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Central Angle Theorem for Hyperbolic Angle

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Abstract. In this study, we give a different proof for the central angle theorem in the Lorentz plane. We also obtain beneficial trigonometric results regarding to the interior, exterior, and inscribes angle on the Lorentz circle S_1^1.

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1. Introduction

In Euclidean space, it is known that a negative Gaussian constant curvature surface is a two-sheeted hyperboloid. In [3], Luneburg is based upon one sheeted hyperboloid and this is called as hyperbolic space. For such reasons, it is normal for a visual space to intersect with S_1^2 de Sitter plane embedded in the Lorentz space. Vieth Müller circles taking a role in examining the visual space are related to the circles involved in the Euclidean model. An angle magnitude defined by means of the Euclidean unit circle has a significant role in the use of Vieth Müller circles [4]. We see that the analogy of Vieth Müller circles in the Lorentz space corresponds to the S_1^1 de Sitter line.

For this reason, the computation of the Lorentz analogies of central, exterior, interior, and inscribed angles, which are well known in Euclidean space, and obtaining the relation among such angles in the S_1^1 de Sitter plane was necessary for our doctorate study and they constitute the principal target of this study. A review including these matters revealed that there are no other studies except [1] and the proofs contained in [1] are carried out by means of artificial methods. We obtained simpler results by making such proofs through analytical methods. Theorem 2.6 and Theorem 2.7 are orijinal results which may not be found in [1].

2. Preliminaries

Let R_1^2 be the Lorentzian plane in which the scalar product of two vectors. $x = (x_0, x_1)$ and $y = (y_0, y_1)$ is given by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1.$$

The sets $\left\{x \in R_1^2 \mid \langle x, x\rangle = 1\right\}$ and $\left\{x \in R_1^2 \mid \langle x, x\rangle = -1\right\}$ are called the Lorentzian circle S_1^1 and hyperbolic line H_0^1.

$$(S_1^1)_+ = \left\{x \epsilon S_1^1 \mid x_2 > 1\right\}$$

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and

$$(S_1^1)_- = \left\{x\epsilon S_1^1 \mid x_2 < 1\right\}$$

as two branches of S_1^1. Each $P \in (S_1^1)_+$ can be given by the parametric form $P(\sinh\theta, \cosh\theta)$. Similarly, each $P \in (S_1^1)_-$ can be given as $P(\sinh\theta, -\cosh\theta)$. Branches of hyperbolic line H_0^1 are given by

$$\begin{aligned}(H_0^1)_+ &= \left\{(x_1, x_2)\epsilon H_0^1 \mid x_1 > 0\right\},\\ (H_0^1)_- &= \left\{(x_1, x_2)\epsilon H_0^1 \mid x_1 < 0\right\}.\end{aligned}$$

As in the Euclidean case, the angles in the Lorentz plane are related to the lengths of arcs on the circle of S_1^1.

Remark 2.1. If $P, Q \in S_1^1$, then we have the following facts:
1. $\langle P, Q\rangle > 1$ if and only if P and Q are on the same branch of S_1^1.
2. $\langle P, Q\rangle < -1$ if and only if P and Q are on the different branch of S_1^1 [2].

Remark 2.2. For each pair of points $P, Q \in H_0^1$,
1. $\langle P, Q\rangle < -1$ if and only if P and Q are on the same branch of H_0^1.
2. $\langle P, Q\rangle > 1$ if and only if P and Q are on the different branch of H_0^1 [2].

Let θ be the angle between unit vectors u and v of R_1^2. Then [2],

$$\theta = \begin{cases} \arccos(\langle u, v\rangle) & ; \text{ if } |\langle u, v\rangle| < 1 \\ \operatorname{arccosh}(-\langle u, v\rangle) & ; \text{ if } \langle u, v\rangle < -1 \\ \operatorname{arccosh}(\langle u, v\rangle) & ; \text{ if } \langle u, v\rangle > 1 \\ \operatorname{arcsinh}(|\langle u, v\rangle|) & ; \text{ if } u \text{ timelike and } v \text{ spacelike.} \end{cases}$$

The triangle $P\overset{\Delta}{Q}R$ with the middle vertex in Lorentz plane is called *pure spacelike triangle* if the vectors $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{PR}$ are spacelike [1].

The triangle $P\overset{\Delta}{Q}R$ with the middle vertex in Lorentz plane is called *pure timelike triangle* if the vectors $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{PR}$ are timelike [1].

Let O be center of S_1^1 and let P and Q be same branch of S_1^1. Then the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is called *central angle*. The magnitude of $\widehat{POQ}$ is the length of the arc PQ [1].

