**Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 12(2): 1023-1033, 2022**
**Journal of the Institute of Science and Technology, 12(2): 1023-1033, 2022**

**ISSN: 2146-0574, eISSN: 2536-4618**

**Mathematics** **DOI: 10.21597/jist.1026364**

**Research Article**

**Received:20.11.2021** **Accepted:29.03.2022**

**To Cite:** Kocabıyık M, 2022. Nonstandard Discretization and Stability Analysis of a Novel Type of Malaria-Ross Model. Journal of the Institute of Science and Technology, 12(2): 1023-1033.

# Nonstandard Discretization and Stability Analysis of a Novel Type of Malaria-Ross Model

Mehmet KOCABIYIK[1*]

**ABSTRACT:** Malaria is still a deadly disease in most developing countries. In order to prevent this and many other diseases in all countries, it is necessary to understand the dynamics of the disease well. For this reason, in this study, a new type of Malaria-Ross equation, Distributed order, is discussed. In this new type, the dynamics of the disease can be understood better and quicker in different situations with the density function included in such equations. Numerical discretization of this model is done with the help of a nonstandard finite difference scheme. Afterward, stability analyses of the equilibrium points obtained from the model that were performed. At the same time, comparisons were made with other numerical methods. Finally, the findings are expressed with graphs and tables.

**Keywords:** Distributed order differential equations, Malaria-Ross model, numerical Analysis, discretization, stability Analysis

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## INTRODUCTION

Malaria is an ancient disease, but it still significantly impacts public health. In fact, WHO (World Health Organization) has declared malaria as an endemic disease in some developing countries (WHO, 2017). Malaria is a vector-borne infectious disease caused by the protozoan parasite transmitted to vertebrates by an infected female Anopheles mosquito. The parasite is transmitted to humans by the bite of an infected mosquito. It is very important to understand and predict the transmission dynamics of malaria because of its impact on the world. As in most fields in biology, mathematical modeling is used to study infectious diseases, to understand the relationship between host and parasite, and the dynamics of disease.

In this framework, Ronald Ross reached the life cycle of the malaria parasite in 1890. He subsequently published a series of articles on the transmission of malaria (Ross, 1911; Ross, 1915; Ross 1916a; Ross 1916b; Ross and Hadson, 1916). In the model to be used in this research, the relationship between the number of mosquitoes and humans was examined. This model is also called Malaria Ross model (Ross, 1915). After these studies, many articles were published about the latent infection period of malaria in humans (Macdonald, 1957; Aron and May, 1982; Dietz, 1988; Aron, 1988; Anderson and May, 1991). In recent years, Ngwa and Shu have investigated the dynamics of the model (Nqwa and Shu, 2000; Nqwa, 2004). Chitnis (2005), on the other hand, has studied the spread of malaria.

Mandal et al. (2011) conducted a study in which all the mathematical models of malaria have been found. In addition to these articles, studies on discretization and stability analysis for different malaria models have also gained importance in recent years (Elsheikh et al., 2014; Nyang'inja, 2019).

In this study, the Malaria Ross model has been discussed and its dynamics have been investigated. The Malaria Ross model for these dynamics is expressed as a new type of equation, with Distributed order differential equations. The main purpose for us to create the distributed order of the Malaria Ross model and reach its solution in this way is that, this type of equation is a general state of ordinary and fractional differential equations. The most important factor that provides generalization is that this type of equation contains a density function. Selection of density function provides interpretation capability for both ordinary equations, fractional equations, and different situations. For example, with the selection $u(\alpha)$=1, the differential equation of the distributed order becomes fractional-order differential equations.

Distributed order differential equations have been defined and developed by Caputo (1969; 1995; 2001; 2003). This type of equation defined by Caputo has gained importance afterward and was used in fractional order systems by Hartley and Lorenzo (2003). Bagley and Torvik (2000a; 2000b) worked on the existence and solution of distributed order differential equations. Distributed order differential equations which have gained more importance with these research studies have been studied and researched by many researchers in subjects such as analytical solutions, numerical solutions, and stability analysis (Diethelm and Ford, 2009; Katsikadelis, 2014; Li and Wu, 2016; Aminikhah et al., 2013; Luchko, 2009).

With $\alpha \in (r_1, r_2)$ let $\int_{r_1}^{r_2} u(\alpha) = l > 0$, $D_t^{\alpha}$ be the fractional derivative operator and $g(t)$ be a function that can be chosen as a Caputo or Riemann Liouville fractional derivative. In this case, the distributed order equations and fractional derivative are respectively defined as (Caputo 1969),

$$D_t^{u(\alpha)} g(t) = \sum_{i=1}^{n} \alpha^i \int_{\tau_1}^{\tau_2} u_i(\alpha)\, D_t^{i-\alpha} g(t) d\alpha + \sum_{j=0}^{n} b_j g^i(t). \quad (1)$$

As stated in the previous explanations, with the solution of these types of differential equations, information about the solutions of more than one type of differential equations can be obtained (Caputo, 1995; Caputo, 2001).

