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Research Article

Optimal investment and reinsurance strategies for an insurer with stochastic economic factor

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Abstract

This work considers optimal investment and reinsurance strategies for an insurer with stochastic economic factor. In our mathematical model, a risk-free asset and a risky asset are assumed to rely on a stochastic economic factor which is described by a diffusion process. We generalize the claim process to a compound Poisson process with the stochastic economic factor. Using expected utility maximization, we characterize the optimal strategy of investment-reinsurance under the power utility function. We use dynamic programming principle to derive the Hamilton–Jacobi–Bellman (HJB) equation. Then, by analysing the solution of the HJB equation, the optimal investment-reinsurance strategy is obtained and given in the verification theorem. Finally, sensitivity analysis is given to show the economic behavior of the optimal investment and reinsurance strategies.

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1. Introduction

In the actuarial industry, the problem of optimal investment proportion is the fundamental problem concerned by insurers, who always want to maximize the expected utility of their terminal wealth. The pioneering research in this area is done by [12], who first introduced stochastic control methods to study the optimal portfolio problem. Subsequently, the optimal investment problem has been the subject of a lot of research. Cox and Huang [7] used the martingale method to obtain an explicit solution for optimal control while considering the non-negative constraints of consumption and wealth. Karatzas and Shreve [11] explained Merton's research and its popularization in their monograph. Pham [15] extended the Merton optimal investment problem to a multi-dimensional model with stochastic volatility and portfolio constraints, using logarithmic transformation to express the value function as a solution of a semi-linear parabolic equation. Using stochastic control representation and some approximations, he proved the smooth solution of the semi-linear parabolic equation, thus proving the existence of the optimal investment portfolio and expressing it with the classical solution of the semi-linear equation. Yang and Zhang [20] studied the optimal investment strategy of an insurance company whose risk process obeys the jump-diffusion stochastic differential equation (SDE) and obtained a

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closed-form expression of the optimal strategy when the utility function is exponential. Wang et al. [19] studied the optimal investment problem of insurers and reinsurers, applied stochastic control theory, established the corresponding HJB equation, and deduced the optimal investment reinsurance strategy under the exponential utility function.

In recent years, the issue of optimal reinsurance has also attracted more and more attention. Reinsurance is an essential tool for insurance companies to manage their risk exposure. Schmidli [16] obtained the optimal reinsurance strategy under the diffusion model and the classic risk model under the minimum ruin probability criterion. Golubin [8] considered the criterion of minimizing the expected maximum loss for the classic risk model, and transformed the insurer's insurance and reinsurance strategy problems into static problems without considering the HJB equation. Guan and Liang [9] introduced interest rate and inflation risk, and analyzed an insurer's optimal reinsurance and investment problem. Shen and Zeng [17] maximized and minimized the mean and variance of the insurer's terminal wealth simultaneously, using the method of backward stochastic differential equation (BSDE), and obtained the closed-form expression of the optimal strategy. Sun and Guo [18] considered optimal investment and reinsurance strategies where the volatility of the risky asset is random and follows that of the CIR process under the mean-variance criterion by using BSDE technology. Brachetta and Ceci [3] investigated the optimal proportional reinsurance-investment strategy of an insurance company which wishes to maximize the expected exponential utility of its terminal wealth in a finite time horizon. The model extends the Cramér–Lundbegr model, introduces a stochastic factor affecting the intensity of the claims arrival process, and assumes that financial market is not affected by the stochastic factor. Subsequently, when both the intensity of the claims arrival process and the claim size distribution are influenced by an exogenous stochastic factor, Brachetta and Ceci [4] considered the optimal excess-of-loss reinsurance problem. Cao et al. [6] studied the optimal reinsurance investment problem of the compound dynamic contagion process. The model considers the self-exciting and externally-exciting clustering effect for the claim arrivals. Ceci et al. [5] characterized the optimal investment and proportional reinsurance problem of an insurance company with exponential utility preferences in a stochastic-factor model. The insurance company experiences both ordinary and catastrophic claims and wishes to maximize the expected exponential utility of its terminal wealth.

This paper considers optimal investment and reinsurance strategies for an insurer with the stochastic economic factor. Regarding the stochastic economic factor, we assumed that it is subject to Itô diffusion (see [2]). We generalize the claim process to a compound Poisson process with the stochastic economic factor. We also allow an insurer to purchase the reinsurance contract to reduce their risks and calculate the premium based on the expected value principle. In addition, there are two types of assets available for an insurer to invest in the financial market, namely risk-free assets and risky assets. Under power utility, in order to be able to apply dynamic programming principles, we innovatively define a new variable called the ratio of the retained proportion of risk over surplus. Then we use the power transformation (see [21]) to make the HJB equation linear. We subsequently apply Proposition 2.3 in [1] to analyze the solution of the updated HJB equation, thereby obtaining the classical solution of the original HJB equation. In this way, the optimal investment-reinsurance strategy is found. We thus use the verification theorem to give it.

To summarize, the main contributions of this paper have four aspects: (1) We introduce reinsurance into our model framework and related to the stochastic economic factor, and then skilfully solve the optimal investment-reinsurance problem by define a new variable. (2) We extend the classical claim process to a pure discontinuous process with the stochastic economic factor. (3) The stochastic factor does not influence claims arrival intensity but only claims jump sizes and this should allow to simplify the solution. (4) We assume that the insurer and reinsurer ignore the stochastic economic factor when calculating the premium, which will simplify the net profit condition (See Remark 2.1) and HJB equation, and thereby simplify the problem.

