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Research Article

Harmonic Sections of Tangent Bundles with Horizontal Sasaki Gradient Metric

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Abstract
In this paper, we introduce harmonic sections of tangent bundles with horizontal Sasaki gradient metric, then we establish necessary and sufficient conditions under which a vector field is harmonic with respect to this metric. We also construct some examples of harmonic vector fields. After that, we study the harmonicity of the maps between a Riemannian manifold and the tangent bundle over another Riemannian manifold or vice versa.

Keywords and 2010 Mathematics Subject Classification
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1. Introduction

In this field, the geometry of the tangent bundle equipped with Sasaki metric has been studied by many authors S. Sasaki [1], K. Yano and S. Ishihara [2], P. Dombrowski [3], A.A Salimov, A. Gezer [4] etc. The rigidity of the Sasaki metric has incited some researchers to construct and study other metrics (in form different deformations of the Sasaki metric) on the tangent bundle. We cite them for example, the Cheeger-Gromoll metric [5], the Mus-Sasaki metric [6], the Deformed-Sasaki metric [7] and the Horizontal Sasaki gradient metric [8].

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X,Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y). \tag{1}$$

Here ∇ is the Riemannian connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi) = trace_g \nabla d\phi, \tag{2}$$

is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi) dv_g, \tag{3}$$

such that K is any compact of M, where

$$e(\phi) = \frac{1}{2} trace_g h(d\phi, d\phi), \tag{4}$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_K h(\tau(\phi),V)dv_g \tag{5}$$

Then ϕ is harmonic if and only if $\tau(\phi) = 0$.

See the special issue [9, 10] of harmonic maps. Also, other further developments which concern the harmonicity on tangent bundle are presented in [11–15].

In a previous work [8] we proposed "Geodesics on tangent bundles with horizontal Sasaki gradient metric". In this note we study the harmonicity on tangent bundle with the horizontal Sasaki gradient metric, then we establish necessary and sufficient conditions when a vector field is harmonic with respect to this metric (Theorem 6, Theorem 7 and Theorem 9). We also construct some examples of harmonic vector fields (Example 11 and Example 14). After that we study the harmonicity of the map $\sigma : (M^m, g) \longrightarrow (TN, h^H_f)$ (Theorem 16 and Theorem 17) and the map $\phi : (TM, g^H_f) \longrightarrow (N, h)$ (Theorem 19 and Theorem 20).

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=1,\dots,m}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1,\dots,m}$ on TM. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\mathfrak{T}^r_s(M)$ be the module over $C^\infty(M)$ of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

The Levi-Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM \tag{6}$$

of the tangent bundle to TM at any $(x, u) \in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial y^i}|_{(x,u)},\ \xi^i \in \mathbb{R}\}, \tag{7}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)},\ \xi^i \in \mathbb{R}\}. \tag{8}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{9}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i\{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \tag{10}$$

We have $(\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^i})^V)_{i=1,\dots,m}$ is a local adapted frame on TTM.

Lemma 1. *Let (M^m, g) be a Riemannian manifold. The Lie bracket of vertical and horizontal vector fields is given by the formulas*

1. $[X^H, Y^H] = [X, Y]^H - (R(X, Y)u)^V$,
2. $[X^H, Y^V] = (\nabla_X Y)^V$,
3. $[X^V, Y^V] = 0$,

for all vector fields $X, Y \in \mathfrak{T}^1_0(M)$, where ∇ and R denotes the Levi-Civita connection and the curvature tensor of (M^m, g), respectively [2, 3].

3. Horizontal Sasaki gradient metric and harmonicity

Definition 2. *Let (M^m,g) be a Riemannian manifold and $f\in C^\infty(M)$, $f>0$ be a strictly positive smooth function on M. On the tangent bundle TM, we define a horizontal Sasaki gradient metric noted g^H_f by*

1. $g^H_f(X^H,Y^H)=g(X,Y)+X(f)Y(f)$,
2. $g^H_f(X^H,Y^V)=0$,
3. $g^H_f(X^V,Y^V)=g(X,Y)$,

for all $X,Y\in\Im^1_0(M)$ [8].