Another important definition for discretization of the model after the definitions of distributed order differential equations required is the approximate Grünwald-Letnikov derivative formula. This formula can be defined as follows (Meerschaert and Tadjeran, 2004),

$$D_{GL}^{\alpha} g(t) = \lim_{v \to 0} v^{-\alpha} \sum_{i=0}^{n} (-1)^i \binom{\alpha}{i} g(t - iv). \tag{2}$$

If necessary arrangements are made in this formula due to its ease of use, the equation

$$D_t^{\alpha} g(t) = \sum_{i=0}^{n} p_i^{\alpha} g(t_{n-r}), \quad n = 1,2,3,,\dots,\frac{t-\alpha}{h} \tag{3}$$

is found for, $i = 1,2,3,\dots,n$, $p_i^{\alpha} = \left(1 - \frac{1+\alpha}{i}\right) p_{i-1}^{\alpha}$, $p_0^{\alpha} = h^{-\alpha}$and $h$ has been selected quite small (Dorciak, 1994).

In this article, we examine a new type of model, the distributed order, which expresses the relationship between the number of mosquitoes and the number of malaria cases in humans, as described by Ross (1911). Distributed order Malaria Ross model can be defined as follows,

$$D_t^{u(\alpha)} I^h = a\, b\, m\ I^m\,(1 - I^h) - r\, I^h, \tag{4}$$

$$D_t^{u(\alpha)} I^m = a\, c\, I^h\,(1 - I^m) - \mu_2\, I^m\ , \tag{5}$$

where;

$I^h =$ the time evolution of the infected classes in human,

$I^m =$ the time evolution of the infected classes in mosquito,

$a =$ Man biting rate [0.01-0.5] day$^{-1}$,

$b =$ Proportion of bites that produce infection in human [0.2-0.5],

$c =$ Proportion of bites by which one susceptiple mosquito becomes infected [0.5],

$m =$ Ratio of number of female mosquitoes to that of humans [0.5-40],

$r =$ Avarage recovery rate of human [0.005-0.05] day$^{-1}$,

$\mu_2 =$ Per capita rate of mosquito mortality [0.05-0.5] day$^{-1}$.

The main purpose here is to understand the rate of progression and development of infected humans and mosquitoes. Knowing these advanced developments is very important to understand the dynamics of drugs on the disease. The most important problem for these dynamics is to know the effects of different factors and situations. Instead of finding such effects with different equation systems, Distributed order differential equations are used.

This article consists of four sections. In the first section, basic information and definitions about Malaria Ross model and Distributed order differential equations are given. Afterward, in this section, a new type of Malaria Ross model of distributed order is defined and its purpose of use is stated. In the second section, discretization of the given new type model is done with Nonstandard finite difference scheme (NSFD). In addition, equilibrium points of the discretized system are found in this section. In the third section, the stability analysis of the equilibrium points is made by substituting the parameter values. The numerical simulations of this system are also included. In the fourth and last section, there is the conclusion part of the outcomes.

## MATERIALS AND METHODS

For discretization, the Nonstandard finite difference method defined by Mickens in 1989 was chosen (Mickens, 1989). If $\varphi$ is considered a parameter and $\frac{dg}{dt} = H(\varphi, g)$ is considered the ordinary differential equation, the NSFD scheme is in the form below;

$$t \to t_n, \qquad R(g) \to R(g_n), \quad g(t) \to g(t_n), \quad \frac{dg}{dt} \to \frac{g_{n+1} - g_n}{\phi}, \tag{6}$$

where $\phi$: denominator function and $\frac{1-e^{-ch}}{c}$ , $h$ depends on the variable $c$ which can be achieved with the help of the step range and the equilibrium point. The NSFD scheme can also be used in fractional order differential equations with the approximate Grünwald -Letnikov derivative formula expressed in the previous section (Mickens, 1994; Mickens, 2002; Ongun and Turhan, 2012; Ongun et al., 2013; Ongun and Arslan, 2018; Kocabıyık et al., 2020).

If the distributed order Malaria Ross model which is expressed by Equations (4)-(5) is discretized with NSFD scheme, it takes the form:

$$\sum_{k=1}^{T} \frac{u(\alpha_k)}{T} \sum_{i=0}^{n+1} q_i^{\alpha_k} I_{n+1-i}^h = a\ b\ m\ I_n^m \left(1 - I_{n+1}^h\right) - r\ I_{n+1}^h \tag{7}$$

$$\sum_{k=1}^{T} \frac{u(\alpha_k)}{T} \sum_{i=0}^{n+1} q_i^{\alpha_k} I_{n+1-i}^m = a\ c\ I_n^h\ (1 - I_{n+1}^m) - \mu_2\ I_{n+1}^m \tag{8}$$

In this discretized system for $i = 1,2$ and for $0 < \alpha_k < 1$ , $p_0^{\alpha_k} = (\phi_i(h))^{-\alpha_k}$ . For the NSFD scheme, the denominator functions were chosen in the form:

$$\phi_1(h) = \frac{1 - e^{rh}}{r}, \qquad \phi_2(h) = \frac{1 - e^{\mu_2 h}}{\mu_2}. \tag{9}$$

The left side of the discretized system can be arranged and for $i = 1,2$ if the abbreviations $\sum_{k=1}^{N} \frac{u(\alpha_k)}{N} = K$ and $\sum_{k=1}^{N} \frac{u(\alpha_k)}{N}\ \phi_i(h) = L_i$ are used, the discretized form will be Equations (10)-(11),

$$I_{n+1}^h = \frac{a\ b\ m\ I_n^m - K\ (\sum_{i=1}^{n+1} q_i^{\alpha_k} I_{n+1-i}^h)}{((L_1)^{-\alpha_k} + a\ b\ m\ I_n^m + r)}, \tag{10}$$

$$I_{n+1}^m = \frac{a\ c\ I_n^h - K\left(\sum_{i=1}^{n+1} q_i^{\alpha_k} I_{n+1-i}^m\right)}{((L_2)^{-\alpha_k} + a\ c\ I_n^h + \mu_2)}. \tag{11}$$

