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On Triple Difference Sequences of Real Numbers in Neutrosophic Normed Spaces

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Abstract
The aim of this article is to investigate triple Δ-statistical convergent sequences in a neutrosophic normed space (NNS). Also, we examine the notions of Δ-statistical limit points and Δ-statistical cluster points and prove their important features.

Keywords: Difference sequence, Lacunary sequence, Neutrosophic normed space, Triple sequence
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1. Introduction

The initial work on fuzzy sets was established by Zadeh [1]. Then, several authors have advanced the theory of fuzzy set. Park [2] defined intuitionistic fuzzy metric space and also intuitionistic fuzzy normed space was examined by Lael and Nourouzi [3]. Some beneficial results on this topic can be found in [4]-[7].

The neutrosophic set (NS) was worked by F. Smarandache [8] who defined the degree of indeterminacy (i) as indepedent component. In [9], neutrosophic logic was firstly examined. It is a logic where each proposition is identified to have a degree of truth (T), falsity (F), and indeterminacy (I). A Neutrosophic set (NS) is specified as a set where each component of the universe has a degree of T, F and I. Kirişçi and Şimşek [10] discussed neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. The theory of NNS and statistical convergence in NNS were first developed by Kirişci and Şimşek [11]. Neutrosophic set and neutrosophic logic has utilized by applied sciences and theoretical science for instance summability theory, decision making, robotics. Some remarkable results on this topic can be reviewed in [12]-[15].

The concept of statistical convergence was investigated under the name almost convergence by Zygmund [16]. It was formally introduced by Fast [17]. Later the idea was associated with summability theory by Fridy [18], and many others (see [19]-[22]). The studies of triple sequences have seen rapid growth. The initial work on the statistical convergence of triple sequences was establised by Şahiner et al. [23] and the other researches continued by [24, 25]. The idea of difference sequences was given by Kızmaz [26] where $\Delta x = (\Delta x_k) = x_k - x_{k+1}$. Başarır [27] investigated the Δ -statistical convergence of sequences. Also, the generalized difference sequence spaces were worked by various authors [28]-[30].

Since sequence convergence plays a very significant role in the fundamental theory of mathematics, there are many convergence notions in summability theory, in approximation theory, in classical measure theory, in probability theory, and the relationships between them are discussed. The interested reader may consult Hazarika et al. [31], the monographs [32] and [33] for the background on the sequence spaces and related topics. Inspired by this, in this study, a further investigation into the mathematical features of triple sequences will be thought. Section 2 recalls some definitions in summability theory and NNS.

In Section 3, we study triple Δ-statistical convergent sequences in a NNS. Also, we examine the notions of Δ-statistical limit point and Δ-statistical cluster point and prove their important features.

2. Definitions and preliminaries

Now, we remember essential definitions required in this study.

Triangular norms (t-norms) (TN) were considered by Menger [34]. Triangular conorms (t-conorms) (TC) recognized as dual operations of TNs. TNs and TCs are significant for fuzzy operations.

Definition 2.1. *([34]) Let* $* : [0,1]\times[0,1]\to[0,1]$ *be an operation. If* $*$ *provides subsequent cases, it is named continuous TN. Take* $a,b,c,d\in[0,1]$,

(a) $a*1=a$,
(b) *If* $a\le c$ *and* $b\le d$, *then* $a*b\le c*d$,
(c) $*$ *is continuous,*
(d) $*$ *associative and commutative.*

Definition 2.2. *([34]) Let* $\diamondsuit : [0,1]\times[0,1]\to[0,1]$ *be an operation. If* $\diamondsuit$ *provides subsequent cases, it is named to be continuous TC.*

(a) $a\diamondsuit 0=a$,
(b) *If* $a\le c$ *and* $b\le d$, *then* $a\diamondsuit b\le c\diamondsuit d$,
(c) $\diamondsuit$ *is continuous,*
(d) $\diamondsuit$ *associative and commutative.*

Definition 2.3. *([11]) Let* F *be a vector space,* $\mathscr{N}=\{\langle\varpi,\mathscr{G}(\varpi),\mathscr{B}(\varpi),\mathscr{Y}(\varpi)\rangle:\varpi\in F\}$ *be a normed space (NS) such that* $\mathscr{N}:F\times\mathbb{R}^+\to[0,1]$. *While subsequent situations hold,* $V=(F,\mathscr{N},*,\diamondsuit)$ *is called to be NNS. For each* $\varpi,\kappa\in F$ *and* $\lambda,\mu>0$ *and for all* $\sigma\neq0$,