Theorem 3. *Let (M^m,g) be a Riemannian manifold and (TM,g^H_f) its tangent bundle equipped with the horizontal Sasaki gradient metric. If ∇ (resp. $\widetilde{\nabla}$) denote the Levi-Civita connection of (M^m,g) (resp. (TM,g^H_f)), then we have:*

$$\begin{aligned}
\widetilde{\nabla}_{X^H}Y^H &= (\nabla_XY)^H+\frac{1}{\alpha}Hess_f(X,Y)(grad\,f)^H-\frac{1}{2}(R(X,Y)u)^V,\\
\widetilde{\nabla}_{X^H}Y^V &= (\nabla_XY)^V-\frac{1}{2\alpha}g(R(u,Y)X,grad\,f)(grad\,f)^H+\frac{1}{2}(R(u,Y)X)^H,\\
\widetilde{\nabla}_{X^V}Y^H &= \frac{1}{2}(R(u,X)Y)^H-\frac{1}{2\alpha}g(R(u,X)Y,grad\,f)(grad\,f)^H,\\
\widetilde{\nabla}_{X^V}Y^V &= 0,
\end{aligned}$$

for all $X,Y\in\Im^1_0(M)$, $\alpha=1+\|grad\,f\|^2$ and $Hess_f(X,Y)=g(\nabla_X grad\,f,Y)$ is the Hessian of f, where R denote the curvature tensor of (M^m,g) [8].

3.1 Harmonicity of a vector field $X:(M^m,g)\longrightarrow(TM,g^H_f)$

Lemma 4. *Let (M^m,g) be a Riemannian manifold. If $X,Y\in\Im^1_0(M)$ are vector fields on M and $(x,u)\in TM$ such that $Y_x=u$, then we have:*

$$d_xY(X_x)=X^H_{(x,u)}+(\nabla_XY)^V_{(x,u)}.$$

Lemma 5. *Let (M^m,g) be a Riemannian manifold and (TM,g^H_f) its tangent bundle equipped with the horizontal Sasaki gradient metric. If $X\in\Im^1_0(M)$, then the energy density associated to X is given by:*

$$e(X)=\frac{m+\alpha-1}{2}+\frac{1}{2}trace_g g(\nabla X,\nabla X), \tag{11}$$

where $\alpha=1+\|grad\,f\|^2$ [14, 16].

Proof. Let $X\in\Im^1_0(M)$ and $(E_i)_{i=1,\dots,m}$ be a local orthonormal frame on M, then:

$$e(X)=\frac{1}{2}trace_g g^H_f(dX,dX)=\frac{1}{2}\sum_{i=1}^m g^H_f(dX(E_i),dX(E_i)).$$

Using Lemma 4, we obtain:

$$\begin{aligned}
e(X) &= \frac{1}{2}\sum_{i=1}^m g^H_f(E^H_i+(\nabla_{E_i}X)^V,E^H_i+(\nabla_{E_i}X)^V)\\
&= \frac{1}{2}\sum_{i=1}^m \big((g^H_f(E^H_i,E^H_i)+g^H_f((\nabla_{E_i}X)^V,(\nabla_{E_i}X)^V))\big)\\
&= \frac{1}{2}\sum_{i=1}^m \big((g(E_i,E_i)+(E_i(f))^2+g(\nabla_{E_i}X,\nabla_{E_i}X)\big)\\
&= \frac{1}{2}(m+\|grad\,f\|^2)+\frac{1}{2}trace_g g(\nabla X,\nabla X)\\
&= \frac{1}{2}(m+\alpha-1)+\frac{1}{2}trace_g g(\nabla X,\nabla X).
\end{aligned}$$

■

Theorem 6. *Let (M^m, g) be a Riemannian manifold and (TM, g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric. If $X \in \Im_0^1(M)$, then the tension field associated to X is given by:*

$$\tau(X) = \big(\frac{1}{\alpha}\Delta(f) grad\, f + trace_g A(X)\big)^H + \big(trace_g \nabla^2 X\big)^V. \tag{12}$$

where $A(X)$ is a bilinear map defined by:

$$A(X) = R(X, \nabla X)* - \frac{1}{\alpha} g(R(X, \nabla X)*, grad\, f) grad\, f,$$

and $\Delta(f) = trace_g Hess_f$ is the Laplacian of f.

