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The Representation and Finite Sums of the Padovan-p Jacobsthal Numbers

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Abstract. In this paper, we regard the Padovan-p Jacobsthal sequence and then we discuss the connection of the Padovan-p Jacobsthal numbers and Jacobsthal numbers. Furthermore, we give the permanental, determinantal, combinatorial, and exponential representations, and the sums of the Padovan-p Jacobsthal numbers by the aid of the generating function and generating matrix of this sequence.

1. Introduction

The well-known Jacobsthal sequence $\{J_n\}$ is defined by the following recurrence relation:

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 2$ in which $J_0 = 0$ and $J_1 = 1$. It is easy to see that the characteristic polynomial of the Jacobsthal sequence is $j(x) = x^2 - x - 2$.

In [2], Aküzüm defined the Padovan-p Jacobsthal sequence $\left\{J_n^p\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(3, 4, 5, \ldots)$ and $n \geq 0$

$$J_{n+p+4}^p = J_{n+p+3}^p + 3J_{n+p+2}^p - J_{n+p+1}^p - 2J_{n+p}^p + J_{n+2}^p - J_{n+1}^p - 2J_n^p$$

in which $J_0^p = \cdots = J_{p+2}^p = 0$ and $J_{p+3}^p = 1$.

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Also in [2], she gave the generating matrix of the Padovan-p Jacobsthal sequence $\left\{J_n^p\right\}$ as follows:

$$PJ_p = \begin{bmatrix} 1 & 3 & -1 & -2 & 0 & \cdots & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+4)\times(p+4)}.$$

The matrix PJ_p is entitled a Padovan-p Jacobsthal matrix. By an inductive argument, she obtained that

$$\left(PJ_p\right)^n = \begin{bmatrix} J_{n+p+3}^p & J_{n+p+4}^p - J_{n+p+3}^p & Pap(n+p+3) - J_{n+p+3}^p & Pap(n+p+4) - J_{n+p+4}^p - J_{n+p+3}^p & \\ J_{n+p+2}^p & J_{n+p+3}^p - J_{n+p+2}^p & Pap(n+p+2) - J_{n+p+2}^p & Pap(n+p+3) - J_{n+p+3}^p - J_{n+p+2}^p & \\ J_{n+p+1}^p & J_{n+p+2}^p - J_{n+p+1}^p & Pap(n+p+1) - J_{n+p+1}^p & Pap(n+p+2) - J_{n+p+2}^p - J_{n+p+1}^p & PJ_p^* \\ \vdots & \vdots & \vdots & \vdots & \\ J_{n+1}^p & J_{n+2}^p - J_{n+1}^p & Pap(n+1) - J_{n+1}^p & Pap(n+2) - J_{n+2}^p - J_{n+1}^p & \\ J_n^p & J_{n+1}^p - J_n^p & Pap(n) - J_n^p & Pap(n+1) - J_{n+1}^p - J_n^p & \end{bmatrix},$$

where PJ_p^* is a $(p+4)\times(p)$ matrix as follows:

$$PJ_p^* = \begin{bmatrix} Pap(n+3) & Pap(n+4) & \cdots & Pap(n+p) & -J_{n+p+2}^p - 2J_{n+p+1}^p & -2J_{n+p+2}^p \\ Pap(n+2) & Pap(n+3) & \cdots & Pap(n+p-1) & -J_{n+p+1}^p - 2J_{n+p}^p & -2J_{n+p+1}^p \\ Pap(n+1) & Pap(n+2) & \cdots & Pap(n+p-2) & -J_{n+p}^p - 2J_{n+p-1}^p & -2J_{n+p}^p \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ Pap(n-p+1) & Pap(n-p+2) & \cdots & Pap(n-2) & -J_n^p - 2J_{n-1}^p & -2J_n^p \\ Pap(n-p) & Pap(n-p+1) & \cdots & Pap(n-3) & -J_{n-1}^p - 2J_{n-2}^p & -2J_{n-1}^p \end{bmatrix}$$

for $n \geq p$.

In the literature, many authors studied number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper; see for example, [5, 7, 8, 14, 15]. In [1, 3, 4, 10–13, 16–20, 23], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we investigate the Padovan-p Jacobsthal sequence. Firstly, we discuss connections between the Jacobsthal and Padovan-p Jacobsthal numbers. Furthermore, we derive the permanental and determinantal representations of the Padovan-p Jacobsthal numbers by using certain matrices which are obtained from the generating matrix of this sequence. Finally, we acquire the combinatorial and exponential representations and the sums of the Padovan-p Jacobsthal numbers by the aid of the generating function and the generating matrix of this sequence.