In order to find the equilibrium point of Equations (10)-(11), the solutions of the equations:

$$I_n^h = \frac{a\ b\ m\ I_n^m - K\ v\ I_n^h}{((L_1)^{-\alpha_k} + a\ b\ m\ I_n^m + r)}, \tag{12}$$

$$I_n^m = \frac{a\ c\ I_n^h - K\ v\ I_n^m}{((L_2)^{-\alpha_k} + a\ c\ I_n^h + \mu_2)}, \tag{13}$$

are required, where $v = \sum_{i=1}^{n+1} q_i^{\alpha_k}$. There are two different situations for this solution. With these solutions, equilibrium points are found as;

$$E_1 = (I_n^h, I_n^m) = (0,0), \tag{14}$$

$$E_2 = (I_n^h, I_n^m) = \left(\frac{a^2\ b\ c\ m - K^2\ v^2 - K\ v\ ((L_1)^{-\alpha_k} + (L_2)^{-\alpha_k} + r + \mu_2) - (L_1)^{-\alpha_k}(L_2)^{-\alpha_k} - (L_1)^{-\alpha_k}\mu_2 - (L_2)^{-\alpha_k} r - \mu_2\ r}{ac((L_1)^{-\alpha_k} + a\ b\ m\ + K\ v + r)}, \right.$$
$$\left. \frac{a^2\ b\ c\ m - K^2\ v^2 - K\ v\ ((L_1)^{-\alpha_k} + (L_2)^{-\alpha_k} + r + \mu_2) - (L_1)^{-\alpha_k}(L_2)^{-\alpha_k} - (L_1)^{-\alpha_k}\mu_2 - (L_2)^{-\alpha_k} r - \mu_2\ r}{abm((L_2)^{-\alpha_k} + a\ c + K\ v + \mu_2)}\right). \tag{15}$$

For the analysis of these equilibrium points, $J$ Jacobian matrix of the discretized system is obtained in the following form:

$$J(I_{n+1}^h, I_{n+1}^m) = \begin{pmatrix} \frac{-K\ q_1^{\alpha_k}}{((L_1)^{-\alpha_k} + a\ b\ m\ I_n^m + r)} & \frac{a\ b\ m\ ((L_1)^{-\alpha_k} + r + K q_1^{\alpha_k} I_n^h))}{((L_1)^{-\alpha_k} + a\ b\ m\ I_n^m + r)^2} \\ \frac{a\ c\ ((L_2)^{-\alpha_k} + \mu_2 + K q_1^{\alpha_k} I_n^m)}{((L_2)^{-\alpha_k} + a\ c\ I_n^h + \mu_2)^2} & \frac{-K\ q_1^{\alpha_k}}{((L_2)^{-\alpha_k} + a\ c\ I_n^h + \mu_2)} \end{pmatrix}. \tag{16}$$

## RESULTS AND DISCUSSION

In this section, the stability analysis of the obtained equilibrium points and numerical simulations of the discretized system will be given with the help of parameter values. For the simulations in this section, the parameters $a = 0.2\ \text{day}^{-1}$, $b = 0.5$, $c = 0.5$, $m = 20$, $r = 0.01\ \text{day}^{-1}$ and $\mu_2 = 0.12\ \text{day}^{-1}$ are used (Mandal et al., 2011).

### Stability analysis of Distributed order Malaria-Ross Model

**Lemma 3.1.** Let the $E$ point be the equilibrium point of the discretized system. In this case, the absolute values of all eigenvalues obtained when substituted in the Jacobian matrix must be less than 1 in order for the equilibrium point to be stable. Otherwise, if the absolute value of at least one eigenvalue is not less than 1, the equilibrium point is not stable (Dimitrov and Kojouharov, 2007; Dimitrov and Kojouharov, 2008).

In some special cases, it is not possible to obtain eigenvalues in the stability analysis section in terms of processing difficulty. When such a situation is encountered, the criteria developed by Schur-Cohn can be used. These criteria, also called Jury Conditions, depend on the coefficients of the characteristic equation obtained from the matrix. For the definition of Jury Conditions, if we consider the characteristic polynomial $P(\lambda) = \lambda^2 + a_1\lambda + a_0$ ,

i) $1 + a_1 + a_0 > 0,$

ii) $1 - a_1 + a_0 > 0,$

iii) $|a_0| < 1,$

if the conditions are satisfied, the equilibrium point is asymptotically stable (Dimitrov and Kojouharov, 2007).