The remainder of this paper is organized as follows. In Section 2, we describe the model. In Section 3, we derive the HJB equation and optimal strategies. In Section 4, we prove the existence and uniqueness of the classical solution of HJB equation and the corresponding verification theorem. Finally, in Section 5, we carry out a sensitivity analysis to study the impact of the market parameters on the optimal strategies.

2. Formulation of the model

Throughout this paper, $\mathbb{N}$ denotes the family of natural numbers, $\mathbb{R}_+$ denotes the family of nonnegative real numbers and $\mathbb{R}$ denotes the family of real numbers. We fix $T > 0$ to be the time horizon. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, namely, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. This space also supports a 2-dimensional Brownian motion $(W_t, \widetilde{W}_t)_{t\in[0.T]}$, a series of positive-valued, independent and identically distributed random variables $\{Z_i\}_{i=1,2,\cdots}$ and a time homogeneous Poisson process $N(t)_{t\in[0.T]}$ with a positive constant intensity λ. Here we assume that W_t, $\widetilde{W}_t$, $\{Z_i\}_{i=1,2,\cdots}$ and $N(t)$ are mutually independent.

Following the classical risk theory, the claim (risk) process is given by a compound Poisson process, i.e., $\sum_{i=1}^{N(t)} Z_i$. Here $N(t)$ represents the number of claims up to time t, and Z_i represents the size of the ith claim. Denote by C_t the premium income up to time t. Then, the insurer's surplus process (wealth process) $(X_t)_{t\in[0,T]}$ is given by $X_t = x + C_t - \sum_{i=1}^{N(t)} Z_i$, where $x > 0$ is the initial wealth.

In this paper, we also assume that the insurer is allowed to purchase reinsurance, which is a business contract. Specifically, an insurer purchases insurance from a reinsurer against any losses that result from claims that are made against it. Suppose that the expected value principle is adopted to calculate the insurance (reinsurance) premium (see [17]), we then obtain that the insurance (reinsurance) premium P for one claim is given by $P(Z) = (1+\delta)\mathbb{E}(Z)$, where $\delta > 0$ is safety loading associated with the expectation of risk with respect to the insurer (reinsurer); $Z > 0$ is a random variable and represents one claim. Thus, the total premium income up to time t can be expressed as

$$C_t = (1+\delta)\mathbb{E}\left(\sum_{i=1}^{N(t)} Z_i\right) = (1+\delta)\lambda\theta t,$$

where $\theta = \mathbb{E}(Z_1) > 0$. Let $\bar{a}_t \in [0, 1]$ for all $t \in [0, T]$. We denote $1 - \bar{a}_t$ as the ceded proportion of risk at time t. Then for each claim Z_i, the insurer's risk retains $\bar{a}_t Z_i$ and the remaining risk $(1 - \bar{a}_t)Z_i$ is ceded to the reinsurer. According to reinsurance contract, the total reinsurance premium up to time t can be describe as

$$\bar{C}_t = (1+\delta)\mathbb{E}\left(\sum_{i=1}^{N(t)} (1-\bar{a}_t)Z_i\right) = (1+\delta)(1-\bar{a}_t)\lambda\theta t.$$

Then the insurer's surplus process can be rewritten as

$$X_t = x + C_t - \bar{C}_t - \sum_{i=1}^{N(t)} \bar{a}_t Z_i = x + (1+\delta)\lambda\theta\bar{a}_t t - \sum_{i=1}^{N(t)} \bar{a}_t Z_i.$$

Since a Lévy process can be represented by a compound Poisson process if and only if its Lévy measure is finite (see, e.g., [14]), we can find that there exists a Poisson random measure $N(dt,dz)$ on $\Omega\times[0,T]\times(0,\infty)$ such that

$$\sum_{i=1}^{N(t)} Z_i = \int_0^t\int_0^\infty zN(ds,dz), \quad \forall t\in[0,T].$$

So the insurer's surplus process $(X_t)_{t\in[0,T]}$ under an $\{\mathcal{F}_t\}$-predictable retention level, $(\bar a_t)_{t\in[0,T]}$, becomes

$$X_t = x + (1+\delta)\lambda\theta\bar a_t t - \int_0^t\int_0^\infty \bar a_t zN(ds,dz). \tag{2.1}$$

Additionally, there are two primitive assets available to investment in the financial market. One of the assets is a risk-free asset with price process $(B_t)_{t\in[0,T]}$ and the other is a risky asset with price process $(S_t)_{t\in[0,T]}$. The dynamics of $(B_t)_{t\in[0,T]}$ and $(S_t)_{t\in[0,T]}$ are given by

$$dB_t = r(Y_t)B_t dt, \quad B_0 = 1 \tag{2.2}$$

and

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t, \quad S_0 > 0, \tag{2.3}$$

where $r(\cdot)>0$ is a bounded C^1-function, $\mu(\cdot)$ and $\sigma(\cdot)>0$ are C^1-functions, Y_t is referred to as the stochastic economic factor governed by

$$dY_t = \beta(Y_t)dt + \alpha(Y_t)\left(\rho dW_t + \sqrt{1-\rho^2}d\widetilde{W}_t\right), \quad Y_0 = y\in\mathbb{R}. \tag{2.4}$$

Here $\beta(\cdot)$ is Lipschitz continuous, $\alpha(\cdot)>0$ is bounded Lipschitz continuous, $\rho\in(-1,1)$. Thus, from classical results (see [13]) it follows that for any initial condition $y\in\mathbb{R}$, there exists a unique strong solution. We further assume that $\frac{\mu(y)-r(y)}{\sigma(y)}>0$ is bounded for all $y\in\mathbb{R}$.