Proof. Let $x \in M$ and $\{E_i\}_{i=1,\dots,m}$ be a local orthonormal frame on M such that $(\nabla_{E_i} E_i)_x = 0$ and $X_x = u$, then

$$\begin{aligned}
\tau(X)_x &= \sum_{i=1}^{m} \{\nabla^X_{E_i} dX(E_i) - dX(\nabla_{E_i} E_i)\}_x \\
&= \sum_{i=1}^{m} \{\widetilde{\nabla}_{dX(E_i)} dX(E_i)\}_{(x,u)} \\
&= \sum_{i=1}^{m} \{\widetilde{\nabla}_{(E_i^H + (\nabla_{E_i} X)^V)} (E_i^H + (\nabla_{E_i} X)^V)\}_{(x,u)} \\
&= \sum_{i=1}^{m} \{\widetilde{\nabla}_{E_i^H} E_i^H + \widetilde{\nabla}_{E_i^H} (\nabla_{E_i} X)^V + \widetilde{\nabla}_{(\nabla_{E_i} X)^V} (E_i)^H + \widetilde{\nabla}_{(\nabla_{E_i} X)^V} (\nabla_{E_i} X)^V\}_{(x,u)}.
\end{aligned}$$

Using Theorem 3, we obtain

$$\begin{aligned}
\tau(X) &= \sum_{i=1}^{m} \big((\nabla_{E_i} E_i)^H + \frac{1}{\alpha} Hess_f(E_i, E_i)(grad\, f)^H - \frac{1}{2}(R(E_i, E_i)X)^V + (\nabla_{E_i}\nabla_{E_i} X)^V + \frac{1}{2}(R(X, \nabla_{E_i} X)E_i)^H \\
&\quad - \frac{1}{2\alpha} g(R(X, \nabla_{E_i} X)E_i, grad\, f)(grad\, f)^H + \frac{1}{2}(R(X, \nabla_{E_i} X)E_i)^H - \frac{1}{2\alpha} g(R(X, \nabla_{E_i} X)E_i, grad\, f)(grad\, f)^H\big) \\
&= \sum_{i=1}^{m} \big(\frac{1}{\alpha} Hess_f(E_i, E_i)(grad\, f)^H + (R(X, \nabla_{E_i} X)E_i)^H - \frac{1}{\alpha} g(R(X, \nabla_{E_i} X)E_i, grad\, f)(grad\, f)^H + (\nabla_{E_i}\nabla_{E_i} X)^V\big) \\
&= \big(\frac{1}{\alpha}\Delta(f) grad\, f + trace_g A(X)\big)^H + \big(trace_g \nabla^2 X\big)^V,
\end{aligned}$$

where

$$trace_g \nabla^2 X = \sum_{i=1}^{m} \{\nabla_{E_i}\nabla_{E_i} X - \nabla_{(\nabla_{E_i} E_i)} X\} = \sum_{i=1}^{m} \nabla_{E_i}\nabla_{E_i} X,$$

$$trace_g Hess_f = \sum_{i=1}^{m} Hess_f(E_i, E_i) = \Delta(f)$$

and $A(X) = R(X, \nabla X)* - \frac{1}{\alpha} g(R(X, \nabla X)*, grad\, f) grad\, f$. ∎

Theorem 7. *Let (M^m, g) be a Riemannian manifold and (TM, g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric. If $X \in \Im_0^1(M)$, then X is a harmonic vector field if and only if the following conditions are verified*

$$trace_g\big(\frac{1}{\alpha} g(R(X, \nabla X)*, grad\, f) grad\, f - R(X, \nabla X)*\big) = \frac{1}{\alpha}\Delta(f) grad\, f, \tag{13}$$

and

$$trace_g \nabla^2 X = 0. \tag{14}$$

Proof. The statement is a direct consequence of Theorem 6. ∎

Corollary 8. *Let (M^m, g) be a Riemannian manifold, (TM, g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric and $\Delta(f) grad\, f = 0$. If $X \in \Im_0^1(M)$ is a parallel vector field (i.e. $\nabla X = 0$), then X is harmonic.*

Theorem 9. *Let (M^m, g) be a Riemannian compact manifold and (TM, g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric. If $X \in \Im_0^1(M)$ is a harmonic vector field then X is parallel.*