(a) $0\le\mathscr{G}(\varpi,\lambda)\le1,\ 0\le\mathscr{B}(\varpi,\lambda)\le1,\ 0\le\mathscr{Y}(\varpi,\lambda)\le1\ \forall\lambda\in\mathbb{R}^+$,
(b) $\mathscr{G}(\varpi,\lambda)+\mathscr{B}(\varpi,\lambda)+\mathscr{Y}(\varpi,\lambda)\le3$ (*for* $\lambda\in\mathbb{R}^+$),
(c) $\mathscr{G}(\varpi,\lambda)=1$ (*for* $\lambda>0$) *iff* $\varpi=0$,
(d) $\mathscr{G}(\sigma\varpi,\lambda)=\mathscr{G}\left(\varpi,\frac{\lambda}{|\sigma|}\right)$,
(e) $\mathscr{G}(\varpi,\mu)*\mathscr{G}(\kappa,\lambda)\le\mathscr{G}(\varpi+\kappa,\mu+\lambda)$,
(f) $\mathscr{G}(\varpi,.)$ *is non-decreasing continuous function,*
(g) $\lim_{\lambda\to\infty}\mathscr{G}(\varpi,\lambda)=1$,
(h) $\mathscr{B}(\varpi,\lambda)=0$ (*for* $\lambda>0$) *iff* $\varpi=0$,
(i) $\mathscr{B}(\sigma\varpi,\lambda)=\mathscr{B}\left(\varpi,\frac{\lambda}{|\sigma|}\right)$,
(j) $\mathscr{B}(\varpi,\mu)\diamondsuit\mathscr{B}(\kappa,\lambda)\ge\mathscr{B}(\varpi+\kappa,\mu+\lambda)$,
(k) $\mathscr{B}(\varpi,.)$ *is non-decreasing continuous function,*
(l) $\lim_{\lambda\to\infty}\mathscr{B}(\varpi,\lambda)=0$,
(m) $\mathscr{Y}(\varpi,\lambda)=0$ (*for* $\lambda>0$) *iff* $\varpi=0$,
(n) $\mathscr{Y}(\sigma\varpi,\lambda)=\mathscr{Y}\left(\varpi,\frac{\lambda}{|\sigma|}\right)$,
(o) $\mathscr{Y}(\varpi,\mu)\diamondsuit\mathscr{Y}(\kappa,\lambda)\ge\mathscr{Y}(\varpi+\kappa,\mu+\lambda)$,
(p) $\mathscr{Y}(\varpi,.)$ *is non-decreasing continuous function,*
(r) $\lim_{\lambda\to\infty}\mathscr{Y}(\varpi,\lambda)=0$,
(s) *If* $\lambda\le0$, *then* $\mathscr{G}(\varpi,\lambda)=0,\mathscr{B}(\varpi,\lambda)=1$ *and* $\mathscr{Y}(\varpi,\lambda)=1$.
Then $\mathscr{N}=(\mathscr{G},\mathscr{B},\mathscr{Y})$ *is called Neutrosophic norm (NN).*

We recall the notions of convergence, statistical convergence, lacunary statistical convergence for single sequences in a NNS.

Definition 2.4. *([11]) Take* V *as an NNS. Let* $\varepsilon\in(0,1)$ *and* $\lambda>0$. *Then, a sequence* (x_k) *is converges to* $L\in F$ *iff there is* $N\in\mathbb{N}$ *such that* $\mathscr{G}(x_k-L,\lambda)>1-\varepsilon$, $\mathscr{B}(x_k-L,\lambda)<\varepsilon$, $\mathscr{Y}(x_k-L,\lambda)<\varepsilon$. *That is,*

$$\lim_{k\to\infty}\mathscr{G}(x_k-L,\lambda)=1,\ \lim_{k\to\infty}\mathscr{B}(x_k-L,\lambda)=0 \text{ and } \lim_{k\to\infty}\mathscr{Y}(x_k-L,\lambda)=0$$

as $\lambda>0$. *The convergent in NNS is signified by* $\mathscr{N}-\lim x_k=L$.

Definition 2.5. *([11]) A sequence (x_k) is named to be statistically convergent to $L \in F$ with regards to NN (SC-NN), provided that, for each $\lambda > 0$ and $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : \mathscr{G}(x_k - L, \lambda) \le 1-\varepsilon \text{ or } \mathscr{B}(x_k - L, \lambda) \ge \varepsilon,\ \mathscr{Y}(x_k - L, \lambda) \ge \varepsilon\}\right| = 0.$$

It is demonstrated by $S_{\mathscr{N}}$-$\lim x_k = L$.

Now we give the following notion.

Definition 2.6. *([23]) A subset K of $\mathbb{N}^3$ is said to have natural density $\delta_3(K)$ if*

$$\delta_3(K) = P - \lim_{n,l,k\to\infty} \frac{|K_{nlk}|}{nlk}$$

exists, where the vertical bars denote the number of (n,l,k) in K such that $p \le n$, $q \le l$, $r \le k$. Then, a real triple sequence $x = (x_{pqr})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : p \le n, q \le l, r \le k,\ \left|x_{pqr} - L\right| \ge \varepsilon\right\}\right) = 0.$$

3. Main results

Definition 3.1. *A triple sequence $w = (w_{nlk})$ in V is named to be Δ-convergent to $\zeta \in F$ with regards to (w.r.t in short) NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ on condition that for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is a positive integer k_0 such that*

$$\mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) > 1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) < \varepsilon$$

for every $n \ge k_0, l \ge k_0, k \ge k_0$ where $n,l,k \in \mathbb{N}$ and $\Delta w_{nlk} = w_{nlk} - w_{n,l+1,k} - w_{n,l,k+1} + w_{n,l+1,k+1} - w_{n+1,l,k} + w_{n+1,l+1,k} + w_{n+1,l,k+1} - w_{n+1,l+1,k+1}$. We indicate $(\mathscr{G},\mathscr{B},\mathscr{Y}) - \lim \Delta w = \zeta$ or $\Delta w \to \zeta\,((\mathscr{G},\mathscr{B},\mathscr{Y}))$ as $n,l,k \to \infty$.