Due to the introduction of the stochastic economic factor Y_t, we generalize the claim process as $\sum_{i=1}^{N(t)} g(Y_t,Z_i)$, where $g(y,z)>0$ is a C^1-function in $y\in\mathbb{R}$ such that $\mathbb{E}[g(Y_t,Z_1)]<\infty$, $\forall t\in[0,T]$. Let us observe that, since Y and N are independent, we get that

$$\mathbb{E}\left[\sum_{i=1}^{N(t)} g(Y_t,Z_i)\right] = \lambda t\mathbb{E}[g(Y_t,Z_1)].$$

Thus, under the expected value principle, Eq. (2.1) becomes

$$dX_t = (1+\delta)\bar a_t\lambda\theta(t)dt - \int_0^\infty \bar a_t g(Y_t,z)N(dt,dz),$$

where $\theta(t)=\mathbb{E}[g(Y_t,Z_1)]$. Since it is difficult for the insurer to evaluate $\theta(t)$ at any time $t\in[0,T]$, we replace $\theta(t)$ with θ. Thus, we have

$$dX_t = (1+\delta)\lambda\theta\bar a_t dt - \int_0^\infty \bar a_t g(Y_t,z)N(dt,dz).$$

Remark 2.1. This simplified premium calculation method is not only convenient for us to use the net profit condition (see Remark 3.2), but also simplifies the analysis of the solvability of HJB equation (See Theorem 4.1). In fact, if $g(y,z)\le z$ for $y\in\mathbb{R}$, it means that the risk of the insurer is reduced and the wealth is increased when the premium is calculated without considering the stochastic economic factor Y. According to (3.7), we know that the net profit condition is obviously true. On the contrary, if $g(y,z)>z$ for $y\in\mathbb{R}$, with our simplified method of calculating premiums, it will reduce wealth. Therefore, for the insurer, if the jump size of the claim $g(y,z)$ is relatively large, the wealth will be relatively small when applying our model, so the insurer needs to increase the premium correspondingly. This is also consistent with our intuition.

At time t, an insurer chooses $\bar{\pi}_t$, the amount of capital allocated in the risky asset. Denote by $(X_t^{\bar{\pi},\bar{a}})_{t\in[0,T]}$ the surplus process with a investment strategy $\bar{\pi} = (\bar{\pi}_t)_{t\in[0,T]}$ and an reinsurance strategy $\bar{a} = (\bar{a}_t)_{t\in[0,T]}$. To simplify our notation, we write $\bar{u} := (\bar{\pi}, \bar{a})$, which is called an investment-reinsurance strategy. Here $(\pi, a) = (\bar{\pi}_t, \bar{a}_t)_{t\in[0,T]}$. Correspondingly, we let $\bar{u}_t = (\bar{\pi}_t, \bar{a}_t)$. Then using the self-financing trading strategy, we have from (2.1) – (2.3) that the dynamics of the surplus process $(X_t^{\bar{u}})_{t\in[0,T]}$ is driven by the following Lévy SDE:

$$dX_t^{\bar{u}} = \Big[r(Y_t)X_t^{\bar{u}} + (\mu(Y_t) - r(Y_t))\bar{\pi}_t + (1+\delta)\lambda\theta\bar{a}_t\Big]\,dt + \bar{\pi}_t\sigma(Y_t)dW_t - \int_0^\infty \bar{a}_t g(Y_t,z)N(dt,dz). \tag{2.5}$$

For greater convenience, at time t, we denote π_t as the proportion of wealth invested in the risky asset and a_t as the ratio of the retained proportion of risk over surplus. Then we obtain $\pi_t = \frac{\bar{\pi}_t}{X_t^{\bar{u}}}$ and $a_t = \frac{\bar{a}_t}{X_{t-}^{\bar{u}}}$. Thus, for a control $u_t = (\pi_t, a_t)$, we have $\bar{u}_t = u_t X_t$. We then rewrite (2.5) as

$$\frac{dX_t^u}{X_{t-}^u} = [r(Y_t) + (\mu(Y_t) - r(Y_t))\pi_t + (1+\delta)\lambda\theta a_t]\,dt + \pi_t\sigma(Y_t)dW_t - \int_0^\infty a_t g(Y_t,z)N(dt,dz). \tag{2.6}$$

Applying the Doléans-Dade formula (see [10]), SDE (2.6) yields a unique solution X_t^u given by

$$\begin{aligned}
X_t^u &= X_0^u \exp\Bigg\{\int_0^t \Big[r(Y_s) + (\mu(Y_s) - r(Y_s))\pi_s + (1+\delta)\lambda\theta a_s - \int_0^\infty a_s g(Y_s,z)\nu(dz)\Big]\,ds \\
&\quad - \frac{1}{2}\int_0^t \pi_s^2\sigma^2(Y_s)ds + \int_0^t \pi_s\sigma(Y_s)dW_s - \int_0^t\int_0^\infty a_s g(Y_s,z)\widetilde{N}(ds,dz) \\
&\quad + \int_0^t\int_0^\infty [\log(1 - a_s g(Y_s,z)) + a_s g(Y_s,z)]\,N(ds,dz)\Bigg\} \\
&= X_0^u \exp\Bigg\{\int_0^t [r(Y_s) + (\mu(Y_s) - r(Y_s))\pi_s + (1+\delta)\lambda\theta a_s]\,ds - \frac{1}{2}\int_0^t \pi_s^2\sigma^2(Y_s)ds \\
&\quad + \int_0^t \pi_s\sigma(Y_s)dW_s + \int_0^t\int_0^\infty \log(1 - a_s g(Y_s,z))\,N(ds,dz)\Bigg\},
\end{aligned}$$

where $\widetilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is the compensated Poisson random measure.