Let P, Q, R be three points of S_1^1, then the angle between vectors $\overrightarrow{RP}, \overrightarrow{RQ}$ is called *inscribed angle* if $\langle P, Q\rangle > 1$ and $\langle R, Q\rangle < -1$.

In what follows we will denote with $\overset{\frown}{PQ}$ to the arc between the points P and Q if $\langle P, Q\rangle > 1$.

The sets $(S_1^1)^I = \left\{r \in R_1^2 \mid \langle r, r\rangle < 1\right\}$ and $(S_1^1)^E = \left\{r \in R_1^2 \mid \langle r, r\rangle > 1\right\}$ are called the *interior of* S_1^1 and *exterior of* S_1^1.

The angle $\widehat{PRQ}$ is called *an exterior angle* of S_1^1 for $P, Q \in S_1^1$ and $r \in (S_1^1)^E$.

The angle $\widehat{PRQ}$ is called *an interior angle* of S_1^1 for $P, Q \in S_1^1$ and $r \in (S_1^1)^I$.

Let P, Q, R be three points on the same branch of S_1^1 then the triangle $P\overset{\Delta}{Q}R$ is a pure timelike triangle.

If $P\overset{\Delta}{Q}R$ is a pure timelike triangle then the angle $\widehat{PRQ}$ is called the middle angle at the middle vertex of $P\overset{\Delta}{Q}R$.

Theorem 2.3 (Central angle theorem)**.** *The central angle subtended by two points on same part of* S_1^1 *is twice the inscribed angle subtended by those points.*

Proof. If the point R is on the first zone of coordinate axis, then we can write $P(\cosh\theta_1, \sinh\theta_1)$, $Q(\cosh\theta_2, \sinh\theta_2)$, and $R(\cosh\theta_3, \sinh\theta_3)$. Therefore,

$$\begin{aligned}\overrightarrow{RP} &= (\sinh\theta_1 - \sinh\theta_3, \cosh\theta_1 - \cosh\theta_3),\\ \overrightarrow{RQ} &= (\sinh\theta_2 - \sinh\theta_3, \cosh\theta_2 - \cosh\theta_3).\end{aligned}$$

Since,

$$\begin{aligned}\langle\overrightarrow{RP},\overrightarrow{RP}\rangle &= -4\sinh^2\left(\frac{\theta_3-\theta_1}{2}\right),\\ \langle\overrightarrow{RQ},\overrightarrow{RQ}\rangle &= -4\sinh^2\left(\frac{\theta_3-\theta_2}{2}\right)\end{aligned}$$

$\overrightarrow{RP}$ and $\overrightarrow{RQ}$ are timelike. Thus we can see that

$$\frac{\langle\overrightarrow{RP},\overrightarrow{RQ}\rangle}{\left\|\overrightarrow{RP}\right\|\left\|\overrightarrow{RQ}\right\|} = -\cosh\left(\frac{\theta_2-\theta_1}{2}\right).$$

ii) If the point R is on the second zone of the coordinate axis, then we have $P(\cosh\theta_1,\sinh\theta_1)$, $Q(\cosh\theta_2,\sinh\theta_2)$, and $R(-\cosh\theta_3,\sinh\theta_3)$. Thus,

$$\begin{aligned}\overrightarrow{RP} &= (\sinh\theta_1-\sinh\theta_3,\cosh\theta_1-\cosh\theta_3),\\ \overrightarrow{RQ} &= (\sinh\theta_2-\sinh\theta_3,\cosh\theta_2+\cosh\theta_3).\end{aligned}$$

Because

$$\begin{aligned}\langle\overrightarrow{RP},\overrightarrow{RP}\rangle &= 4\cosh^2\left(\frac{\theta_1+\theta_3}{2}\right),\\ \langle\overrightarrow{RQ},\overrightarrow{RQ}\rangle &= 4\cosh^2\left(\frac{\theta_2+\theta_3}{2}\right)\end{aligned}$$

$\overrightarrow{RP}$ and $\overrightarrow{RQ}$ are spacelike, then we have

$$\frac{\langle\overrightarrow{RP},\overrightarrow{RQ}\rangle}{\left\|\overrightarrow{RP}\right\|\left\|\overrightarrow{RQ}\right\|} = \cosh\left(\frac{\theta_2-\theta_1}{2}\right).$$

iii) If the point R is on the third zone of the coordinate axis, $P(\cosh\theta_1,\sinh\theta_1)$, $Q(\cosh\theta_2,\sinh\theta_2)$ and $R(-\cosh\theta_3,-\sinh\theta_3)$. We see that

$$\begin{aligned}\overrightarrow{RP} &= (\sinh\theta_1+\sinh\theta_3,\cosh\theta_1+\cosh\theta_3),\\ \overrightarrow{RQ} &= (\sinh\theta_2+\sinh\theta_3,\cosh\theta_2+\cosh\theta_3).\end{aligned}$$