2. Main Results

First, we derive a relationship between the above-described Padovan-p Jacobsthal sequence and Jacobsthal sequence.

Theorem 2.1. *Let $J(n)$ and J_n^p be the nth the Jacobsthal number and Padovan-p Jacobsthal numbers, respectively. Then,*

$$J(n) = J_{n+p+2}^p - J_{n+p}^p - J_n^p$$

for $n \geq 0$ and $p \geq 3$.

Proof. The assertion may be proved by induction method on n. It is clear that $J(0) = J^p_{p+2} - J^p_p - J^p_0 = 0$. Assume that the equation holds for $n \geq 1$. Then we must show that the equation holds for $n+1$. Since the characteristic polynomial of the Jacobsthal sequence $\{J(n)\}$, is

$$j(x) = x^2 - x - 2$$

we obtain the following relations:

$$J(n+p+4) = J(n+p+3) + 3J(n+p+2) - J(n+p+1) - 2J(n+p) + J(n+2) - J(n+1) - 2J(n)$$

for $n \geq 1$. Hence, by a simple calculation, we have the conclusion. □

Now we take into account the relationship between the Padovan-p Jacobsthal numbers and the permanents of a certain matrix which is obtained using the Padovan-p Jacobsthal matrix $\left(PJ_p\right)^n$.

Definition 2.2. *A $u \times v$ real matrix $M = \left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [6], Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Padovan-p Jacobsthal numbers and the permanents of certain matrices which are obtained by using the generating matrix of this sequence. Let $F^{Pa,J}_{m,p} = \left[f^{(p)}_{i,j}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$f^{(p)}_{i,j} = \begin{cases} 3 & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq m-1, \\ 1 & \begin{array}{c} \text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m, \\ i = \tau \text{ and } j = \tau + p + 1 \text{ for } 1 \leq \tau \leq m-p-1 \\ \text{and} \\ i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m-1, \end{array} \\ -1 & \begin{array}{c} \text{if } i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \leq \tau \leq m-2 \\ \text{and} \\ i = \tau \text{ and } j = \tau + p + 2 \text{ for } 1 \leq \tau \leq m-p-2, \end{array} \\ -2 & \begin{array}{c} \text{if } i = \tau \text{ and } j = \tau + 3 \text{ for } 1 \leq \tau \leq m-3 \\ \text{and} \\ i = \tau \text{ and } j = \tau + p + 3 \text{ for } 1 \leq \tau \leq m-p-3, \end{array} \\ 0 & \text{otherwise.} \end{cases},$$

for $m \geq p+4$. Then we have the following Theorem.

Theorem 2.3. *For $m \geq p+4$,*

$$perF^{Pa,J}_{m,p} = J^p_{m+p+3}.$$

Proof. Let us keep in view matrix $F^{Pa,J}_{m,p}$ and let the equation be hold for $m \geq p+4$. Then we show that the equation holds for $m+1$. If we expand the $perF^{Pa,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perF^{Pa,J}_{m+1,p} = perF^{Pa,J}_{m,p} + 3perF^{Pa,J}_{m-1,p} - perF^{Pa,J}_{m-2,p} - 2perF^{Pa,J}_{m-3,p} + perF^{Pa,J}_{m-p-1,p} - perF^{Pa,J}_{m-p-2,p} - 2perF^{Pa,J}_{m-p-3,p}.$$

Since

$$perF^{Pa,J}_{m,p} = J^{p}_{m+p+3},$$
$$perF^{Pa,J}_{m-1,p} = J^{p}_{m+p+2},$$
$$perF^{Pa,J}_{m-2,p} = J^{p}_{m+p+1},$$
$$perF^{Pa,J}_{m-3,p} = J^{p}_{m+p},$$
$$perF^{Pa,J}_{m-p-1,p} = J^{p}_{m+2},$$
$$perF^{Pa,J}_{m-p-2,p} = J^{p}_{m+1}$$

and

$$perF^{Pa,J}_{m-p-3,p} = J^{p}_{m},$$

we easily obtain that $perF^{Pa,J}_{m+1,p} = J^{p}_{m+p+4}$. So the proof is complete. □