**Theorem 3.2.** The equilibrium point $E_1$ is locally asymptotically stable if the following condition is satisfied, if not unstable.

$$\left|\frac{-1}{2}(q_1^{\alpha_k}K[(L_1)^{-\alpha_k} + (L_2)^{-\alpha_k} + \mu_2 + r]\right| + \left|\sqrt{\begin{matrix}a^2bcm(4(L_1)^{-\alpha_k}(L_2)^{-\alpha_k} + 4(L_1)^{-\alpha_k}\mu_2 + (L_1)^{-\alpha_k}(L_2)^{-\alpha_k}r + 4\mu_2 r) \\ +(q_1^{\alpha_k})^2K^2(((L_1)^{-\alpha_k} - (L_2)^{-\alpha_k})^2 \\ -2(L_1)^{-\alpha_k}\mu_2 + 2(L_1)^{-\alpha_k}r + 2(L_2)^{-\alpha_k}\mu_2 - 2(L_2)^{-\alpha_k}r + (\mu_2 - r)^2)\end{matrix}}\right| < |((L_1)^{-\alpha_k} + r)((L_2)^{-\alpha_k} + \mu_2)|,$$

**Proof**

If equilibrium point $E_1$ is placed in the Jacobian matrix:

$$J\left(I_{n+1}^h, I_{n+1}^m\right) = \begin{pmatrix} \dfrac{-K\ q_1^{\alpha_k}}{(L_1)^{-\alpha_k} + r} & \dfrac{a\ b\ m}{(L_1)^{-\alpha_k} + r} \\ \dfrac{a\ c}{(L_2)^{-\alpha_k} + \mu_2} & \dfrac{-K\ q_1^{\alpha_k}}{(L_2)^{-\alpha_k} + \mu_2)} \end{pmatrix}.$$

If the determinant $|J - \lambda I| = 0$ is used in this obtained Jacobian matrix, the characteristic equation will be found as follows:

$$P(\lambda) = \lambda^2 + \left(\frac{q_1^{\alpha_k}K[(L_1)^{-\alpha_k} + (L_2)^{-\alpha_k} + \mu_2 + r]}{((L_1)^{-\alpha_k} + r)((L_2)^{-\alpha_k} + \mu_2)}\right)\lambda + \left(\frac{-a^2bcm + (q_1^{\alpha_k})^2K^2}{((L_1)^{-\alpha_k} + r)((L_2)^{-\alpha_k} + \mu_2)}\right) = 0.$$

With the solution of the characteristic polynomial, the eigenvalues are as follows;

$\lambda_1$

$$= \frac{\frac{-1}{2}(q_1^{\alpha_k}K[(L_1)^{-\alpha_k}+(L_2)^{-\alpha_k}+\mu_2+r] - \sqrt{\begin{matrix}a^2bcm(4(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}+4(L_1)^{-\alpha_k}\mu_2+(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}r+4\mu_2 r)\\ +(q_1^{\alpha_k})^2K^2(((L_1)^{-\alpha_k}-(L_2)^{-\alpha_k})^2\\ -2(L_1)^{-\alpha_k}\mu_2+2(L_1)^{-\alpha_k}r+2(L_2)^{-\alpha_k}\mu_2-2(L_2)^{-\alpha_k}r+(\mu_2-r)^2)\end{matrix}}}{((L_1)^{-\alpha_k}+r)((L_2)^{-\alpha_k}+\mu_2)},$$

$\lambda_2$

$$= \frac{\frac{-1}{2}(q_1^{\alpha_k}K[(L_1)^{-\alpha_k}+(L_2)^{-\alpha_k}+\mu_2+r] + \sqrt{\begin{matrix}a^2bcm(4(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}+4(L_1)^{-\alpha_k}\mu_2+(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}r+4\mu_2 r)\\ +(q_1^{\alpha_k})^2K^2(((L_1)^{-\alpha_k}-(L_2)^{-\alpha_k})^2\\ -2(L_1)^{-\alpha_k}\mu_2+2(L_1)^{-\alpha_k}r+2(L_2)^{-\alpha_k}\mu_2-2(L_2)^{-\alpha_k}r+(\mu_2-r)^2)\end{matrix}}}{((L_1)^{-\alpha_k}+r)((L_2)^{-\alpha_k}+\mu_2)}.$$

Because of the stability condition expressed in Lemma 3.1, the absolute values of the eigenvalues must be less than 1. Then the following conditions must be satisfied:

$$\left|\frac{\frac{-1}{2}(q_1^{\alpha_k}K[(L_1)^{-\alpha_k}+(L_2)^{-\alpha_k}+\mu_2+r] - \sqrt{\begin{matrix}a^2bcm(4(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}+4(L_1)^{-\alpha_k}\mu_2+(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}r+4\mu_2 r)\\ +(q_1^{\alpha_k})^2K^2(((L_1)^{-\alpha_k}-(L_2)^{-\alpha_k})^2\\ -2(L_1)^{-\alpha_k}\mu_2+2(L_1)^{-\alpha_k}r+2(L_2)^{-\alpha_k}\mu_2-2(L_2)^{-\alpha_k}r+(\mu_2-r)^2)\end{matrix}}}{((L_1)^{-\alpha_k}+r)((L_2)^{-\alpha_k}+\mu_2)}\right| < 1,$$

$$\left|\frac{\frac{-1}{2}(q_1^{\alpha_k}K[(L_1)^{-\alpha_k}+(L_2)^{-\alpha_k}+\mu_2+r] + \sqrt{\begin{matrix}a^2bcm(4(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}+4(L_1)^{-\alpha_k}\mu_2+(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}r+4\mu_2 r)\\ +(q_1^{\alpha_k})^2K^2(((L_1)^{-\alpha_k}-(L_2)^{-\alpha_k})^2\\ -2(L_1)^{-\alpha_k}\mu_2+2(L_1)^{-\alpha_k}r+2(L_2)^{-\alpha_k}\mu_2-2(L_2)^{-\alpha_k}r+(\mu_2-r)^2)\end{matrix}}}{((L_1)^{-\alpha_k}+r)((L_2)^{-\alpha_k}+\mu_2)}\right| < 1.$$

