (X, σ) be a partial metric space and A be a subset of X. We say that A is sequentially dense in X if for any $x \in X$, there is a sequence $\{a_n\}$ in A converging to x.

Definition 3.4. Let (X, σ) and $(\bar{X}, \bar{\sigma})$ be partial metric spaces. Then we write $(X, \sigma) \subset (\bar{X}, \bar{\sigma})$ to mean that $X \subset \bar{X}$ and $\bar{\sigma}|_{X \times X} = \sigma$.

In the sequal, we will say $(\bar{X}, \bar{\sigma})$ contains (X, σ) to mean that $(X, \sigma) \subset (\bar{X}, \bar{\sigma})$. We will sometimes call $(\bar{X}, \bar{\sigma})$ an extension of (X, σ).

Theorem 3.5. *For every partial metric space (X, σ) there exits a partial metric space $(\bar{X}, \bar{\sigma})$ that is 0-Cauchy complete and contains (X, σ) as a sequentially dense subset.*

Proof. (*i*) Let $C = \{x_n : \{x_n\}$ is a 0-Cauchy sequence in $(X, \sigma)\}$. If $x_n, y_n \in C$, write

$$x_n \sim y_n \Leftrightarrow \lim_n \sigma(x_n, y_n) = 0.$$

Then $\sim$ is an equivalence relation on C.

(*ii*) Next, let $\mathcal{K} = \{x :$ where $\{x\}$ is a constant sequence which is not a 0-Cauchy sequence in $(X, \sigma)\}$. If $x, y \in \mathcal{K}$, write

$$x \sim y \Leftrightarrow x = y.$$

Then $\sim$ is an equivalence relation on $\mathcal{K}$.

(iii) If $x \in \mathcal{K}$ and $x_n \in C$, write

$$x \sim x_n \Leftrightarrow \lim_n \sigma(x, x_n) = 0.$$

That is, if and only if the sequence $\{x_n\}$ 0-converges to x. Put $\bar{X}$ to be the set of all the equivalence classes in $\mathcal{K}$ together with the set of all the equivalence classes in C, that is,

$$\bar{X} = \{[\{x_n\}] : x \in \mathcal{K}\} \cup \{[\{x_n\}] : \{x_n\} \in C\}.$$

Now for every $\bar{x}, \bar{y} \in \bar{X}$, define $\bar{\sigma} : \bar{X} \times \bar{X} \to [0, \infty)$ by

$$\bar{\sigma}(\bar{x}, \bar{y}) = \lim_n \sigma(x_n, y_n),$$

where $\bar{x} = [\{x_n\}]$ and $\bar{y} = [\{y_n\}]$. Then $\bar{\sigma}$ is well defined. Next, we show that $(\bar{X}, \bar{\sigma})$ is a partial metric space. We need only show that for all $\bar{x}, \bar{y}$ and $\bar{z} \in \bar{X}$, the following holds $\bar{\sigma}(\bar{x}, \bar{y}) \leq \bar{\sigma}(\bar{x}, \bar{z}) + \bar{\sigma}(\bar{z}, \bar{y}) - \bar{\sigma}(\bar{z}, \bar{z})$. Now, $\bar{\sigma}(\bar{x}, \bar{y}) = \lim_n \sigma(x_n, y_n) \leq \lim_n \sigma(x_n, z_n) + \lim_n \sigma(z_n, y_n) - \lim_n \sigma(z_n, z_n)$, so,

$$\bar{\sigma}(\bar{x}, \bar{y}) \leq \bar{\sigma}(\bar{x}, \bar{z}) + \bar{\sigma}(\bar{z}, \bar{y}) - \bar{\sigma}(\bar{z}, \bar{z}).$$

Consider a map $f : (X, \sigma) \to (\bar{X}, \bar{\sigma})$ defined by $f(x) = [\{x\}]$. Then f is an isometry. This shows that $f(X)$ is embedded in $\bar{X}$. Next we show that $f(X)$ is sequentially dense in $\bar{X}$. Let $\bar{x} \in \bar{X}$. Without loss for generality we assume that $\bar{x} \in C$. That is, $\bar{x} = [\{x_n\}]$. So that $\lim_{n,m} \sigma(x_n, x_m) = 0$. Now

$$\lim_n \bar{\sigma}(f(x_n), \bar{x}) = \lim_n [\lim_m \bar{\sigma}(x_n, x_m)] = \lim_{n,m} \sigma(x_n, x_m) = 0.$$