We now define the admissible control set.

Definition 2.2. Let $t \in [0,T]$ and $s \in [t,T]$. An investment-reinsurance strategy u_s is said to be admissible if the surplus process (2.6) associated with $\{\mathcal{F}_t\}$-predictable strategy (π_s, a_s) has a unique positive solution X_s^u while $X_t^u = x > 0$. Denote by $\mathcal{A}$ the set of all admissible investment-reinsurance strategies.

To avoid the possibility of bankruptcy at jumps, that is $X_t^u > 0$ for all $t \geq 0$, we need to assume that $1 - a_t g(y,z) > 0$ for all $y \in \mathbb{R}$ and $z > 0$ if $u \in \mathcal{A}$. In addition, we assume that the utility function is given by power utility, i.e., $U(x) = \frac{1}{\gamma}x^\gamma$, where $\gamma \in (0,1)$ is the risk-aversion parameter. The power utility function belongs to the class of hyperbolic absolute risk aversion (HARA) utility functions. Then the value function is given by

$$V(t,x,y) := \sup_{u\in\mathcal{A}} \mathbb{E}\,[U(X_T^u) | X_t^u = x, Y_t = y] \tag{2.7}$$

for $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$.

3. Dynamic programming and optimal strategies

In this section, we limit the strategies (π, a) in the admissible control set $\mathcal{A}$. Our aim is to find an admissible control $u^* = (\pi^*, a^*) \in \mathcal{A}$ that attains the value function $V(t,x,y)$. The control u^* is called an optimal control. In our case, it suffices to consider Markov control (see Theorem 11.2.3 in [13]). We first derive the HJB equation.

For $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$, we assume that $V(t,x,y)$ is $C^{1,2,2}$. Using the dynamic programming principle, the HJB equation can be formulated as follows:

$$\begin{aligned} 0 =& \frac{\partial V(t,x,y)}{\partial t} + \beta(y)\frac{\partial V(t,x,y)}{\partial y} + \frac{1}{2}\alpha^2(y)\frac{\partial^2 V(t,x,y)}{\partial y^2} \\ &+ \sup_{(\pi,a)\in\mathbb{R}\times\mathbb{R}_+}\left\{\frac{\partial V(t,x,y)}{\partial x}x[r(y) + (\mu(y) - r(y))\pi + (1+\delta)\lambda\theta a]\right. \\ &+ \rho\frac{\partial^2 V(t,x,y)}{\partial x\partial y}x\pi\alpha(y)\sigma(y) + \frac{1}{2}\frac{\partial^2 V(t,x,y)}{\partial x^2}x^2\pi^2\sigma^2(y) \\ &\left. + \int_0^\infty [V(t,x(1-ag(y,z)),y) - V(t,x,y)]\,\nu(dz)\right\} \end{aligned} \tag{3.1}$$

with terminal condition $V(T,x,y) = U(x)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$.

Since the utility function is power utility, we conjecture that $V(t,x,y)$ has the form $V(t,x,y) = U(x)B(t,y)$, where $B(t,y)$ is a positive $C^{1,2}$-function. Direct substitution in the HJB equation (3.1) yields that

$$0 = \frac{\partial B(t,y)}{\partial t} + \beta(y)\frac{\partial B(t,y)}{\partial y} + \frac{1}{2}\alpha^2(y)\frac{\partial^2 B(t,y)}{\partial y^2} + B(t,y)\sup_{(\pi,a)\in\mathbb{R}\times\mathbb{R}_+}\mathcal{H}(\pi,a;y,\xi) \tag{3.2}$$

with $B(T,y) = 1$ for all $y \in \mathbb{R}$. Here

$$\xi = \frac{\partial B(t,y)}{\partial y}\frac{1}{B(t,y)},$$

and the Hamiltonian $\mathcal{H}(\pi,a;y,\xi) : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as

$$\begin{aligned} \mathcal{H}(\pi,a;y,\xi) :=& \gamma[r(y) + (\mu(y) - r(y))\pi + (1+\delta)\lambda\theta a] + \gamma\rho\pi\alpha(y)\sigma(y)\xi \\ &+ \frac{1}{2}\gamma(\gamma-1)\pi^2\sigma^2(y) + \int_0^\infty [(1-ag(y,z))^\gamma - 1]\nu(dz). \end{aligned}$$

The first-order condition is then given by

$$\frac{\partial\mathcal{H}(\pi,a;y,\xi)}{\partial\pi} = \gamma(\mu(y) - r(y)) + \gamma\rho\alpha(y)\sigma(y)\xi + \gamma(\gamma-1)\pi\sigma^2(y) = 0 \tag{3.3}$$

and

$$\frac{\partial\mathcal{H}(\pi,a;y,\xi)}{\partial a} = \gamma(1+\delta)\lambda\theta - \gamma\int_0^\infty [1-ag(y,z)]^{\gamma-1}g(y,z)\nu(dz) = 0. \tag{3.4}$$

Hence, we derive that a candidate for π^* from Eq. (3.3) as

$$\pi^*(y,\xi) = \frac{\mu(y) - r(y)}{(1-\gamma)\sigma^2(y)} + \frac{\rho\alpha(y)\xi}{(1-\gamma)\sigma(y)}. \tag{3.5}$$

By Eq. (3.4), we have

$$G(a;y) := (1+\delta)\lambda\theta - \int_0^\infty (1-ag(y,z))^{\gamma-1}g(y,z)\nu(dz) = 0. \tag{3.6}$$

Next we may reveal the properties of the solution of Eq. (3.6).