In this case, the equilibrium point $E_1$ is locally asymptotically stable, provided the following condition is satisfied.

$$\left|\frac{-1}{2}(q_1^{\alpha_k}K[(L_1)^{-\alpha_k}+(L_2)^{-\alpha_k}+\mu_2+r]\right| + \left|\sqrt{\begin{matrix}a^2bcm(4(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}+4(L_1)^{-\alpha_k}\mu_2+(L_1)^{-\alpha_k}(L_2)^{-\alpha_k}r+4\mu_2 r)\\ +(q_1^{\alpha_k})^2K^2(((L_1)^{-\alpha_k}-(L_2)^{-\alpha_k})^2\\ -2(L_1)^{-\alpha_k}\mu_2+2(L_1)^{-\alpha_k}r+2(L_2)^{-\alpha_k}\mu_2-2(L_2)^{-\alpha_k}r+(\mu_2-r)^2)\end{matrix}}\right| < |((L_1)^{-\alpha_k}+r)((L_2)^{-\alpha_k}+\mu_2)|.$$

**Remark 3.3** In addition, stability analysis can be found using the denominator function. For this analysis, let $((L_1)^{-\alpha_k}+r)((L_2)^{-\alpha_k}+\mu_2)=k_1$, $q_1^{\alpha_k}K[(L_1)^{-\alpha_k}+(L_2)^{-\alpha_k}+\mu_2+r]=U$ and $-a^2bcm+(q_1^{\alpha_k})^2K^2=V$. We need to find the constants $D_{E_1}$, which must satisfy the Jury Conditions for all $0<k_1<D_{1E_1}$. For this reason, the characteristic function obtained as;

$$\lambda^2+\left(\frac{U}{k_1}\right)\lambda+\left(\frac{V}{k_1}\right)=0.$$

When the Jury conditions are examined;

i) $1+a_1+a_0=1+\frac{U}{k_1}+\frac{V}{k_1}>0,$

ii) $1 - a_1 + a_0 = 1 - \frac{U}{k_1} + \frac{U}{k_1} > 0.$

So, $D_{1E_1}$ can be chosen as below:

$$D_{1E_1} \begin{cases} \sqrt{|V|}, & U = 0 \\ \min\left(\left|U, \frac{\sqrt{|V|}}{|U|}\right|\right), & otherwise. \end{cases}$$

iii) $|a_0| = \left|\frac{V}{k_1}\right| < 1.$

With this iii condition $D_{2E_1}$ is selected as $\sqrt{|V|}$. So, if $k_1 < \min(D_{1E_1}, D_{2E_1})$, then the Jury conditions are satisfied and $E_1$ is locally asymptotically stable.

**Remark 3.4** Analysis for equilibrium point $E_2$ contains quite complex operations. For this reason, the stability of the $E_2$ point has been investigated with the Schur-Cohn test.

After this information, if it is desired to examine the stability with the help of Jury condition criteria, by writing the $E_1 = (I_n^h, I_n^m) = (0,0)$ equilibrium point in its place in the Jacobian matrix, $J(I_{n+1}^h, I_{n+1}^m) = \begin{pmatrix} 0.00109 & 1.25399 \\ 0.05865 & 0.00102 \end{pmatrix}$ with $|J - \lambda I| = 0$, eigenvalues are found in the form, $\lambda_1 = 0.27226$, $\lambda_2 = -0.27013$. In this case, with Lemma 3.1, $E_1$ equilibrium point locally asymptotically stable. On the other hand, the characteristic equation of the system is obtained as, $P(\lambda) = \lambda^2 + a_1\lambda + a_0$, and the coefficients are in the form $a_1 = -0.00212, a_0 = -0.07354$. According to the Jury criteria

i) $1 + a_1 + a_0 = 0.92432 > 0,$

ii) $1 - a_1 + a_0 = 0.92857 > 0,$

iii) $|a_0| = 0.07354 < 1,$

all conditions are satisfied and this will lead to $E_1$ equilibrium point is locally asymptotically stable.

Like $E_1$ equilibrium point analysis, by writing the $E_2$ equilibrium point in its place in the Jacobian matrix, $J(I_{n+1}^h, I_{n+1}^m) = \begin{pmatrix} -0.00001 & 0.00068 \\ 0.00001 & -0.00001 \end{pmatrix}$ with $|J - \lambda I| = 0$, eigenvalues are found in the form, $\lambda_1 = 0.00009$, $\lambda_2 = -0.00013$. So, $E_2$ equilibrium point locally asymptotically stable. Again, if the Jury conditions are controlled, coefficients are found in the form, $a_1 = 0.00003, a_0 = -0.12870.10^{-7}$,

i) $1 + a_1 + a_0 = 1.00003 > 0,$

ii) $1 - a_1 + a_0 = 0.99996 > 0,$

iii) $|a_0| = 0.12870.10^{-7} < 1,$

as seen all Jury conditions are satisfied therefore $E_2$ equilibrium point is locally asymptotically stable.

**Numerical Simulations**

Using the parameters given at the beginning of the section, in Figure 1, the effect of different $u(\alpha)$ density functions on the solutions is seen when $h = 0.01$ and $\alpha = 1$. As can be seen in this figure, it is easier to determine the dynamics of the disease with different $u(\alpha)$ selection.