Proof. Let φ_t be a compactly supported variation of X defined by:

$$\begin{array}{rcl} \mathbb{R}\times M & \longrightarrow & T_xM \\ (t,x) & \longmapsto & \varphi_t(x)=(1+t)X_x. \end{array}$$

From Lemma 5 we have:

$$e(\varphi_t) = \frac{1}{2}(m+\alpha-1)+\frac{(1+t)^2}{2}trace_g g(\nabla X,\nabla X),$$

$$E(\varphi_t) = \frac{1}{2}(m+\alpha-1)Vol(M)+\frac{(1+t)^2}{2}\int_M trace_g g(\nabla X,\nabla X)dv_g.$$

X is harmonic, then we have:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big(\frac{1}{2}(m+\alpha-1)Vol(M)+\frac{(1+t)^2}{2}\int_M trace_g g(\nabla X,\nabla X)dv_g\Big)_{t=0} \\ &= \int_M trace_g g(\nabla X,\nabla X)dv_g \end{aligned}$$

which gives

$$g(\nabla X,\nabla X)=0,$$

from which the result follows. ■

Corollary 10. *Let (M^m,g) be a Riemannian compact manifold, (TM,g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric and $\Delta(f)=0$ or $grad\, f=0$. If $X\in\Im_0^1(M)$, then X is a harmonic vector field if and only if X is parallel.*

Proof. The statement is a direct consequence of Corollary 8 and Theorem 7. ■

Example 11. *Let $\mathbb{S}^1$ (Riemannian compact manifold) equipped with the metric:*

$$g_{\mathbb{S}^1}=e^x dx^2.$$

The Christoffel symbols of the Reimannian connection are given by:

$$\Gamma_{11}^1=\frac{1}{2}g^{11}\Big(\frac{\partial g_{11}}{\partial x_1}+\frac{\partial g_{11}}{\partial x_1}-\frac{\partial g_{11}}{\partial x_1}\Big)=\frac{1}{2}.$$

Using the Corollary 10, if $\Delta(f).grad\, f=0$, the vector field $X=h(x)\frac{d}{dx}$, $h\in\mathscr{C}^\infty(\mathbb{S}^1)$ is harmonic if and only if X is parallel, then

$$grad\, f=0\Leftrightarrow f \text{ is constant}$$

or

$$\Delta(f)=0\Leftrightarrow f''(x)-\frac{1}{2}f'(x)=0\Leftrightarrow f(x)=a.\exp(\frac{x}{2})+b,$$

where $a,b\in\mathbb{R}$, $a\neq 0$. Hence, X is parallel gives,

$$\nabla X=0\Leftrightarrow h'(x)+\frac{1}{2}h(x)=0\Leftrightarrow h(x)=k.\exp(-\frac{x}{2})\ \Leftrightarrow X=k\exp(-\frac{x}{2})\frac{d}{dx},$$

where $k\in\mathbb{R}$. Finally for f is constant or $f(x)=a.\exp(\frac{x}{2})+b$, the vector field $X=k.\exp(-\frac{x}{2})\frac{d}{dx}$ is harmonic.

Remark 12. *In general , using Corollary 8 and Corollary 10 , we can construct many examples for harmonic vector fields.*

Corollary 13. *Let (M^m,g) be a flat Riemannian manifold and (TM,g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric. If $X\in\Im_0^1(M)$, then X is harmonic vector field if and only if the following conditions are verified*

$$\Delta(f)=0 \text{ or } grad\, f=0, \tag{15}$$

and

$$trace_g\nabla^2X=0. \tag{16}$$

Proof. The statement is a direct consequence of Theorem 7. ■

Example 14. *Let $\mathbb{R}^m$ equipped with the canonical metric (flat manifold and not compact), $T\mathbb{R}^m$ its tangent bundle equipped with the horizontal Sasaki gradient metric g_f^H and the vector field :*

$$\begin{aligned} X:\mathbb{R}^m &\longrightarrow T\mathbb{R}^m\\ x=(x_1,\cdots,x_m) &\longmapsto X_x=(X_x^1,\cdots,X_x^m). \end{aligned}$$

Using the Corollary 13, for $\Delta(f)=0$ or $grad\, f=0$ we have:

$$trace_g\nabla^2X=(\sum_{i=1}^m\frac{\partial^2X^1}{\partial x_i^2},\cdots,\sum_{i=1}^m\frac{\partial^2X^n}{\partial x_i^2})=(\Delta(X^1),\cdots,\Delta(X^m))$$