Definition 3.2. *A triple sequence $w = (w_{nlk})$ is named to be Δ-Cauchy in V w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$, there are positive integers t_0, t_1, t_2 such that $\mathscr{G}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) > 1-\varepsilon$ and $\mathscr{B}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) < \varepsilon$, $\mathscr{Y}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) < \varepsilon$, whenever $n,p \ge t_0$, $l,q \ge t_1$, $k,r \ge t_2$.*

Definition 3.3. *A triple sequence $w = (w_{nlk})$ is named to be Δ-statistical convergent to ζ in V w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$,*

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \le 1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \ge \varepsilon\right\}\right) = 0.$$

In this case, we denote $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$.

Definition 3.4. *A triple sequence $w = (w_{nlk})$ is named to be Δ-statistically Cauchy in V w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ if for each $\varepsilon \in (0,1)$ and $\lambda > 0$, there are positive integers U, V and Y such that*

$$\delta_3\left(\left\{\begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) \le 1-\varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \Delta w_{pqr}, \lambda) \ge \varepsilon \end{array}\right\}\right) = 0$$

for all $n,p \ge U$, $l,q \ge V$, $k,r \ge Y$.

Lemma 3.5. *For each $\varepsilon \in (0,1)$ and $\lambda > 0$, the subsequent cases are equivalent.*

(a) $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$.

(b)

$$\delta_3\left(\left\{\begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \le 1-\varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) \ge \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) \ge \varepsilon \end{array}\right\}\right) = 0,$$

(c)

$$\delta_3\left(\left\{\begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) > 1-\varepsilon \text{ and} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon \end{array}\right\}\right) = 1,$$

(d)

$$\begin{array}{l} st^3_{N(\Delta)} - \lim \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) = 1 \text{ and} \\ st^3_{N(\Delta)} - \lim \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) = 0, \\ st^3_{N(\Delta)} - \lim \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) = 0. \end{array}$$

Proof. $(a) \Rightarrow (b)$ Presume that $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta$. Then, we get for each $\varepsilon \in (0,1)$ and $\lambda > 0$,

$$\delta_3 \left(\left\{ \begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \zeta, \lambda) \leq 1 - \varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \zeta, \lambda) \geq \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \zeta, \lambda) \geq \varepsilon \end{array} \right\} \right) = 0.$$

$(b) \Rightarrow (c)$ Take $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, we acquire

$$\begin{aligned} &\delta_3 \left(\left\{ \begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) > 1 - \varepsilon \text{ and} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) < \varepsilon \end{array} \right\} \right) \\ &= 1 - \delta_3 \left(\left\{ \begin{array}{c} (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1 - \varepsilon \text{ or} \\ \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) \geq \varepsilon,\ \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) \geq \varepsilon \end{array} \right\} \right) = 1. \end{aligned}$$

$(c) \Rightarrow (d)$ Take $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, we obtain

$$\begin{aligned} &\left\{ (n,l,k) \in \mathbb{N}^3 : |\mathscr{G}(\Delta w_{nlk} - \xi, \lambda) - 1| \geq \varepsilon \right\} \\ &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1 - \varepsilon \right\} \\ &\cup \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \geq 1 + \varepsilon \right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\delta_3 \left(\left\{ (n,l,k) \in \mathbb{N}^3 : |\mathscr{G}(\Delta w_{nlk} - \xi, \lambda) - 1| \geq \varepsilon \right\} \right) \\ &= \delta_3 \left(\left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1 - \varepsilon \right\} \right) \\ &+ \delta_3 \left(\left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \geq 1 + \varepsilon \right\} \right). \end{aligned}$$

Since,

$$\begin{aligned} &\delta_3 \left(\left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \leq 1 - \varepsilon \right\} \right) = 0 \text{ and} \\ &\delta_3 \left(\left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) \geq 1 + \varepsilon \right\} \right) = 0 \end{aligned}$$

we get

$$\delta_3 \left(\left\{ (n,l,k) \in \mathbb{N}^3 : |\mathscr{G}(\Delta w_{nlk} - \xi, \lambda) - 1| \geq \varepsilon \right\} \right) = 0.$$

So $st^3_{N(\Delta)} - \lim \mathscr{G}(\Delta w_{nlk} - \xi, \lambda) = 1$. Similarly, we obtain $st^3_{N(\Delta)} - \lim \mathscr{B}(\Delta w_{nlk} - \xi, \lambda) = 0$, $st^3_{N(\Delta)} - \lim \mathscr{Y}(\Delta w_{nlk} - \xi, \lambda) = 0$. □