In Figure 2, when $u(\alpha) = \alpha$ and $\alpha = 1$, the effect of the solutions is examined by changing the step size, that is, the $h$ value. Here, the difference in the solutions of choosing the different $h$ values is seen. Finally, in Figure 3, if $u(\alpha) = \alpha$ and $h = 0.01$, this time the effect on the solutions is seen by changing the $\alpha$ values. After these graphics, in Table 1, CPU times are compared for numerical methods. As seen in Table 1, we can say that the numerical methods evaluated among themselves are not very different. The qualitative results of the Malaria Ross model for different time step sizes are given in Table 2.

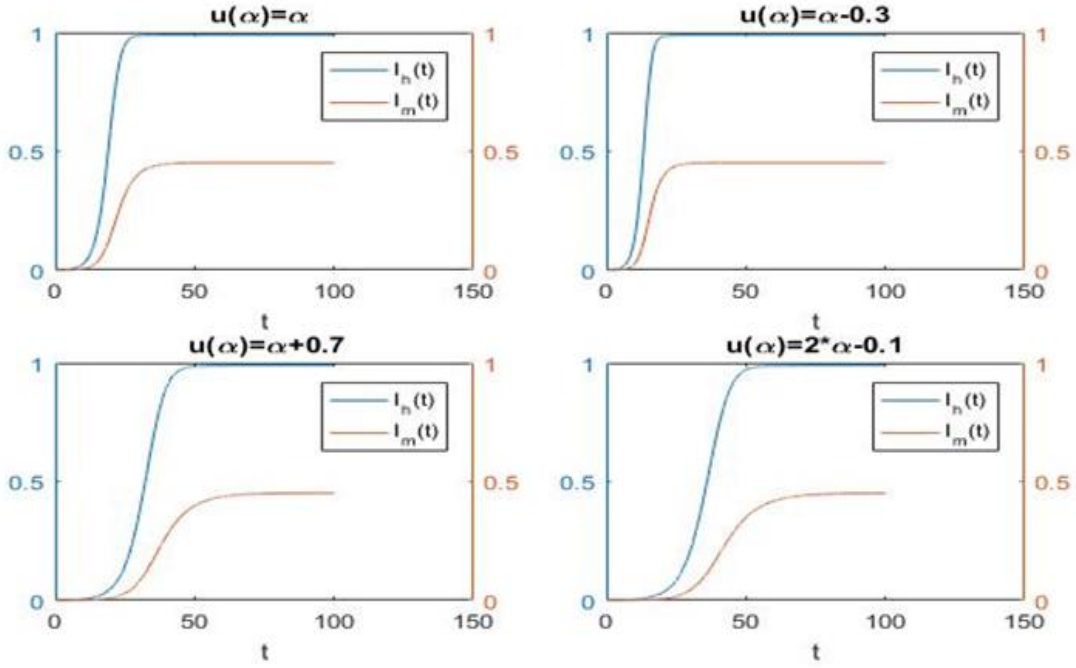


**Figure 1.** Different u(α) solutions for Malaria Ross model $(h = 0.01, \alpha = 1)$

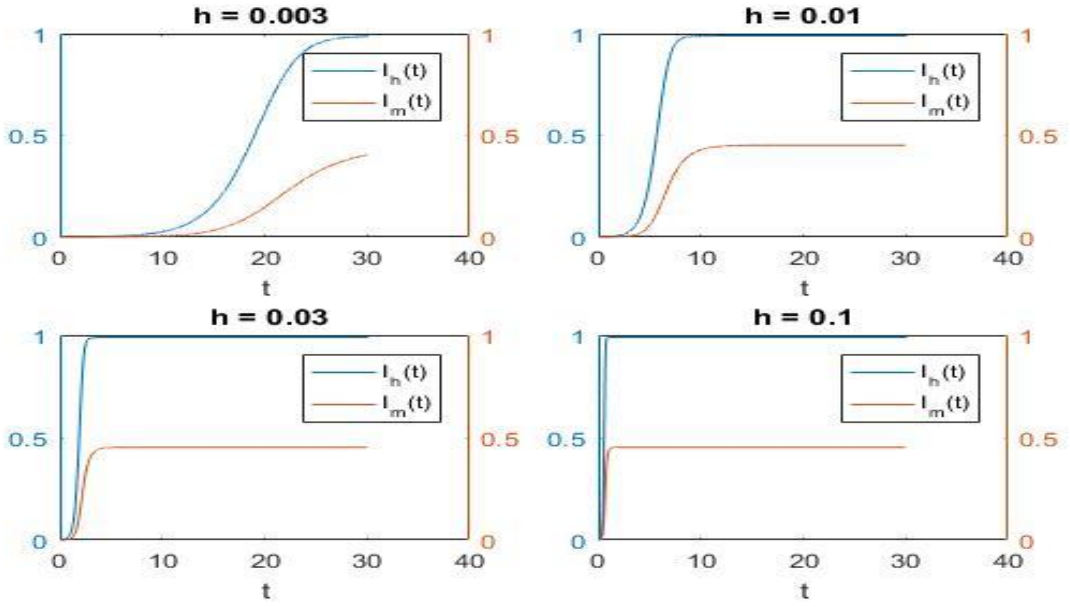


**Figure 2.** Different h solutions for Malaria Ross model $(u(\alpha) = \alpha, \alpha = 1)$

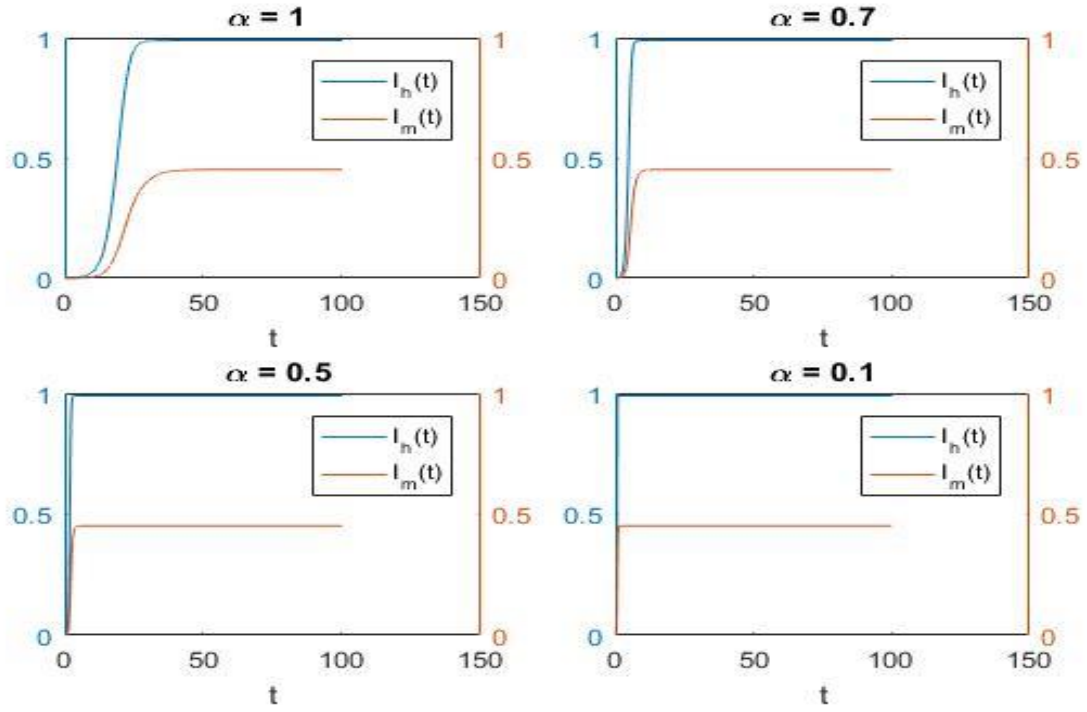


**Figure 3.** Different α solutions for Malaria Ross model $(u(\alpha) = \alpha, h = 0.01)$

**Table 1.** CPU Times (seconds) for $u(\alpha) = \alpha$, $h = 0.01$

| $\alpha$ | Theta Method | NSFD |
|---|---|---|
| 0.1 | 0.6320 | 0.6555 |
| 0.3 | 0.5416 | 0.6233 |
| 0.8 | 0.5595 | 0.5535 |
| 1 | 0.7010 | 0.6014 |

**Table 2.** Qualitative results for different time step sizes $h$ in Malaria Ross model with $u(\alpha) = \alpha, \alpha = 1$

| $h$ | Theta Method | Runge Kutta | NSFD |
|---|---|---|---|
| 0.00001 | Convergence | Convergence | Convergence |
| 0.0001 | Convergence | Convergence | Convergence |
| 0.001 | Convergence | Convergence | Convergence |
| 0.01 | Convergence | Convergence | Convergence |
| 0.1 | Convergence | Convergence | Convergence |
| 1 | Convergence | Convergence | Convergence |
| 2 | Divergence | Divergence | Convergence |
| 3 | Divergence | Divergence | Convergence |