Lemma 3.1. *For each $y \in \mathbb{R}$, there exists a unique nonnegative solution $a^* = a^*(y)$ satisfying $1 - a^*g(y,z) > 0$ of Eq. (3.6) if and only if*

$$(1+\delta)\lambda\theta - \int_0^\infty g(y,z)\nu(dz) \geq 0. \tag{3.7}$$

In addition, the solution $a^* = a^*(y)$ is C^1 on $y \in \mathbb{R}$.

Proof. For $y \in \mathbb{R}$ fixed, it is easy to see that $G(a;y)$ is C^1 on $a \in \mathbb{R}_+$. Since $0 < \gamma < 1$, we observe that $G(a;y)$ is a decreasing function on $a \in \mathbb{R}_+$. It is not difficult to find that $\lim_{a\uparrow\frac{1}{g(y,z)}} \int_0^\infty (1 - ag(y,z))^{\gamma-1} g(y,z)\nu(dz) = \infty$. This implies that $\lim_{a\uparrow\frac{1}{g(y,z)}} G(a;y) = -\infty$. But $\lim_{a\downarrow 0} G(a;y) = (1+\delta)\lambda\theta - \int_0^\infty g(y,z)\nu(dz)$. The argument above indicates that there is a unique nonnegative solution a^* lying in $[0, \frac{1}{g(y,z)})$ of Eq. (3.6) if and only if Eq. (3.7) holds. This also means that $1 - a^*g(y,z) > 0$ holds. In addition, by the implicit function theorem and $G(a;y)$ is C^1 on $a \in [0, \frac{1}{g(y,z)})$ for $y \in \mathbb{R}$, $a^* = a^*(y)$ is C^1 on $y \in \mathbb{R}$ follows immediately. The proof is therefore complete. □

Remark 3.2. Inequality (3.7) implies the so called net-profit condition (see, for example, [3]), which is usually assumed in insurance risk models to ensure that the expected total risk premium covers the expected losses. In fact, under (3.7) we have that

$$\begin{aligned}(1+\delta)\lambda\theta t &\geq \mathbb{E}\left[\int_0^t \int_0^\infty g(Y_s,z)\nu(dz)ds\right] \\ &= \mathbb{E}\left[\int_0^t \int_0^\infty g(Y_s,z)N(ds,dz)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{N(t)} g(Y_t,Z_i)\right].\end{aligned}$$

In the light of Lemma 3.1, the solutions to Eq.s (3.3) and (3.4) are given by

$$\begin{cases}\pi^*(y,\xi) = \frac{\mu(y)-r(y)}{(1-\gamma)\sigma^2(y)} + \frac{\rho\alpha(y)\xi}{(1-\gamma)\sigma(y)}, \\ a^* = a^*(y) \text{ is the unique nonnegative solution obtained in Lemma 3.1.}\end{cases} \tag{3.8}$$

It is not hard to verify that the Hessian matrix of $\mathcal{H}(\pi, a; y, \xi)$ is negative definite. So, $(\pi^*(y,\xi), a^*(y))$ is a candidate to be the optimal strategy.

4. Solvability of HJB equation and verification theorem

In this section, we will prove the solvability of HJB equation (3.2) and the corresponding verification theorem.

If we can show that HJB equation (3.2) admits a unique classical solution with the terminal condition $B(T,y) = 1$ for all $y \in \mathbb{R}$, then we can conclude that $V(t,x,y) = U(x)B(t,y)$ is the unique classical solution of Eq. (3.1) with terminal condition $V(T,x,y) = U(x)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$. Substituting (3.8) into (3.2), we have

$$\begin{aligned}0 = \frac{\partial B(t,y)}{\partial t} &+ \left[\beta(y) + \frac{\gamma\rho\alpha(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)}\right]\frac{\partial B(t,y)}{\partial y} + \frac{1}{2}\alpha^2(y)\frac{\partial^2 B(t,y)}{\partial y^2} \\ &+ \frac{\gamma\rho^2\alpha^2(y)}{2(1-\gamma)}\left(\frac{\partial B(t,y)}{\partial y}\right)^2 \frac{1}{B(t,y)} + \gamma B(t,y)\Bigg[r(y) + \frac{(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} \\ &+ (1+\delta)\lambda\theta a^*\Bigg] + B(t,y)\int_0^\infty [(1-a^*g(y,z))^\gamma - 1]\nu(dz),\end{aligned} \tag{4.1}$$

the terminal condition is still $B(T,y) = 1$ for all $y \in \mathbb{R}$. Then we have the following theorem.