Theorem 3.6. *If $w = (w_{nlk})$ is Δ-statistically convergent to ζ in V w.r.t NN $(\mathscr{G}, \mathscr{B}, \mathscr{Y})$, then $st^3_{N(\Delta)} - \lim w_{nlk}$ is determined unique.*

Proof. Let $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_1$ and $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_2$, where $\zeta_1 \neq \zeta_2$. For a given $\varepsilon \in (0,1)$ select $\nu \in (0,1)$ such that $(1-\nu) * (1-\nu) > 1 - \varepsilon$ and $\nu \Diamond \nu < \varepsilon$. For any $\lambda > 0$, we identify the subsequent sets:

$$\begin{aligned} F_{\mathscr{G},1}(\nu, \lambda) &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi_1, \lambda) \leq 1 - \nu \right\} \\ F_{\mathscr{G},2}(\nu, \lambda) &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk} - \xi_2, \lambda) \leq 1 - \nu \right\} \\ F_{\mathscr{B},1}(\nu, \lambda) &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{B}(\Delta w_{nlk} - \xi_1, \lambda) \geq \nu \right\} \\ F_{\mathscr{B},2}(\nu, \lambda) &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{B}(\Delta w_{nlk} - \xi_2, \lambda) \geq \nu \right\} \\ F_{\mathscr{Y},1}(\nu, \lambda) &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{Y}(\Delta w_{nlk} - \xi_1, \lambda) \geq \nu \right\} \\ F_{\mathscr{Y},2}(\nu, \lambda) &= \left\{ (n,l,k) \in \mathbb{N}^3 : \mathscr{Y}(\Delta w_{nlk} - \xi_2, \lambda) \geq \nu \right\} \end{aligned}$$

Since $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_1$, we say

$$\delta_3 \left(F_{\mathscr{G},1}(\nu, \lambda) \right) = \delta_3 \left(F_{\mathscr{B},1}(\nu, \lambda) \right) = \delta_3 \left(F_{\mathscr{Y},1}(\nu, \lambda) \right) = 0$$

for all $\lambda > 0$. In addition, utilizing $st^3_{N(\Delta)} - \lim w_{nlk} = \zeta_2$, we acquire

$$\delta_3 \left(F_{\mathscr{G},2}(\nu, \lambda) \right) = \delta_3 \left(F_{\mathscr{B},2}(\nu, \lambda) \right) = \delta_3 \left(F_{\mathscr{Y},2}(\nu, \lambda) \right) = 0$$

for all $\lambda > 0$.

Now, take

$$F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda) := \left(F_{\mathscr{G},1}(\nu,\lambda)\cup F_{\mathscr{G},2}(\nu,\lambda)\right)\cap\left(F_{\mathscr{B},1}(\nu,\lambda)\cup F_{\mathscr{B},2}(\nu,\lambda)\right)\cap\left(F_{\mathscr{Y},1}(\nu,\lambda)\cup F_{\mathscr{Y},2}(\nu,\lambda)\right).$$

Then, contemplate that $\delta_3\left(F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda)\right)=0$ that implies $\delta_3\left(\mathbb{N}^3\setminus F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda)\right)=1$. If $(n,l,k)\in\mathbb{N}^3\setminus F_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\nu,\lambda)$, then we get three possible situations. The former is the situation of $(n,l,k)\in\mathbb{N}^3\setminus\left(F_{\mathscr{G},1}(\nu,\lambda)\cup F_{\mathscr{G},2}(\nu,\lambda)\right)$, the second is $(n,l,k)\in\mathbb{N}^3\setminus\left(F_{\mathscr{B},1}(\nu,\lambda)\cup F_{\mathscr{B},2}(\nu,\lambda)\right)$ and the third is $(n,l,k)\in\mathbb{N}^3\setminus\left(F_{\mathscr{Y},1}(\nu,\lambda)\cup F_{\mathscr{Y},2}(\nu,\lambda)\right)$. First think that $(n,l,k)\in\mathbb{N}^3\setminus\left(F_{\mathscr{G},1}(\nu,\lambda)\cup F_{\mathscr{G},2}(\nu,\lambda)\right)$. Then, we acquire

$$\mathscr{G}(\xi_1-\xi_2,\lambda)\geq\mathscr{G}\left(\Delta w_{nlk}-\xi_1,\frac{\lambda}{2}\right)*\mathscr{G}\left(\Delta w_{nlk}-\xi_2,\frac{\lambda}{2}\right)>(1-\nu)*(1-\nu).$$