## CONCLUSION

In this study, the mathematical model of Malaria developed by Ross is defined by distributed order differential equations. The dynamics of the Malaria Ross model have been investigated using the density function included in the distributed order differential equations. In this way, the acts of the model can be interpreted under different conditions. Thanks to this interpretation, the effects of the disease on people can be predicted clearly and the use of drugs can be determined accordingly. In addition, it is seen that these solutions which stability analysis is performed, are also mathematically reliable. It has been seen that distributed order differential equations can be used not only for the Malaria Ross model but also for many endemic models and they are very useful in the interpretation phase.

## ACKNOWLEDGEMENTS

This research was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) with the 2211-E Program. I am also grateful for the exceptional support of my supervisor, Prof. Dr. Mevlüde YAKIT ONGUN.

### Conflict of Interest

The author declares that there is no conflict of interest.

## REFERENCES

Aminikhah H, Refahi A, Rezazadeh H, 2013. Stability analysis of distributed order fractional Chen system. The Scientific World Journal, 2013.

Anderson RM, May RM, 1991. Infectious diseases of humans: dynamics and control London: Oxford University Press.

Aron JL, 1988. Mathematical modeling of immunity to malaria. Math Bioscience, 90: 385-396.

Aron JL, May RM, 1982. The population dynamics of malaria. In Population Dynamics of Infectious Disease. Edited by: Anderson RM. London: Chapmanand Hall, pp. 139-179.

Bagley RL, Torvik PJ, 2000 a. On the existence of the order domain and the solution of distributed order equations-Part I. International Journal of Applied Mathematics, 2(7): 865-882.