Theorem 4.1. *Under the net-profit condition (3.7), for $(t, y) \in [0, T] \times \mathbb{R}$, the updated HJB equation (4.1) has a unique positive and bounded classical solution.*

Proof. We use the method of [21] to transform $B(t, y)$. We let $B(t, y) = E^m(t, y)$ for a parameter m to be determined. We may easily observe that

$$0 = mE^{m-1}(t,y)\frac{\partial E(t,y)}{\partial t} + m\varphi(y)E^{m-1}(t,y)\frac{\partial E(t,y)}{\partial y} + \frac{1}{2}m\alpha^2(y)E^{m-1}(t,y)\frac{\partial^2 E(t,y)}{\partial y^2} + \alpha^2(y)\left[\frac{m(m-1)}{2} + \frac{\gamma\rho^2 m^2}{2(1-\gamma)}\right]E^{m-2}(t,y)\left(\frac{\partial E(t,y)}{\partial y}\right)^2 + \tau(y)E^m(t,y),$$

where

$$\varphi(y) := \beta(y) + \frac{\gamma\rho\alpha(y)(\mu(y)-r(y))}{(1-\gamma)\sigma(y)},$$

$$\tau(y) := \gamma\left[r(y) + \frac{(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + (1+\delta)\lambda\theta a^*\right] + \int_0^\infty [(1-a^* g(y,z))^\gamma - 1]\nu(dz).$$

By Lemma 3.1, we have $0 \le a^* < \frac{1}{g(y,z)}$. Recall that $\frac{\mu(y)-r(y)}{\sigma(y)} > 0$ is bounded. We then easily see that

$$\begin{aligned}\tau(y) &\le \gamma\left[r(y) + \frac{(\mu(y)-r(y))^2}{2(1-\gamma)\sigma^2(y)} + (1+\delta)\lambda\theta a^*\right]\\ &\le C\end{aligned}$$

for $y \in \mathbb{R}$, where $C > 0$ is a constant. If we choose $m = \frac{1-\gamma}{1-\gamma+\gamma\rho^2} \in (0, 1]$, then we have

$$0 = \frac{\partial E(t,y)}{\partial t} + \varphi(y)\frac{\partial E(t,y)}{\partial y} + \frac{1}{2}\alpha^2(y)\frac{\partial^2 E(t,y)}{\partial y^2} + \frac{\tau(y)}{m}E(t,y) \tag{4.2}$$

with the terminal condition $E(T, y) = 1$ for all $y \in \mathbb{R}$. Therefore, Eq. (4.1) becomes a linear parabolic differential equation. We next prove that Eq. (4.2) has a unique bounded classical solution. For this purpose, we should apply Girsanov's Theorem (see, e.g., [13]). Consider the following SDE:

$$d\widetilde{Y}_t = \varphi(\widetilde{Y}_t)dt + \alpha(\widetilde{Y}_t)\left(\rho dW_t + \sqrt{1-\rho^2}d\widetilde{W}_t\right), \quad \widetilde{Y}_0 = y \in \mathbb{R}, \tag{4.3}$$

where $\varphi(y)$ is defined as before. Let

$$q(Y_t) = \frac{\gamma\rho(\mu(Y_t) - r(Y_t))}{(\gamma-1)\sigma(Y_t)}$$

and

$$M(t) = \exp\left\{-\int_0^t q(Y_s)\left(\rho dW_s + \sqrt{1-\rho^2}d\widetilde{W}_s\right) - \frac{1}{2}\int_0^t q^2(Y_s)ds\right\}.$$

Define the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ and the process $W_t^{\mathbb{Q}}$ by $d\mathbb{Q} = M(T)d\mathbb{P}$ and $dW_t^{\mathbb{Q}} = q(t)dt + \rho dW_t + \sqrt{1-\rho^2}d\widetilde{W}_t$. Then $W_t^{\mathbb{Q}}$ is a Brownian motion with respect to $W_t^{\mathbb{Q}}$. Since $\frac{\mu(y)-r(y)}{\sigma(y)}$ is bounded for all $y \in \mathbb{R}$, $\mathbb{Q} \sim \mathbb{P}$ is well defined. Therefore, by Girsanov's Theorem, in terms of $W_t^{\mathbb{Q}}$ SDE (2.4) can easily be rewritten as

$$dY_t = \varphi(Y_t)dt + \alpha(Y_t)dW_t^{\mathbb{Q}}, \quad Y_0 = y \in \mathbb{R}. \tag{4.4}$$

This implies that SDE (4.4) has a unique solution under $\mathbb{Q}$. So we have proved that SDE (4.3) has a unique solution under $\mathbb{P}$.

Let $D_n = (-n, n)$ for $n \in \mathbb{N}$. Clearly, $E(T, y) = 1$ is C^2 and bounded for all $y \in \mathbb{R}$. It is not hard to verify that the coefficients of SDE (4.2) satisfy the conditions in Proposition 2.3 of [1]. Furthermore, for $(t, y) \in [0, T] \times \mathbb{R}$, by the Feynman–Kac formula, we have

$$E(t, y) = \mathbb{E}\left[e^{\int_t^T \frac{\tau(\widetilde{Y}_s)}{m} ds} | \widetilde{Y}_t = y\right].$$

So $E(t, y)$ is positive and bounded. Then SDE (4.2) has a unique positive and bounded classical solution.

Since $B(t, y) = E^m(t, y)$, $m = \frac{1-\gamma}{1-\gamma+\gamma\rho^2} \in (0, 1]$, we know that $B(t, y)$ is the unique positive and bounded classical solution to the updated HJB equation (4.1). Thus the proof is complete. □

We now state the following verification theorem.