Since,

$$\begin{aligned}\langle\overrightarrow{RP},\overrightarrow{RP}\rangle &= 4\cosh^2\left(\frac{\theta_2-\theta_1}{2}\right),\\ \langle\overrightarrow{RQ},\overrightarrow{RQ}\rangle &= 4\cosh^2\left(\frac{\theta_3-\theta_2}{2}\right)\end{aligned}$$

$\overrightarrow{RP}$ and $\overrightarrow{RQ}$ are spacelike. Then we have

$$\frac{\langle\overrightarrow{RP},\overrightarrow{RQ}\rangle}{\left\|\overrightarrow{RP}\right\|\left\|\overrightarrow{RQ}\right\|} = \cosh\left(\frac{\theta_2-\theta_1}{2}\right).$$

iv) If the point R is on the fourth zone of the coordinate axis, $P(\cosh\theta_1,\sinh\theta_1)$, $Q(\cosh\theta_2,\sinh\theta_2)$, and $R(\cosh\theta_3,-\sinh\theta_3)$. Thus,

$$\begin{aligned}\overrightarrow{RP} &= (\sinh\theta_1+\sinh\theta_3,\cosh\theta_1-\cosh\theta_3),\\ \overrightarrow{RQ} &= (\sinh\theta_2+\sinh\theta_3,\cosh\theta_2-\cosh\theta_3).\end{aligned}$$

Because

$$\begin{aligned} \langle \overrightarrow{RP}, \overrightarrow{RP} \rangle &= -4\sinh^2\left(\frac{\theta_3+\theta_1}{2}\right), \\ \langle \overrightarrow{RQ}, \overrightarrow{RQ} \rangle &= -4\sinh^2\left(\frac{\theta_3+\theta_2}{2}\right) \end{aligned}$$

$\overrightarrow{RP}$ and $\overrightarrow{RQ}$ are timelike. Then we have

$$\frac{\langle \overrightarrow{RP}, \overrightarrow{RQ} \rangle}{\left\|\overrightarrow{RP}\right\| \left\|\overrightarrow{RQ}\right\|} = -\cosh\left(\frac{\theta_2-\theta_1}{2}\right).$$

□

By the proof of Theorem 2.3, we have the following corollary.

Corollary 2.4. *An inscribed angle is half of a central angle that has the subtends of the same arc.*

Theorem 2.5. *The sum of two interior angles is equal to an exterior angle of a triangle.*

Proof. Let V be a point in $(S^1_1)^I$ and let A,B be a point of the same branch of S^1_1. The lines $\overrightarrow{VA}$ and $\overrightarrow{VB}$ cut the other branch of S^1_1 points at C and D. Then we can write points $A(\cosh\theta_0, \sinh\theta_0)$, $B(\cosh\theta_1, \sinh\theta_1)$, $C(-\cosh\theta_3, \sinh\theta_3)$, $D(-\cosh\theta_2, \sinh\theta_2)$, which are taken on the hyperbola.

We must show the following equation for the proof of theorem:

$$\operatorname{arcsinh}\left(\frac{\langle \overrightarrow{AB}, \overrightarrow{AD} \rangle}{\left\|\overrightarrow{AB}\right\| \left\|\overrightarrow{AD}\right\|}\right) + \operatorname{arccosh}\left(\frac{\langle \overrightarrow{DA}, \overrightarrow{DB} \rangle}{\left\|\overrightarrow{DA}\right\| \left\|\overrightarrow{DB}\right\|}\right) = -\operatorname{arcsinh}\left(\frac{\langle \overrightarrow{BA}, \overrightarrow{BD} \rangle}{\left\|\overrightarrow{BA}\right\| \left\|\overrightarrow{BD}\right\|}\right).$$

We can write

$$\begin{aligned} \overrightarrow{AB} &= (\sinh\theta_1 - \sinh\theta_0, \cosh\theta_1 - \cosh\theta_0), \\ \overrightarrow{AD} &= (\sinh\theta_2 - \sinh\theta_0, -\cosh\theta_2 - \cosh\theta_0), \\ \overrightarrow{DB} &= (\sinh\theta_1 - \sinh\theta_2, \cosh\theta_1 + \cosh\theta_2), \end{aligned}$$

.