1. If X is constant, then X is harmonic ($trace_g\nabla^2X=0$).
2. If $X^i=a_ix_i$ and $a_i\neq0$, then X is harmonic ($trace_g\nabla^2X=0$) but

$$\nabla X=\sum_i a_i\frac{\partial}{\partial x_i}\otimes dx_i\neq0.$$

indeed

$$\nabla X(\frac{\partial}{\partial x_j})=\sum_i a_i(\frac{\partial}{\partial x_i}\otimes dx_i)(\frac{\partial}{\partial x_j})=\sum_i\delta_{ij}a_i\frac{\partial}{\partial x_i}=a_j\frac{\partial}{\partial x_j}\neq0.$$

3.2 Harmonicity of the map $\sigma:(M^m,g)\longrightarrow(TN,h_f^H)$

Lemma 15. *Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between Riemannian manifolds and*

$$\begin{aligned} \sigma: M &\longrightarrow TN\\ x &\longmapsto (Y\circ\varphi)(x)=(\varphi(x),Y_{\varphi(x)}) \end{aligned} \tag{17}$$

a smooth map, such that Y be a vector field on N. Then

$$d\sigma(X)=(d\varphi(X))^H+(\nabla_X^\varphi\sigma)^V$$

for all $X\in\Im_0^1(M)$.

Proof. Let $x\in M$ and $v\in T_{\varphi(x)}N$, such that $v=Y_{\varphi(x)}$, for any vector field X on M. Using Lemma 5, we obtain

$$\begin{aligned} d_x\sigma(X_x) &= d_x(Y\circ\varphi)(X_x)\\ &= d_{\varphi(x)}Y(d_x\varphi(X_x))\\ &= (d\varphi(X))^H_{(\varphi(x),v)}+(\nabla_{d\varphi(X)}Y)^V_{(\varphi(x),v)}\\ &= (d\varphi(X))^H_{(\varphi(x),v)}+(\nabla_X^\varphi\sigma)^V_{(\varphi(x),v)}. \end{aligned}$$

■

Theorem 16. *Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between the Riemannian manifolds (M^m,g), (N^n,h) and $f\in C^\infty(N)$, $f>0$ be a strictly positive smooth function on N. Let (TN,h_f^H) the tangent bundle of N equipped with the horizontal Sasaki gradient metric. The tension field of the map $\sigma:(M^m,g)\longrightarrow(TN,h_f^H)$ defined by (17) is given by*

$$\tau(\sigma)=\big(\tau(\varphi)+trace_gA(\sigma)\big)^H+\big(trace_g(\nabla^\varphi)^2\sigma\big)^V, \tag{18}$$

where $A(\sigma)$ is a bilinear map defined by

$$A(\sigma)=\frac{1}{\alpha}\big(Hess_f(d\varphi(*),d\varphi(*))-h(R^N(\sigma,\nabla^\varphi\sigma)d\varphi(*),grad\, f)\big)grad\, f+R^N(\sigma,\nabla^\varphi\sigma)d\varphi(*).$$

Proof. Let $x \in M$ and $\{E_i\}_{i=1,\ldots,m}$ be a local orthonormal frame on M such that $(\nabla^M_{E_i} E_i)_x = 0$ and $\sigma(x) = Y_{\varphi(x)} = v \in T_{\varphi(x)}N$. We have

$$\begin{aligned}
\tau(\sigma)_x &= trace_g(\nabla d\sigma)_x \\
&= \sum_{i=1}^{m} \{\widetilde{\nabla}_{d\sigma(E_i)} d\sigma(E_i)\}_{(\varphi(x),v)} \\
&= \sum_{i=1}^{m} \{\widetilde{\nabla}_{(d\varphi(E_i))^H} (d\varphi(E_i))^H + \widetilde{\nabla}_{(d\varphi(E_i))^H} (\nabla^{\varphi}_{E_i}\sigma)^V + \widetilde{\nabla}_{(\nabla^{\varphi}_{E_i}\sigma)^V} (d\varphi(E_i))^H + \widetilde{\nabla}_{(\nabla^{\varphi}_{E_i}\sigma)^V} (\nabla^{\varphi}_{E_i}\sigma)^V\}_{(\varphi(x),v)}.
\end{aligned}$$