Theorem 4.2. *Under the net-profit condition (3.7), for $(t, x, y) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}$, the value function satisfies*

$$V(t, x, y) = \frac{1}{\gamma} x^\gamma \left\{\mathbb{E}\left[e^{\frac{1-\gamma+\gamma\rho^2}{1-\gamma} \int_t^T \tau(\widetilde{Y}_s) ds} | \widetilde{Y}_t = y\right]\right\}^{\frac{1-\gamma}{1-\gamma+\gamma\rho^2}}, \tag{4.5}$$

where $\widetilde{Y}_t$, $0 \le t \le T$ is the unique strong solution to SDE (4.3). Moreover, for $y \in \mathbb{R}$ and $z > 0$, the optimal strategies are given by

$$\pi_t^* = \pi^*\left(Y_t, \frac{\partial B(t, Y_t)}{\partial y} \frac{1}{B(t, Y_t)}\right), \quad a_t^* = a^*(Y_t). \tag{4.6}$$

Here $(\pi^*(y, \xi), a^*)$ is given by (3.8) and Y_t, $0 \le t \le T$ is the unique strong solution to SDE (2.4).

Proof. Applying Theorem 3.2 of [21], one easily concludes that the value function $V(t, x, y)$ is given by $\frac{1}{\gamma} x^\gamma E^m(t, y)$. Since $m = \frac{1-\gamma}{1-\gamma+\gamma\rho^2}$, we may easily get (4.5). Using the fact that $\frac{\mu(y)-r(y)}{\sigma(y)}$ and $\alpha(y)$ are bounded, $\sigma(y)$ is locally bounded and Lemma 3.1, it follows that $\pi^*\left(y, \frac{\partial B(t,y)}{\partial y} \frac{1}{B(t,y)}\right)$ and $a^*(y)$ are locally bounded. So we have (4.6) is admissible by Definition 2.2. Clearly, (4.6) is the optimum. Therefore, the proof is completed. □

5. Sensitivity analysis

In this section, we analyze the impact of some important parameters on optimal investment-reinsurance strategies.

To conduct the sensitivity analysis, we proposed the model as follows. The stochastic economic factor model is chosen to be of the mean-reverting Ornstein–Uhlenbeck process

$$dY_t = (m - Y_t)dt + \alpha\left(\rho dW_t + \sqrt{1 - \rho^2} d\widetilde{W}_t\right), \quad Y_0 = y \in \mathbb{R},$$

where $m \in \mathbb{R}$ and $\alpha > 0$ are constants. Thus there exists a unique solution Y_t. We assume that the riskless bond and the price processes of the risky asset are given by $B_t = e^{rt}$ and

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_t, \quad S_0 > 0.$$

Here $r > 0$ and μ are constants, the volatility $\sigma(\cdot) > 0$ is C^1-function. Then the surplus process $(X_t^u)_{t\in[0,T]}$ is given by

$$\frac{dX_t^u}{X_{t-}^u} = [r + (\mu - r)\pi_t + (1 + \delta)\lambda\theta a_t]\, dt + \pi_t \sigma(Y_t) dW_t - \int_0^\infty a_t g(Y_t, z) N(dt, dz).$$

We assume that $\nu(dz) = j\delta_1(dz)$, where $j > 0$, δ_x is the Dirac measure concentrated at x. For the volatility function, we choose the uniformly elliptic Scott volatility; see for instance [15]. In other words, $\sigma(y) = \sqrt{\varepsilon + e^{\varsigma y}}$ for $\varepsilon, \varsigma > 0$. Clearly, $\frac{|\mu-r|}{\sqrt{\varepsilon+e^{\varsigma y}}} < \frac{|\mu-r|}{\sqrt{\varepsilon}}$. By

Eq. (3.6), we have $(1+\delta)\lambda\theta - (1-a^*(y)g(y,1))^{\gamma-1} jg(y,1) = 0$. Then the solution is given by

$$a^*(y) = \frac{1}{g(y,1)}\left\{1 - \left[\frac{(1+\delta)\lambda\theta}{jg(y,1)}\right]^{\frac{1}{\gamma-1}}\right\}. \tag{5.1}$$

In addition, the net-profit condition (3.7) implies that $jg(y,1) \le (1+\delta)\lambda\theta$ for $y \in \mathbb{R}$. Recall that the optimal strategy $(\pi^*(y,\xi), a^*(y))$ given by (3.8), then for $(y,\xi) \in \mathbb{R}\times\mathbb{R}$,

$$\pi^*(y,\xi) = \frac{\mu(y) - r(y)}{(1-\gamma)(\varepsilon + e^{\varsigma y})} + \frac{\rho\alpha(y)\xi}{(1-\gamma)\sqrt{\varepsilon + e^{\varsigma y}}}. \tag{5.2}$$

Therefore, for $0 \le t \le T$, the optimal strategies are given by

$$\pi_t^* = \pi^*\left(Y_t, \frac{\partial B(t,Y_t)}{\partial y}\frac{1}{B(t,Y_t)}\right), \quad a_t^* = a^*(Y_t).$$

Throughout this section, unless otherwise stated, the market parameters are given as follows: $r = 0.2$, $\lambda = 1$, $\delta = 0.2$, $\theta = 0.5$, $\mu = 0.6$, $\alpha = 0.01$, $\varepsilon = 0.2$, $\varsigma = 0.02$. Graphs for the impact of y, ρ, j and γ on the optimal strategies are presented in Fig. 1.