Since $(1-\nu)*(1-\nu)>1-\varepsilon$, we have $\mathscr{G}(\xi_1-\xi_1,\lambda)>1-\varepsilon$. Since $\varepsilon\in(0,1)$ was arbitrary, we get $\mathscr{G}(\xi_1-\xi_2,\lambda)=1$ for all $\lambda>0$ which means that $\xi_1=\xi_2$. At the same time, if $(n,l,k)\in\mathbb{N}^3\setminus\left(F_{\mathscr{B},1}(\nu,\lambda)\cup F_{\mathscr{B},2}(\nu,\lambda)\right)$, we can see

$$\mathscr{B}(\xi_1-\xi_2,\lambda)<\mathscr{B}\left(\Delta w_{nlk}-\xi_1,\frac{\lambda}{2}\right)\Diamond\mathscr{B}\left(\Delta w_{nlk}-\xi_2,\frac{\lambda}{2}\right)<\nu\Diamond\nu.$$

Since $\nu\Diamond\nu<\varepsilon$, we get $\mathscr{B}(\xi_1-\xi_2,\lambda)<\varepsilon$. Since $\varepsilon\in(0,1)$ was arbitrary, we acquire $\mathscr{B}(\xi_1-\xi_2,\lambda)=0$ for all $\lambda>0$ which means that $\xi_1=\xi_2$. If we observe the third case, we see that $\xi_1=\xi_2$. Hence, in all conditions, we obtain $st^3_{N(\Delta)}-\lim w_{nlk}$ is determined unique. □

Theorem 3.7. *Let $w=(w_{nlk})$ be a sequence in V. If $\Delta w\to\zeta\ ((\mathscr{G},\mathscr{B},\mathscr{Y}))$, then $st^3_{N(\Delta)}-\lim w=\zeta$.*

Proof. By supposition, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is a $k_0\in\mathbb{N}$ such that

$$\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon$$

for every $n\geq k_0, l\geq k_0, k\geq k_0$. This assurances that the set

$$\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon\right\}$$

has at most finitely many terms. Every finite subset of the $\mathbb{N}$ has density zero, so we acquire

$$\delta_3\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\geq\varepsilon\right\}\right)=0$$

gives the result. Hence, $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$. □

The subsequent example indicates that the converse of Theorem 3.7 is not valid.

Example 3.8. *Let $(F,\|.\|)$ be a NS. For each $a,b\in[0,1]$, select the TN $a*b=ab$ and the TC $a\Diamond b=\min\{a+b,1\}$. For every $w=(w_{nlk})\in F$ and each $\lambda>0$, we contemplate $\mathscr{G}(w,\lambda)=\frac{\lambda}{\lambda+\|w\|}$, $\mathscr{B}(w,\lambda)=\frac{\|w\|}{\lambda+\|w\|}$ and $\mathscr{Y}(w,\lambda)=\frac{\|w\|}{\lambda}$. Then V is an NNS. We identify a sequence (w_{nlk}) by*

$$w_{nlk}=\begin{cases}1, & n=k^2, l=v^2, k=t^2\ (k,v,t\in\mathbb{N})\\ 0, & \text{otherwise.}\end{cases}$$

Consider

$$A_{pqr}(\varepsilon,\lambda)=\left\{\begin{array}{c} n\leq p, l\leq q, k\leq r:\mathscr{G}(w_{nlk}-\xi,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(w_{nlk}-\xi,\lambda)\geq\varepsilon,\ \mathscr{Y}(w_{nlk}-\xi,\lambda)\geq\varepsilon\end{array}\right\}$$

for every $\varepsilon\in(0,1)$ and for any $\lambda>0$. Then we acquire

$$\begin{aligned}A_{pqr}(\varepsilon,\lambda)&=\left\{n\leq p,l\leq q,k\leq r:\tfrac{\lambda}{\lambda+\|w_{nlk}\|}\leq 1-\varepsilon \text{ or } \tfrac{\|w_{nlk}\|}{\lambda+\|w_{nlk}\|}\geq\varepsilon,\ \tfrac{\|w_{nlk}\|}{\lambda}\geq\varepsilon\right\}\\ &=\left\{n\leq p,l\leq q,k\leq r:\|w_{nlk}\|\geq\tfrac{\lambda\varepsilon}{1-\varepsilon},\text{ or } \|w_{nlk}\|\geq\lambda\varepsilon\right\}\\ &=\left\{n\leq p,l\leq q,k\leq r:\|w_{nlk}\|=1\right\}\\ &=\left\{n\leq p,l\leq q,k\leq r:n=k^2,l=v^2,k=t^2\ (k,v,t\in\mathbb{N})\right\}\end{aligned}$$

we get

$$\frac{1}{pqr}\left|A_{pqr}(\varepsilon,\lambda)\right| = \frac{1}{pqr}\left|\left\{n\le p, l\le q, k\le r : n=k^2, l=v^2, k=t^2\ (k,v,t\in\mathbb{N})\right\}\right| \le \frac{\sqrt{pqr}}{pqr}$$

which means that $\lim_{pqr\to\infty}\frac{1}{pqr}\left|A_{pqr}(\varepsilon,\lambda)\right| = 0$*. Hence, we have* $st^3_{N(\Delta)}-\lim w_{nlk}=0$*. However, the sequence* $w=(w_{nlk})$ *is not* Δ*-convergent in the space* $(F,\|.\|)$*.*