Let $G^{Pa,J}_{m,p} = \left[g^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$g^{(p)}_{i,j} = \begin{cases} 3 & \text{if } i = \tau \text{ and } j = \tau + 1 \text{ for } 1 \le \tau \le m-2, \\ 1 & \begin{array}{c}\text{if } i = \tau \text{ and } j = \tau \text{ for } 1 \le \tau \le m, \\ i = \tau \text{ and } j = \tau + p + 1 \text{ for } 1 \le \tau \le m-p-2 \\ \text{and} \\ i = \tau + 1 \text{ and } j = \tau \text{ for } 1 \le \tau \le m-2,\end{array} \\ -1 & \begin{array}{c}\text{if } i = \tau \text{ and } j = \tau + 2 \text{ for } 1 \le \tau \le m-3 \\ \text{and} \\ i = \tau \text{ and } j = \tau + p + 2 \text{ for } 1 \le \tau \le m-p-3,\end{array} \\ -2 & \begin{array}{c}\text{if } i = \tau \text{ and } j = \tau + 3 \text{ for } 1 \le \tau \le m-4 \\ \text{and} \\ i = \tau \text{ and } j = \tau + p + 3 \text{ for } 1 \le \tau \le m-p-3,\end{array} \\ 0 & \text{otherwise.} \end{cases},$$

for $m \ge p + 4$. Then we have the following Theorem.

Theorem 2.4. *For $m \ge p + 4$,*

$$perG^{Pa,J}_{m,p} = J^{p}_{m+p+2}.$$

Proof. Let us keep in view matrix $G^{Pa,J}_{m,p}$ and let the equation be hold for $m \ge p + 4$. Then we show that the equation holds for $m + 1$. If we expand the $perG^{Pa,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perG^{Pa,J}_{m+1,p} = perG^{Pa,J}_{m,p} + 3perG^{Pa,J}_{m-1,p} - perG^{Pa,J}_{m-2,p} - 2perG^{Pa,J}_{m-3,p} + perG^{Pa,J}_{m-p-1,p} - perG^{Pa,J}_{m-p-2,p} - 2perG^{Pa,J}_{m-p-3,p}.$$

Since

$$perG^{Pa,J}_{m,p} = J^{p}_{m+p+2},$$
$$perG^{Pa,J}_{m-1,p} = J^{p}_{m+p+1},$$
$$perG^{Pa,J}_{m-2,p} = J^{p}_{m+p},$$
$$perG^{Pa,J}_{m-3,p} = J^{p}_{m+p-1},$$

$$perG^{Pa,J}_{m-p-1,p} = J^{p}_{m+1},$$

$$perG^{Pa,J}_{m-p-2,p} = J^{p}_{m}$$

and

$$perG^{Pa,J}_{m-p-3,p} = J^{p}_{m-1},$$

we easily obtain that $perG^{Pa,J}_{m+1,p} = J^{p}_{m+p+3}$. So the proof is complete. □

Suppose that $H^{Pa,J}_{m,p} = \left[h^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$H^{Pa,J}_{m,p} = \begin{bmatrix} & & (m-1)\text{ th} & \\ & & \downarrow & \\ 1 & \cdots & 1 & 0 \\ 1 & & & \\ 0 & & G^{Pa,J}_{m-1,p} & \\ \vdots & & & \\ 0 & & & \end{bmatrix},$$

for $m > p + 4$, then we have the following results:

Theorem 2.5. *For $m > p + 4$,*

$$perH^{Pa,J}_{m,p} = \sum_{i=0}^{m+p+1} J^{p}_{i}.$$

Proof. If we extend $per\, H^{Pa,J}_{m,p}$ with respect to the first row, we write

$$perH^{Pa,J}_{m,p} = perH^{Pa,J}_{m-1,p} + perG^{Pa,J}_{m-1,p}.$$

Thence, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1, -1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Padovan-p Jacobsthal numbers and the determinants of certain matrices which are obtained by using the matrices $F^{Pa,J}_{m,p}$, $G^{Pa,J}_{m,p}$ and $H^{Pa,J}_{m,p}$. Let $m > p + 4$ and let R be the $m \times m$ Hadamard matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.6. *For $m > p + 4$,*

$$\det\left(F^{Pa,J}_{m,p} \circ R\right) = J^{p}_{m+p+3},$$

$$\det\left(G^{Pa,J}_{m,p} \circ R\right) = J^{p}_{m+p+2}$$

and

$$\det\left(H^{Pa,J}_{m,p} \circ R\right) = \sum_{i=0}^{m+p+1} J^{p}_{i}.$$

Proof. *Since* $perF^{Pa,J}_{m,p} = \det\left(F^{Pa,J}_{m,p} \circ R\right)$, $perG^{Pa,J}_{m,p} = \det\left(G^{Pa,J}_{m,p} \circ R\right)$ *and* $perH^{Pa,J}_{m,p} = \det\left(H^{Pa,J}_{m,p} \circ R\right)$ *for* $m > p+4$, *by Theorem 2.3, Theorem 2.4 and Theorem 2.5, we have the conclusion.* □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For more details on the companion type matrices, see [21, 22].