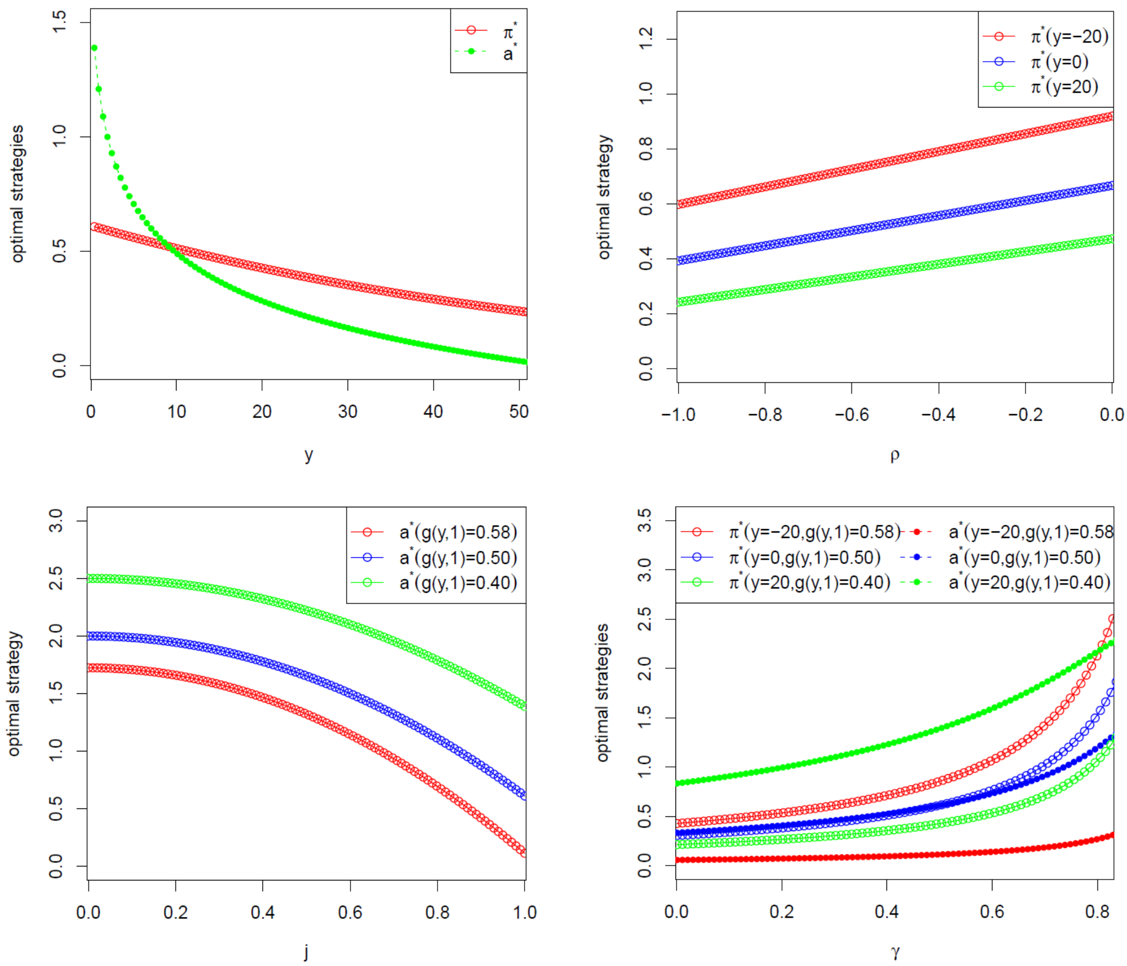

Figure 1. Impact of y, ρ, j and γ on the optimal strategies.

First, we study the impact of y, the stochastic economic factor, on the optimal strategy. We fix $\gamma = 0.5$, $\rho = -0.2$, $j = 1$ and $\xi = 15$. Based on the left top figure in Fig. 1, which plots the optimal strategies with respect to the stochastic economic factor y, we observe that the optimal investment proportion π^* in the risky asset is a decreasing function of y. This implies that when the stochastic economic factor y becomes larger, the optimal investment proportion in the risky asset would be smaller. The figure also shows that the optimal retained proportion of risk over surplus a^* is a decreasing function of y, but the decreasing speed is faster than π^*. We use the volatility function to explain the behavior of the optimal investment proportion. Since the volatility $\sigma(y) = \sqrt{\varepsilon + e^{\varsigma y}}$ is increasing in y, it means that when y becomes larger, $\sigma(y)$ would also be larger. Hence, by (5.2), when y increases, an insure reduces his/her investment proportion in the risky asset.

The graph of optimal strategy π^* with respect to different values of ρ in the right top figure in Fig. 1 is considered. For this analysis, we fix the stochastic economic factor $y = 0, \pm 20$. In terms of π^* in (5.2), we know that π^* is an increasing function of ρ. This is consistent with the graphic line. This behavior is also supported by the economic interpretation of ρ. When ρ takes values near zero it means that the price processes and the stochastic economic factor are almost uncorrelated. In other words, when ρ takes values near zero, there is little uncertainty in the financial market. Hence, π^* takes a maximum value in that region.

We further analyze the impact of the jump's size j on the optimal strategy a^*. For this analysis, we fix $y = 20$. The left bottom figure in Fig. 1 shows that the optimal retained proportion of risk over surplus a^* is a decreasing function of j. It can be explained that when the jump's size j becomes larger, the risk exposure to the insurer would be larger. It is natural for the insurer to increase the ceded proportion of risk. Therefore, the optimal retained proportion of risk over surplus a^* would be small.

The last variable we analyze is γ, the insurer's risk aversion. The corresponding optimal strategies are presented in the right bottom figure in Fig. 1. Notice that $0 < \gamma < 1$. In this case, the insurer with greater γ are more risk seeking. When γ increases, it is natural for an insurer to invest more money in the risky asset. Hence π^* increases as well. This can also explain the optimal retained proportion of risk over surplus a^*. Because an insurer invests more money in the risky asset, the risk is naturally greater.

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