From the Theorem 3, we obtain:

$$\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^{m} \Big((\nabla^{\varphi}_{E_i} d\varphi(E_i))^H + \frac{1}{\alpha} Hess_f(d\varphi(E_i), d\varphi(E_i))(grad\, f)^H + (\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\sigma)^V \\
&\quad - \frac{1}{\alpha} h(R^N(\sigma, \nabla^{\varphi}_{E_i}\sigma) d\varphi(E_i), grad\, f)(grad\, f)^H + (R^N(\sigma, \nabla^{\varphi}_{E_i}\sigma) d\varphi(E_i))^H \Big) \\
&= \big(\tau(\varphi) + trace_g \frac{1}{\alpha} Hess_f(d\varphi(*), d\varphi(*)) grad\, f - trace_g \frac{1}{\alpha} h(R^N(\sigma, \nabla^{\varphi}\sigma) d\varphi(*), grad\, f) grad\, f \\
&\quad + trace_g \big(R^N(\sigma, \nabla^{\varphi}\sigma) d\varphi(*)\big)\big)^H + \big(trace_g (\nabla^{\varphi})^2 \sigma\big)^V \\
&= \big(\tau(\varphi) + trace_g A(\sigma)\big)^H + \big(trace_g (\nabla^{\varphi})^2 \sigma\big)^V,
\end{aligned}$$

where $A(\sigma)$ is a bilinear map defined by

$$A(\sigma) = \frac{1}{\alpha}\big(Hess_f(d\varphi(*), d\varphi(*)) - h(R^N(\sigma, \nabla^{\varphi}\sigma) d\varphi(*), grad\, f)\big) grad\, f + R^N(\sigma, \nabla^{\varphi}\sigma) d\varphi(*).$$

∎

From Theorem 16 we obtain the following theorem:

Theorem 17. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between the Riemannian manifolds (M^m, g), (N^n, h) and $f \in C^{\infty}(N)$, $f > 0$ be a strictly positive smooth function on N. Let (TN, h^H_f) the tangent bundle of N equipped with the horizontal Sasaki gradient metric. The map $\sigma : (M^m, g) \longrightarrow (TN, h^H_f)$ defined by (17) is a harmonic if and only if the following conditions are verified*

$$\tau(\varphi) = -trace_g \Big(\frac{1}{\alpha}\big(Hess_f(d\varphi(*), d\varphi(*)) - h(R^N(\sigma, \nabla^{\varphi}\sigma) d\varphi(*), grad\, f)\big) grad\, f + R^N(\sigma, \nabla^{\varphi}\sigma) d\varphi(*)\Big), \tag{19}$$

and

$$trace_g (\nabla^{\varphi})^2 \sigma = 0. \tag{20}$$

3.3 Harmonicity of the map $\phi : (TM, g^H_f) \longrightarrow (N, h)$

Lemma 18. *Let (M^m, g) be a Riemannian manifold and (TM, g^H_f) its tangent bundle equipped with the horizontal Sasaki gradient metric. The tension field of the canonical projection $\pi : (TM, g^H_f) \longrightarrow (M^m, g)$ is given by:*

$$\tau(\pi) = \frac{1}{\alpha}\big(\frac{1}{\alpha} Hess_f(grad\, f, grad\, f) - \Delta(f)\big)(grad\, f) \circ \pi, \tag{21}$$

and $\alpha = 1 + \|grad\, f\|^2$.

Proof. Let $\{E_i\}_{i=1,\ldots,m}$ be a local orthonormal frame on M and $\{\widetilde{E}_j\}_{j=1,\ldots,2m}$ be a local frame on TM, where

$$\widetilde{E}_j = \begin{cases} E^H_j & , \quad 1 \le j \le m \\ E^V_{j-m} & , \quad m+1 \le j \le 2m \end{cases} \tag{22}$$