Clearly, $f(x_n) \in f(X)$. Furthermore, $\lim_n \bar{\sigma}(f(x_n), \bar{x}) = \lim_n \sigma(x_n, x_n) = \bar{\sigma}(\bar{x}, \bar{x}) = 0$. This shows that for every $\bar{x} \in \bar{X}$ there exists a sequence $\{f(x_n)\}$ in $f(X)$ such that $\lim_n \bar{\sigma}(f(x_n), \bar{x}) = \bar{\sigma}(\bar{x}, \bar{x}) = 0$. Now we show that $(\bar{X}, \bar{\sigma})$ is 0-Cauchy complete. Consider an arbitrary 0-Cauchy sequence $\bar{x}_n \in \bar{X}$, then $\bar{x}_n \in C$, hence, $\lim_{n,m} \sigma(x_n, x_m) = 0 = \lim_n \sigma(x_n, x_n)$. We have: $\sigma(x_n, x_m) = \bar{\sigma}(f(x_n), f(x_m))$ and $\bar{\sigma}(f(x_n), f(x_m)) \leq \bar{\sigma}(f(x_n), \bar{x}_n) + \bar{\sigma}(\bar{x}_n, \bar{x}_m) + \bar{\sigma}(\bar{x}_m, f(x_m)) - \bar{\sigma}(x_n, x_n) - \bar{\sigma}(x_m, x_m)$. Using sequential denseness of $f(X)$ in $\bar{X}$, the definition of f, and the fact that f is an isometry, we get that $\lim_{n,m} \sigma(x_n, x_m) = 0$, so $\{x_n\}$ is a 0-Cauchy sequence in X. Let $\bar{x} = [\{x_n\}]$. Then $\bar{x} \in C$. Finally, we show that $\lim_n \bar{\sigma}(\bar{x}_n, \bar{x}) = 0$. We note that for $\bar{x} \in \bar{X}$, we have $\lim_n \bar{\sigma}(f(x_n), \bar{x}) = 0$. Also,

$$\bar{\sigma}(\bar{x}_n, \bar{x}) \leq \bar{\sigma}(\bar{x}_n, f(x_n)) + \bar{\sigma}(f(x_n), \bar{x}_n) - \bar{\sigma}(f(x_n), f(x_n)).$$

Now $\bar{\sigma}(f(x_n), f(x_n)) = \bar{\sigma}(x_n, x_n)$. Hence,

$$\lim_n \bar{\sigma}(\bar{x}_n, \bar{x}) \leq \lim_n \bar{\sigma}(\bar{x}_n, f(x_n)) + \lim_n \bar{\sigma}(f(x_n), \bar{x}_n) - \lim_n \bar{\sigma}(x_n, x_n) = 0.$$

Clearly, $\bar{\sigma}(\bar{x}, \bar{x}) = \lim_n \bar{\sigma}(x_n, x_n) = 0$. Therefore $(\bar{X}, \bar{\sigma})$ is 0-Cauchy complete. Furthermore, $\bar{\sigma}|_{X \times X} = \sigma$. This complete the proof. □

The proof of the following theorem is the same as that of Proposition 1 in [5], for Cauchy complete partial metric spaces, so we omit the proof.

Theorem 3.6. *Let (X, σ_X) and (Y, σ_Y) be 0-Cauchy complete partial metric spaces, and A and B be sequentially dense in X and Y, respectively. An isometry $f : A \to B$, admits a unique map $F : X \to Y$, such that F is an isometry, and $F_A = f$.*

Theorem 3.7. *Every partial metric space (X, σ) admits a partial metric 0-Cauchy completion $(\bar{X}, \bar{\sigma})$, which is unique up to isometry.*

By slightly modifying some examples from the literature [3, 4, 7] and [8] for our purposes we present the following examples to justify our results:

Example 3.8.

Let $X = \{\frac{1}{n} : n \geq 1\}$. Define $\sigma : X \times X \to [0, \infty)$ by

$$\sigma(x, y) = max\{x, y\}.$$

Then (X, σ) is a partial metric space, which is not 0-Cauchy complete. To see this, the sequence $\{x_n = \frac{1}{n}, n \geq 1\}$ is a 0-Cauchy sequence in X but there is no $x \in X$, such that $\lim_n \sigma(x_n, x) = \sigma(x, x) = \lim_n \sigma(x_n, x_n)$. The 0-Cauchy completion of (X, σ) is $(\bar{X}, \bar{\sigma})$, where $\bar{X} = X \cup \{0\}$ and $\bar{\sigma}(\bar{x}, \bar{y}) = max\{\bar{x}, \bar{y}\}$.

Example 3.9.