Theorem 3.9. *Take NNS as* V*. Then,* $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$ *iff there is a subset*

$$K=\left\{(n,l,k)\in\mathbb{N}^3 : n,l,k=1,2,3,...\right\}\subset\mathbb{N}^3$$

such that $\delta_3(K)=1$ *and* $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim_{(n,l,k)\in K, n,l,k\to\infty}\Delta w_{nlk}=\zeta$*.*

Proof. Presume that $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$. Then, for every $\lambda>0$ and $j\ge1$,

$$K(j,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\frac{1}{j} \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\frac{1}{j},\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\frac{1}{j}\right\}$$

and

$$M(j,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\le1-\frac{1}{j} \text{ or } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\ge\frac{1}{j},\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\ge\frac{1}{j}\right\}.$$

Then $\delta_3(M(j,\lambda))=0$ since

$$K(j,\lambda)\supset K(j+1,\lambda) \tag{3.1}$$

and

$$\delta_3(K(j,\lambda))=1 \tag{3.2}$$

for $\lambda>0$ and $j\ge1$. Now we need to show that for $(n,l,k)\in K(j,\lambda)$ the triple sequence $w=(w_{nlk})$ is Δ-convergent to $\zeta\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Suppose $w=(w_{nlk})$ be not Δ-convergent to $\zeta\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Therefore, there are $\beta>0$ and $k_0>0$ such that $\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\le1-\beta$ or $\mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\ge\beta$, $\mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\ge\beta$ for all $n\ge k_0, l\ge k_0, k\ge k_0$. Let $\beta>\frac{1}{j}$ and

$$K(\beta,\lambda)=\left\{(n,l,k)\in\mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\beta \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\beta,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\beta\right\}.$$

Then, we have $\delta_3(K(\beta,\lambda))=0$. Since $\beta>\frac{1}{j}$, by (3.1) we get $\delta_3(K(j,\lambda))=0$, which contradicts by (3.2). Therefore, $w=(w_{nlk})$ is Δ-convergent to $\zeta\in F$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.

Conversely presume that there is a subset $K=\left\{(n,l,k)\in\mathbb{N}^3 : n,l,k=1,2,3,...\right\}\subset\mathbb{N}^3$ such that $\delta_3(K)=1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim_{(n,l,k)\in K,n,l,k\to\infty}\Delta w_{nlk}=L$. Then for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $k_0\in\mathbb{N}$ such that $\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon$ and $\mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon$, $\mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon$ for all $n\ge k_0, l\ge k_0, k\ge k_0$. Let

$$\begin{aligned} M(\varepsilon,\lambda) &:= \left\{(n,l,k)\in\mathbb{N}^3 : \mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)\le1-\varepsilon \text{ or } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)\ge\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)\ge\varepsilon\right\} \\ &\subseteq \mathbb{N}^3-\left\{\left(n_{k_0+1}, l_{k_0+1}k_{k_0+1}\right),\left(n_{k_0+2}, l_{k_0+2}k_{k_0+2}\right),...\right\}\end{aligned}$$

and as a consequence $\delta_3(M(\varepsilon,\lambda))\le1-1=0$. Hence $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$. Then, the desired result has been acquired. □

Definition 3.10. *Let* V *be an NNS, then* ζ *is named a* Δ*-limit point of the sequence* $w=(w_{nlk})$ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$ *on condition that there is a subsequence of the sequence* w *which* Δ*-converges to* ζ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$*. Let* $L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$*, indicate the set of all limit points of the sequence* w *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$*. Let* $\left\{\left(w_{n(j_1)l(j_2)k(j_3)}\right)\right\}$ *be a subsequence of* $w=(w_{nlk})$ *and* $P=\left\{(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3,\ j_1,j_2,j_3\in\mathbb{N}\right\}$*, then we contract* $\left\{\left(w_{n(j_1)l(j_2)k(j_3)}\right)\right\}$ *by* $\{w\}_P$*, which in case* $\delta_3(P)=0$*,* $\{w\}_P$ *is named a thin subsequence or subsequence of density zero. At the same time,* $\{w\}_P$ *is a non-thin subsequence of* w *if* P *does not have density zero.*

Definition 3.11. *Let* V *be an NNS. Then,* ζ *is named a* Δ*-statistical limit point of the sequence* $w=(w_{nlk})$ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$ *on condition that there is a non-thin subsequence of* w *that* Δ*-converges to* $\zeta\in V$ *w.r.t NN* $(\mathscr{G},\mathscr{B},\mathscr{Y})$*. In that case, we say* ζ *is* $st_{N(\Delta)}$*-limit point of sequence* w*. Throughout* $\Lambda^3_{N(\Delta)}(w)$ *demonstrates the set of all* $st^3_{N(\Delta)}$*-limit point of sequence* w*.*

Definition 3.12. *Let V be an NNS. Then, ζ is named a Δ-statistical cluster point of the sequence $w=(w_{nlk})$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$ on condition that for every $\lambda>0$ and $\varepsilon\in(0,1)$,*

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)>0$$

where $\overline{\delta_3}=\limsup\delta_3$. In that case, we say that ζ is $st^3_{N(\Delta)}$-cluster point of sequence w. Throughout $Cl^3_{N(\Delta)}(w)$ indicates the set of all $st^3_{N(\Delta)}$-cluster point of sequence w.