The tension field of π is given by

$$\begin{aligned}
\tau(\pi) &= trace_{g^H_f} \nabla d\pi \\
&= \sum_{i,j=1}^{2m} G^{ij} \big(\nabla^M_{d\pi(\widetilde{E}_i)} d\pi(\widetilde{E}_j) - d\pi(\nabla^{TM}_{\widetilde{E}_i} \widetilde{E}_j)\big)
\end{aligned}$$

where $G = g_f^H$ and its matrix is (G_{ij}) such that:

$$\begin{cases} G_{ij} = \delta_{ij} + E_i(f)E_j(f) & , \quad 1 \le i,j \le m \\ G_{ij} = 0 & , \quad 1 \le i \le m, m+1 \le j \le 2m \\ G_{ij} = \delta_{ij} & , \quad m+1 \le i,j \le 2m \end{cases}$$

and (G^{ij}) is the inverse matrix of the matrix (G_{ij}) such that:

$$\begin{cases} G^{ij} = \delta_{ij} - \frac{1}{\alpha}E_i(f)E_j(f) & , \quad 1 \le i,j \le m \\ G^{ij} = 0 & , \quad 1 \le i \le m, m+1 \le j \le 2m \\ G^{ij} = \delta_{ij} & , \quad m+1 \le i,j \le 2m \end{cases}$$

then

$$\tau(\pi) = \sum_{i,j=1}^{m} \big(\delta_{ij} - \frac{1}{\alpha}E_i(f)E_j(f)\big)\big(\nabla^M_{d\pi(E_i^H)} d\pi(E_j^H) - d\pi(\nabla^{TM}_{E_i^H} E_j^H)\big) + \sum_{i,j=m+1}^{2m} \delta_{ij}\big(\nabla^M_{d\pi(E^V_{i-m})} d\pi(E^V_{j-m}) - d\pi(\nabla^{TM}_{E^V_{i-m}} E^V_{j-m})\big)$$

as $d\pi(X^V) = 0$ and $d\pi(X^H) = X \circ \pi$ for any $X \in \Im^1_0(M)$. So we have:

$$\begin{aligned} \tau(\pi) &= \sum_{i,j=1}^{m} \big(\delta_{ij} - \frac{1}{\alpha}E_i(f)E_j(f)\big)\Big(\nabla^M_{(E_i\circ\pi)}(E_j\circ\pi) - d\pi\big((\nabla^M_{E_i}E_j)^H \\ &\quad + \frac{1}{\alpha}Hess_f(E_i,E_j)(grad\, f)^H - \frac{1}{2}(R(E_i,E_j)u)^V\big)\Big) \\ &= \sum_{i,j=1}^{m} \big(\delta_{ij} - \frac{1}{\alpha}E_i(f)E_j(f)\big)\Big((\nabla^M_{E_i}E_j)\circ\pi - d\pi\big((\nabla^M_{E_i}E_j)^H\big) - \frac{1}{\alpha}Hess_f(E_i,E_j)d\pi\big((grad\, f)^H\big)\Big) \\ &= -\frac{1}{\alpha}\sum_{i,j=1}^{m}(\delta_{ij} - \frac{1}{\alpha}E_i(f)E_j(f))Hess_f(E_i,E_j)(grad\, f)\circ\pi \\ &= -\frac{1}{\alpha}\sum_{i,j=1}^{m}\delta_{ij}Hess_f(E_i,E_j)(grad\, f)\circ\pi + \frac{1}{\alpha^2}\sum_{i,j=1}^{m}E_i(f)E_j(f)Hess_f(E_i,E_j)(grad\, f)\circ\pi \\ &= -\frac{1}{\alpha}\Delta(f)(grad\, f)\circ\pi + \frac{1}{\alpha^2}Hess_f(grad\, f, grad\, f)(grad\, f)\circ\pi \\ &= \frac{1}{\alpha}\big(\frac{1}{\alpha}Hess_f(grad\, f, grad\, f) - \Delta(f)\big)(grad\, f)\circ\pi. \end{aligned}$$

■

Theorem 19. *Let $\varphi : (M^m,g) \to (N^n,h)$ be a smooth map between the Riemannian manifolds (M^m,g), (N^n,h) and $f : M \to]0,+\infty[$ be a strictly positive smooth function on M. Let (TM, g_f^H) the tangent bundle of M equipped with the horizontal Sasaki gradient metric. The tension field of the map*

$$\begin{aligned} \phi : (TM, g_f^H) &\longrightarrow (N,h) \\ (x,y) &\longmapsto \varphi(x) \end{aligned} \qquad (23)$$

is given by:

$$\tau(\phi) = \big(\tau(\varphi) - \frac{1}{\alpha}\nabla d\varphi(grad\, f, grad\, f)\big)\circ\pi + \frac{1}{\alpha}\big(\frac{1}{\alpha}Hess_f(grad\, f, grad\, f) - \Delta(f)\big)d\varphi(grad\, f)\circ\pi \qquad (24)$$

and $\alpha = 1 + \|grad\, f\|^2$.