Let $X = \mathbb{Q} \cap (0, \infty)$. Define $\sigma : X \times X \to [0, \infty)$ by

$$\sigma(x, y) = max\{x, y\}.$$

Then (X, σ) is a partial metric space, which is not 0-Cauchy complete. The 0-Cauchy completion of (X, σ) is $(\bar{X}, \bar{\sigma})$, where $\bar{X} = \mathbb{Q} \cap [0, \infty)$, and $\bar{\sigma}(\bar{x}, \bar{y}) = max\{\bar{x}, \bar{y}\}$. It is well known [11] that $(\bar{X}, \bar{\sigma})$, is not Cauchy complete.

Definition 3.10. Let A be a subset of a partial metric space (X, σ) we shall say that A is sequentially closed if for a sequence $\{a_n\}$ in A and $x \in X$ then $\lim_n \sigma(a_n, x) = \sigma(x, x) = \lim_n \sigma(a_n, a_n)$, imply that $x \in A$.

Definition 3.11. Let (X, σ) be a partial metric space. Then (X, σ) is said to be 0-convergence complete if for every partial metric space $(\bar{X}, \bar{\sigma})$ that contains (X, σ) a sequence $\{x_n\}$ in X that 0-converges to $\bar{x} \in \bar{X}$ with respect to $\bar{\sigma}$ also 0-converges to some point $x \in X$ with respect to σ.

Theorem 3.12. *Let (X, σ) be a partial metric space and A be a sequentially closed subset of X. If (X, σ) is 0-convergence complete, then A is 0-convergence complete.*

Theorem 3.13. *Let (X, σ) be a partial metric space and A be a sequentially closed subset of X. If (X, σ) is Cauchy complete, then A is Cauchy complete.*

We now characterize 0-completeness in partial metric spaces.

Theorem 3.14. *Let (X, σ) be a partial metric space. The following statements are equivalent:*
(a) (X, σ) *is 0-Cauchy complete;*
(b) (X, σ) *is 0-convergence complete.*

Proof. Suppose that the partial metric space (X, σ) is 0-Cauchy complete. We want to show that (X, σ) is 0-convergence complete. Let $\{x_n\}$ be a sequence in X and assume that there exists a partial metric space $(\bar{X}, \bar{\sigma})$ that contains (X, σ) with $\lim_n \bar{\sigma}(x_n, \bar{x}) = \bar{\sigma}(\bar{x}, \bar{x}) = \lim_n \bar{\sigma}(x_n, x_n) = 0$. Then $\{x_n\}$ is a 0-Cauchy sequence, hence, by 0-Cauchy completeness of (X, σ), there exists $x \in X$, such that

$$\lim_n \sigma(x_n, x) = \sigma(x, x) = \lim_n \sigma(x_n, x_n) = 0.$$

We have $x = \bar{x}$. This shows that (X, σ) is 0-convergence complete. Thus $(a) \Rightarrow (b)$. Conversely, assume that (b) is true and let $\{x_n\}$ 0-Cauchy complete. By Theorem 3.5, there exists $(\bar{X}, \bar{\sigma})$ that contains (X, σ) such that

$$\lim_n \bar{\sigma}(x_n, \bar{x}) = \bar{\sigma}(\bar{x}, \bar{x}) = \lim_n \bar{\sigma}(x_n, x_n) = 0,$$

for $\bar{x} \in \bar{X}$. By property (b), there exists $x \in X$ such that $\lim_n \sigma(x_n, x) = \sigma(x, x) = \lim_n \sigma(x_n, x_n) = 0$. This shows that every 0-Cauchy sequence in X converges to a point in X and (X, σ) is 0-Cauchy complete. □

Theorem 3.14 tells us that a 0-Cauchy complete partial metric space cannot be extended as far as the convergence of its 0-Cauchy sequences is concerned.

4. Conclusion

It is well known [5] that every partial metric space (X, σ) admits a unique Cauchy completion $(\bar{X}, \bar{\sigma})$ up to isometry. It follows that such a Cauchy completion is also a 0-Cauchy completion. However, since not every 0-Cauchy complete partial metric space is Cauchy complete, see Example 3.8 it follows that a Cauchy completion of a partial metric space may be larger than its 0-Cauchy completion. Importantly, our completion theory of a partial metric space via 0-Cauchy sequences, the Cauchy completion theory of a partial metric space via Cauchy sequences and the classical Cauchy completion theory of a metric space all coincide in the category of metric spaces. It is worthwhile to study 0-Cauchy completeness in partial metric spaces and the 0-Cauchy completion on partial metric spaces. This is because although a Cauchy completion of partial metric space is a 0-Cauchy completion it is desirable in some applications [11] to work with a smaller completion of a partial metric space, namely, a 0-Cauchy completion instead. In our future work we would like to construct the 0-Cauchy completion of a partial metric space (X, σ) through the associated metric space (X, d), see [6].

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