Definition 3.13. *A NNS V is called to be Δ-complete if every Δ-Cauchy sequence is Δ-convergent in V w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$.*

Theorem 3.14. *Let V be an NNS. Then, for any sequence $w=(w_{nlk})\in V$, $\Lambda^3_{N(\Delta)}(w)\subset Cl^3_{N(\Delta)}(w)$.*

Proof. Let $\zeta\in\Lambda_{N(\Delta)}(w)$, then there is a non-thin subsequence $\left(w_{n(j_1)l(j_2)k(j_3)}\right)$ of w that Δ-converges to $\zeta\in V$ w.r.t NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, i.e.

$$\delta_3\left(\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)>1-\varepsilon\\ \text{and } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon\end{array}\right\}\right)=d>0.$$

Since

$$\begin{aligned}&\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\right\}\\ &\supseteq\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)>1-\varepsilon\\ \text{and } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon\end{array}\right\}.\end{aligned}$$

For every $\lambda>0$ and $\varepsilon\in(0,1)$, we obtain

$$\begin{aligned}&\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\right\}\\ &\supseteq\left\{(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:j_1,j_2,j_3\in\mathbb{N}\right\}\\ &\diagdown\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\le 1-\varepsilon\\ \text{or } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon\end{array}\right\}.\end{aligned}$$

Since $\left\{\left(w_{n(j_i)l(j_2)k(j_3)}\right)\right\}$ is Δ-convergent to ζ w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, the set

$$\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\le 1-\varepsilon\\ \text{or } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon\end{array}\right\}$$

is finite, for any $\varepsilon\in(0,1)$, so

$$\begin{aligned}&\overline{\delta_3}\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\right\}\right)\\ &\ge\overline{\delta_3}\left(\left\{(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:j_1,j_2,j_3\in\mathbb{N}\right\}\right)\\ &-\overline{\delta_3}\left(\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\le 1-\varepsilon\\ \text{or } \mathscr{B}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon,\ \mathscr{Y}\left(\Delta w_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)\ge\varepsilon\end{array}\right\}\right).\end{aligned}$$

Hence

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta w_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta w_{nlk}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)>0,$$

which gives $\zeta\in Cl^3_{N(\Delta)}(w)$. Therefore, we acquire $\Lambda^3_{N(\Delta)}(w)\subset Cl^3_{N(\Delta)}(w)$. □

Theorem 3.15. *For any sequence $w=(w_{nlk})\in V$, $Cl^3_{N(\Delta)}(w)\subset L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$.*

Proof. Let $\zeta\in Cl^3_{N(\Delta)}(w)$, then

$$\delta_3\left(\left\{(n,l,k)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nlk}-\zeta,\lambda)>1-\varepsilon \text{ and } \mathscr{B}(\Delta x_{nlk}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nlk}-\zeta,\lambda)<\varepsilon\right\}\right)>0$$

for every $\lambda>0$ and $\varepsilon\in(0,1)$. Let $\{w\}_P$ be a non-thin subsequence of w such that

$$P=\left\{\begin{array}{c}(n(j_1),l(j_2),k(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon,\mathscr{Y}\left(\Delta x_{n(j_1)l(j_2)k(j_3)}-\zeta,\lambda\right)<\varepsilon\end{array}\right\}$$

for each $\varepsilon\in(0,1)$ and $\delta_3(P)\neq 0$. Since there are infinitely many elements in P, $\zeta\in L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$. Therefore, we obtain $Cl^3_{N(\Delta)}(w)\subset L^3_{(\mathscr{G},\mathscr{B},\mathscr{Y})(\Delta)}(w)$. □

Theorem 3.16. *For any sequence $w=(w_{nlk})\in V$, $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$, gives $\Lambda^3_{N(\Delta)}(w)=Cl^3_{N(\Delta)}(w)=\{\zeta\}$.*

Proof. First we denote that $\Lambda^3_{N(\Delta)}(w)=\{\zeta\}$. Presume that $\Lambda^3_{N(\Delta)}(w)=\{\zeta,\eta\}$ such that $\zeta\neq\eta$. In that case, there are two non-thin subsequences $\left\{\left(w_{n(j_1)l(j_2)k(j_3)}\right)\right\}$ and $\left\{\left(w_{p(j_1)q(j_2)r(j_3)}\right)\right\}$ of $w=(w_{nlk})$ those Δ-converge to ζ and η respectively w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$. Since $\left\{\left(w_{p(j_1)q(j_2)r(j_3)}\right)\right\}$ is Δ-convergent to η w.r.t the NN $(\mathscr{G},\mathscr{B},\mathscr{Y})$, so for every $\lambda>0$ and $\varepsilon\in(0,1)$,

$$P=\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon\end{array}\right\}$$

is a finite set and so $\delta_3(P)=0$. Then, we observe that

$$\begin{aligned}&\left\{(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3\in\mathbb{N}^3,\ j_1,j_2,j_3\in\mathbb{N}\right\}\\ &=\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\\ &\cup\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)\geq\varepsilon\end{array}\right\}\end{aligned}$$

which gives that

$$\delta_3\left(\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\right)\neq 0. \tag{3.3}$$