Theorem 2.7. *(Chen and Louck [9]) The* (i, j) *entry* $k^{(n)}_{i,j}(k_1, k_2, \ldots, k_v)$ *in the matrix* $K^n(k_1, k_2, \ldots, k_v)$ *is given by the following formula:*

$$k^{(n)}_{i,j}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \quad (1)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ *is a multinomial coefficient, and the coefficients in (1) are defined to be* 1 *if* $n = i - j$.

Then we can give combinatorial representations for the Padovan-p Jacobsthal numbers by the following Corollary.

Corollary 2.8. *Let* J^p_n *be the nth the Padovan-p Jacobsthal number for* $n \geq p$. *Then*
i.

$$J^p_n = \sum_{(t_1, t_2, \ldots, t_{p+4})} \binom{t_1 + t_2 + \cdots + t_{p+4}}{t_1, t_2, \cdots, t_{p+4}} 3^{t_2} (-1)^{t_3 + t_{p+3}} (-2)^{t_4 + t_{p+4}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+4)t_{p+4} = n - p - 3$.
ii.

$$F^{Pa,p}_n = -\frac{1}{2} \sum_{(t_1, t_2, \ldots, t_4)} \frac{t_{p+4}}{t_1 + t_2 + \cdots + t_{p+4}} \times \binom{t_1 + t_2 + \cdots + t_{p+4}}{t_1, t_2, \cdots, t_{p+4}} 3^{t_2} (-1)^{t_3 + t_{p+3}} (-2)^{t_4 + t_{p+4}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+4)t_{p+4} = n + 1$.

Proof. If we take $i = p+4$, $j = 1$ for the case i. and $i = p+3$, $j = p+4$ for the case ii. in Theorem 2.7, then we can directly see the conclusions from $\left(PJ_p\right)^n$. □

The generating function of the Padovan-p Jacobsthal sequence $\left\{J^p_n\right\}$ is obtained as follows:

$$g(x) = \frac{x^{p+3}}{1 - x - 3x^2 + x^3 + 2x^4 - x^{p+2} + x^{p+3} + 2x^{p+4}},$$

where $p \geq 3$.

Then, with the following theorem, we can deliver an exponential representation for the Padovan-p Jacobsthal numbers by the aid of the generating function.

Theorem 2.9. *Let* $g(x)$ *be generating function of the Padovan-p Jacobsthal numbers. The following exponential representation for the Padovan-p Jacobsthal numbers as follows::*

$$g(x) = x^{p+3} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^i\right),$$

where $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+3} - \ln\left(1 - x - 3x^2 + x^3 + 2x^4 - x^{p+2} + x^{p+3} + 2x^{p+4}\right)$$

and

$$\begin{aligned} -\ln\left(1 - x - 3x^2 + x^3 + 2x^4 - x^{p+2} + x^{p+3} + 2x^{p+4}\right) &= -[-x\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right) - \\ &\frac{1}{2}x^2\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^2 - \cdots \\ &-\frac{1}{i}x^i\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+3}\exp\left(\sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^i\right)$$

by a simple calculation, we obtain the conclusion. □

Now we consider the sums of the Padovan-p Jacobsthal numbers. Let

$$T_n = \sum_{i=0}^{n} J_i^p$$

for $n \geq p$ and $p \geq 3$, and let $K_p^{Pa,J}$ and $\left(K_p^{Pa,J}\right)^n$ be the $(p+5)\times(p+5)$ matrix such that

$$K_p^{Pa,J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & PJ_p & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}.$$

If we use induction on n, then we obtain

$$\left(K_p^{Pa,J}\right)^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ T_{n+p+2} & & & & & \\ T_{n+p+1} & & & & & \\ \vdots & & & PJ_p & & \\ T_n & & & & & \\ T_{n-1} & & & & & \end{bmatrix}.$$

3. Conclusion

We considered a sequence called the Padovan-p Jacobsthal sequence, which is obtained using polynomials characteristic of the Padovan p-sequence and the Jacobsthal sequence. Furthermore, using the generating matrix of the Padovan-p Jacobsthal sequence, we obtained some new structural properties of the Padovan-p Jacobsthal numbers such as the generating functions, the permanental, combinatorial, determinantal, and exponential representations and the finite sums.

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