Bagley RL, Torvik PJ, 2000b. On the existence of the order domain and the solution of distributed order equations-Part II. International Journal of Applied Mathematics, 2(8): 965-988.

Caputo M, 1969. Elasticita e dissipazione. Zanichelli.

Caputo M, 1995. Mean fractional-order-derivatives differential equations and filters. Annali dell'Universita di Ferrara, 41(1): 73-84.

Caputo M, 2001. Distributed order differential equations modelling dielectric induction and diffusion. Fractional Calculus and Applied Analysis, 4(4): 421-442.

Caputo M, 2003. Diffusion with space memory modelled with distributed order space fractional differential equations. Annals of Geophysics.

Chitnis N, 2005. Using mathematical models in controlling the spread of malaria. PhD thesis University of Arizona, Program in Applied Mathematics.

Diethelm K, Ford NJ, 2009. Numerical analysis for distributed-order differential equations. Journal of Computational and Applied Mathematics, 225(1): 96-104.

Dietz K, 1988. Mathematical models for transmission and control of malaria. In Principles and Practice of Malariology. Edited by: Wernsdorfer W, McGregor Y. Edinburgh: Churchill Livingston, pp. 1091-1133.

Dimitrov DT, Kojouharov HV, 2007. Nonstandard numerical methods for a class of predator-prey models with predator interference. Electronic Journal of Differential Equations (EJDE) pp. 67-75.

Dimitrov DT, Kojouharov HV, 2008. Nonstandard finite-difference methods for predator–prey models with general functional response. Mathematics and Computers in Simulation, 78(1): 1-11.

Dorciak L, 1994. Numerical models for simulation the fractional-order control systems, UEF-04-94, The Academy of Sciences, Institute of Experimental Physic, Kosiice, Slovak Republic.

Elsheikh S, Ouifki R, Patidar KC, 2014. A non-standard finite difference method to solve a model of HIV Malaria co-infection. Journal of Difference Equations and Applications, 20(3): 354–378. doi: 10.1080/10236198.2013.821116.

Hartley TT, Lorenzo CF, 2003. Fractional-order system identification based on continuous order-distributions. Signal processing, 83(11): 2287-2300.

Katsikadelis JT, 2014. Numerical solution of distributed order fractional differential equations. Journal of Computational Physics, 259: 11-22.

Kocabıyık M, Özdoğan N, Ongun MY, 2020. Nonstandard Finite Difference Scheme for a Computer Virus Model. Journal of Innovative Science and Engineering (JISE), 4(2): 96-108.

Li XY, Wu BY, 2016. A numerical method for solving distributed order diffusion equations. Applied Mathematics Letters, 53: 92-99.

Luchko Y, 2009. Boundary value problems for the generalized time-fractional difusion equation of distributed order. Fractional Calculus and Applied Analysis, 12 (4): 409-422.

Macdonald G, 1957. The epidemiology and control of malaria London: Oxford University Press.

Mandal S, Sarkar RR, Sinha S, 2011. Mathematical models of malaria-a review. Malaria journal, 10(1): 1-19.

Meerschaert MM, Tadjeran C, 2004. Finite difference approximations for fractional advection–dispersion flow equations. Journal of computational and applied mathematics, 172(1): 65-77.

Mickens RE, 1989. Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis. Numerical Methods for Partial Differential Equations, 5(4): 313-325.

Mickens RE, 1994. Nonstandard finite difference models of differential equations. World scientific.

Mickens RE, 2002. Nonstandard finite difference schemes for differential equations. Journal of Difference Equations and Applications, 8(9): 823-847.

Ngwa GA, 2004. Modelling the dynamics of endemic malaria in growing populations. Discrete Contin Dyn System- Ser B, 4: 1173-1202.

Ngwa GA, Shu WS, 2000. A mathematical model for endemic malaria with variable human and mosquito populations. Math Comput Model, 32: 747-763.

Nyang'inja R, Lawi G, Okongo M, Orwa A, 2019. Stability analysis of Rotavirus-malaria co-epidemic model with vaccination. Dyn. Syst. Appl, 28: 371-407.

Ongun MY, Arslan D, 2018. Explicit and Implicit Schemes for Fractional orders Hantavirus Model. Iranian Journal of Numerical Analysis and Optimization, 8(2): 75–93.

Ongun MY, Arslan D. Garrappa R, 2013. Nonstandard finite difference schemes for a fractional-order Brusselator system. Advances in Difference equations, 2013(1), 1-13.

Ongun MY, Turhan I, 2012. A numerical comparison for a discrete HIV infection of CD4+ T-Cell model derived from nonstandard numerical scheme. Journal of Applied Mathematics, 2013.4.

Ross R, 1911. The prevention of malaria London: John Murray.

Ross R, 1915. Some a priori pathometric equations. Br Med J, 1: 546-447.

Ross R, 1916. An application of the theory of probabilities to the study of a priori pathometry- I. Proc R Soc, A92: 204-230.

Ross R, 1916. An application of the theory of probabilities to the study of a priori pathometry- II. Proc R Soc, A93: 212-225.

Ross R, Hudson HP, 1916. An application of the theory of probabilities to the study of a priori pathometry- III. Proc R Soc, A93: 225-240.

WHO, 2017. Diarrhoeal disease fact sheet. World Health Organization.