Since $st^3_{N(\Delta)}-\lim w_{nlk}=\zeta$, we get

$$\delta_3\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\right)=0, \tag{3.4}$$

for every $\lambda>0$ and $\varepsilon\in(0,1)$. Therefore, we can write

$$\delta_3\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)\neq 0.$$

For every $\zeta\neq\eta$, we get

$$\begin{aligned}&\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\\ &\cap\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon\end{array}\right\}\right)=\emptyset.\end{aligned}$$

Hence

$$\begin{aligned}&\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\\ &\subset\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\overline{\delta_3}\left(\left\{\begin{array}{c}(p(j_1),q(j_2),r(j_3))\in\mathbb{N}^3:\mathscr{G}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)>1-\varepsilon \text{ and}\\ \mathscr{B}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon,\ \mathscr{Y}\left(\Delta x_{p(j_1)q(j_2)r(j_3)}-\eta,\lambda\right)<\varepsilon\end{array}\right\}\right)\\ &\leq\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\right)=0.\end{aligned}$$

This contradicts (3.3). Hence $\Lambda^3_{N(\Delta)}(w)=\{\zeta\}$.

Next we demonstrate that $Cl^3_{N(\Delta)}(w)=\{\zeta\}$. Presume that $Cl^3_{N(\Delta)}(w)=\{\zeta,\gamma\}$ such that $\zeta\neq\gamma$. Then

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}\right)\neq 0. \tag{3.5}$$

Since

$$\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)<\varepsilon\end{array}\right\}\cap\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}=\emptyset$$

for every $\zeta\neq\gamma$, so

$$\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\supseteq\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}.$$

Therefore

$$\begin{aligned}&\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\zeta,\lambda)\leq 1-\varepsilon \text{ or}\\ \mathscr{B}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\zeta,\lambda)\geq\varepsilon\end{array}\right\}\right)\\ &\geq\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\Delta x_{nkl}-\gamma,\lambda)>1-\varepsilon \text{ and}\\ \mathscr{B}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon,\ \mathscr{Y}(\Delta x_{nkl}-\gamma,\lambda)<\varepsilon\end{array}\right\}\right).\end{aligned}\tag{3.6}$$

From (3.5), the right hand side of (3.6) is greater than zero and from (3.4) the left hand side of (3.6) equals zero. This causes a contradiction. Hence $Cl^3_{N(\Delta)}(w)=\{\zeta\}$. □

Theorem 3.17. *The set $Cl^3_{N(\Delta)}$ is closed in V for each $w=(w_{nlk})$ of elements of V.*

Proof. Let $q\in\overline{Cl^3_{N(\Delta)}(w)}$. Let $r\in(0,1)$ and $\lambda>0$, there is $\sigma\in Cl^3_{N(\Delta)}(w)\cap B(q,r,\lambda)$ such that

$$B(q,r,\lambda)=\{s\in V:\mathscr{G}(q-s,\lambda)>1-r \text{ and } \mathscr{B}(q-s,\lambda)<r,\ \mathscr{Y}(q-s,\lambda)<r\}$$

Select $\xi>0$ such that $B(\xi,\sigma,\lambda)\subset B(q,r,\lambda)$. Then, we get

$$\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(q-\Delta w_{nkl},\lambda)>1-r \text{ and}\\ \mathscr{B}(q-\Delta w_{nkl},\lambda)<r,\ \mathscr{Y}(q-\Delta w_{nkl},\lambda)<r\end{array}\right\}\supset\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\sigma-\Delta w_{nkl},\lambda)>1-\xi \text{ and}\\ \mathscr{B}(\sigma-\Delta w_{nkl},\lambda)<\xi,\ \mathscr{Y}(\sigma-\Delta w_{nkl},\lambda)<\xi\end{array}\right\}.$$

Since $\sigma\in Cl^3_{N(\Delta)}(w)$ so

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(\sigma-\Delta w_{nkl},\lambda)>1-\xi \text{ and}\\ \mathscr{B}(\sigma-\Delta w_{nkl},\lambda)<\xi,\ \mathscr{Y}(\sigma-\Delta w_{nkl},\lambda)<\xi\end{array}\right\}\right)>0.$$

Hence

$$\overline{\delta_3}\left(\left\{\begin{array}{c}(n,k,l)\in\mathbb{N}^3:\mathscr{G}(q-\Delta x_{nkl},\lambda)>1-r \text{ and}\\ \mathscr{B}(q-\Delta x_{nkl},\lambda)<r,\ \mathscr{Y}(q-\Delta x_{nkl},\lambda)<r\end{array}\right\}\right)>0.$$

Thus $q\in Cl^3_{N(\Delta)}(w)$. This ends the proof. □

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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