Proof. Let $\{E_i\}_{i=1,\dots,m}$ be a local orthonormal frame on M and $\{\widetilde{E}_j\}_{j=1,\dots,2m}$ be a local frame on TM defined by (22), as ϕ is defined by $\phi = \varphi\circ\pi$, we have:

$$\begin{aligned} \tau(\phi) &= \tau(\varphi\circ\pi) \\ &= d\varphi(\tau(\pi)) + trace_{g^f}\nabla d\varphi(d\pi, d\pi) \end{aligned}$$

$$\begin{aligned}
trace_{g^f}\nabla d\varphi(d\pi,d\pi) &= \sum_{i,j=1}^{2m} G^{ij}\Big(\nabla^N_{d\varphi(d\pi(\widetilde{E}_i))}d\varphi(d\pi(\widetilde{E}_j))-d\varphi(\nabla^M_{d\pi(\widetilde{E}_i)}d\pi(\widetilde{E}_j))\Big)\\
&= \sum_{i,j=1}^{m}\Big(\delta_{ij}(\nabla^N_{d\varphi(E_i)}d\varphi(E_j)-d\varphi(\nabla^M_{E_i}E_j))-\frac{1}{\alpha}E_i(f)E_j(f)\big(\nabla^N_{d\varphi(E_i)}d\varphi(E_j)-d\varphi(\nabla^M_{E_i}E_j)\big)\Big)\circ\pi\\
&= \sum_{i=1}^{m}\Big(\nabla^N_{d\varphi(E_i)}d\varphi(E_i)-d\varphi(\nabla^M_{E_i}E_i)\Big)\circ\pi-\frac{1}{\alpha}\big(\nabla^N_{d\varphi(grad\,f)}d\varphi(grad\,f)-d\varphi(\nabla^M_{grad\,f}grad\,f)\big)\circ\pi\\
&= \big(\tau(\varphi)-\frac{1}{\alpha}\nabla d\varphi(grad\,f,grad\,f)\big)\circ\pi.
\end{aligned}$$

Using Lemma 18, we obtain:

$$\tau(\phi) = \big(\tau(\varphi)-\frac{1}{\alpha}\nabla d\varphi(grad\,f,grad\,f)\big)\circ\pi+\frac{1}{\alpha}\big(\frac{1}{\alpha}Hess_f(grad\,f,grad\,f)-\Delta(f)\big)d\varphi(grad\,f)\circ\pi.$$

■

Theorem 20. *Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between the Riemannian manifolds (M^m,g), (N^n,h) and $f:M\to]0,+\infty[$ be a strictly positive smooth function on M. Let (TM,g^H_f) the tangent bundle of M equipped with the horizontal Sasaki gradient metric, the map*

$$\begin{aligned}
\phi:(TM,g^H_f) &\longrightarrow (N,h)\\
(x,y) &\longmapsto \varphi(x)
\end{aligned}$$

is a harmonic if and only if

$$\tau(\varphi)=\frac{1}{\alpha}\nabla d\varphi(grad\,f,grad\,f)\circ\pi-\frac{1}{\alpha}\big(\frac{1}{\alpha}Hess_f(grad\,f,grad\,f)-\Delta(f)\big)d\varphi(grad\,f)\circ\pi$$

and $\alpha=1+\|grad\,f\|^2$.

4. Conclusions

In this work, first, we studied the harmonicity of a tangent bundle with the horizontal Sasaki gradient metric and we gave the necessary and sufficient conditions when a vector field is harmonic with respect to this metric. Secondly, we searched the harmonicity of the maps between a Riemannian manifold and the tangent bundle over another Riemannian manifold or vice versa. In future works, we can study the harmonicity of an another metrics on the tangent bundle by deformation in the vertical bundle or in the horizontal bundle.

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