$$\begin{aligned} \langle \overrightarrow{AB}, \overrightarrow{AB} \rangle &= -4\sinh^2\left(\frac{\theta_0-\theta_1}{2}\right), \\ \langle \overrightarrow{AD}, \overrightarrow{AD} \rangle &= 4\cosh^2\left(\frac{\theta_0+\theta_2}{2}\right), \\ \langle \overrightarrow{DB}, \overrightarrow{DB} \rangle &= 4\cosh^2\left(\frac{\theta_1+\theta_2}{2}\right). \end{aligned}$$

Since $\overrightarrow{AD}$, $\overrightarrow{DB}$ are spacelike and $\overrightarrow{AB}$ timelike. We obtain that

$$\begin{aligned} \frac{\langle \overrightarrow{AB}, \overrightarrow{AD} \rangle}{\left\|\overrightarrow{AB}\right\| \left\|\overrightarrow{AD}\right\|} &= \sinh\left(\frac{\theta_1+\theta_2}{2}\right), \\ \frac{\langle \overrightarrow{DA}, \overrightarrow{DB} \rangle}{\left\|\overrightarrow{DA}\right\| \left\|\overrightarrow{DB}\right\|} &= \cosh\left(\frac{\theta_0-\theta_1}{2}\right), \\ \frac{\langle \overrightarrow{BA}, \overrightarrow{BD} \rangle}{\left\|\overrightarrow{BA}\right\| \left\|\overrightarrow{BD}\right\|} &= -\sinh\left(\frac{\theta_0+\theta_2}{2}\right), \end{aligned}$$

$$\begin{aligned}
\operatorname{arcsinh}\left(\frac{\langle \overrightarrow{AB}, \overrightarrow{AD}\rangle}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AD}\right\|}\right) + \operatorname{arccosh}\left(\frac{\langle \overrightarrow{DA}, \overrightarrow{DB}\rangle}{\left\|\overrightarrow{DA}\right\|\left\|\overrightarrow{DB}\right\|}\right) &= \frac{\theta_1+\theta_2}{2} + \frac{\theta_0-\theta_1}{2} \\
&= \frac{\theta_0+\theta_2}{2}
\end{aligned}$$

which implies the proof of theorem .

□

Theorem 2.6. *Inscribed angles where one chord is a diameter are the Lorentz orthogonal angle.*

Proof. If $R(0,-1) \in (S^1_1)_-$, $L(0,1)$ and $P(\sinh\theta, \cosh\theta) \in (S^1_1)_+$, then

$$\begin{aligned}
\overrightarrow{PR} &= -(\sinh\theta, 1+\cosh\theta), \\
\overrightarrow{PL} &= (-\sinh\theta, 1-\cosh\theta).
\end{aligned}$$

Because

$$\begin{aligned}
\langle \overrightarrow{PR}, \overrightarrow{PR}\rangle &= 2(1+\cosh\theta), \\
\langle \overrightarrow{PL}, \overrightarrow{PL}\rangle &= 2(1-\cosh\theta)
\end{aligned}$$

$\overrightarrow{PL}$ is a timelike vector and $\overrightarrow{PR}$ is a spacelike vector. Since

$$\langle \overrightarrow{PL}, \overrightarrow{PR}\rangle = 0$$

and the angle α between $\overrightarrow{PL}$ and $\overrightarrow{PR}$ is calculated by

$$\langle \overrightarrow{PL}, \overrightarrow{PR}\rangle = \left\|\overrightarrow{PR}\right\| \left\|\left|\overrightarrow{PL}\right|\right\| \sinh\alpha$$

we obtain $\alpha = 0$. □

Theorem 2.7. *In a hyperbola, two inscribed angles with the same intercepted arc of S^1_1 are congruent.*

Proof. Let the points A, B be on same branch of S^1_1 and let points C, D be on the other branch of S^1_1. Then by Theorem 2.3, inscribed angles with the same arc $\frown{AB}$ are equal. □

References

[1] Desideri, G.M., *Angles in the Lorentzian circle*, Comptes rendus de l'Académie bulgare des Sciences, **60**(2007), 357–364. 1, 2

[2] Karliga, B., Tokeser, U., Savas, M., *Girard type theorem for de Sitter triangles*, (2016), preprint. 2.1, 2.2, 2

[3] Luneburg, R.K., *Mathematical Analysis of Binocular Vision*, Princeton, New Jersey: Princeton University Press, 1947. 1

[4] Zage, W. M., *Hyperbolic Geometry and Luneburg's Theory of Binocular Visual Space*, University of Illinois at Chicago, D. A., 1977. 1