# Stability Result for a Kirchhoff Beam Equation with Variable Exponent and Time Delay

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## Article Info

## Abstract

This paper is concerned with a stability result for a Kirchhoff beam equation with variable exponents and time delay. The exponential and polynomial stability results are proved based on Komornik's inequality.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. For $(x,t) \in \Omega \times \mathbb{R}^+$ we consider a Kirchhoff beam equation with variable exponents and time delay given by

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) = 0, \tag{1.1}$$

with Dirichlet boundary condition

$$u(x,t) = 0 \text{ in } \partial\Omega \times [0,\infty), \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{1.3}$$
$$u_t(x,0) = u_1(x) \text{ in } \Omega, \tag{1.4}$$
$$u_t(x,t-\tau) = f_0(x,t-\tau) \text{ in } \Omega \times (0,\tau), \tag{1.5}$$

where $\tau > 0$ is time delay term, $\mu_1$ is a positive constant, $\mu_2$ is a real number. $M(s)$ is a positive $C^1$-function like $M(s) = a + bs^\gamma$ for $s \geq 0$, specially we take $a, b = 1$ and $\gamma > 0$. The functions $u_0$, $u_1$, $f_0$ are the initial data to be specified later.
The variable exponent $m(\cdot)$ is a measurable function on $\overline{\Omega}$ satisfying

$$2 \leq m^- \leq m(x) \leq m^+ \leq m^*, \text{ where } m^- = \underset{x\in\Omega}{ess\inf} m(x), \ m^+ = \underset{x\in\Omega}{ess\sup} m(x) \text{ e } m^* = \frac{2(n-2)}{n-4} \text{ if } n \geq 5. \tag{1.6}$$

The problems with variable exponents arise in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [7,8,26]. Time delay often appears in many practical problems such as thermal, economic phenomena, biological, chemical

and physical [12].

One of the first mathematical analysis of beam equation for $\Omega = (0, L) \subset \mathbb{R}$, $L > 0$,

$$u_{tt} + u_{xxxx} - M\left(\int_0^L |u_x|^2 \, dx\right) u_{xx} = 0 \tag{1.7}$$

was done by Ball (1973) [6]. Tucsnak (1996) [29] extended (1.7) for the beam equation in $\Omega \subset \mathbb{R}^n$

$$u_{tt} + \Delta^2 u + M(||\nabla u||^2)(-\Delta u) = 0, \tag{1.8}$$

where

$$||\nabla u||^2 =: \int_\Omega |\nabla u|^2 \, dx.$$

The problem (1.8) is called of Kirchhoff type in reason of the one-dimensional nonlinear equation (1.9) proposed by Kirchhoff [13] (1883),

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{2mL}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.9}$$

where $\tau_0$ is the initial tension, $m$ the mass of the string and $k$ the Young's modulus of the material of the string. This model, in connection with some problems in nonlinear elasticity, describes small vibrations of a stretched string of the length $L$ when only the transverse component of the tension is considered.

This kind of problem (1.9) is obtained from the model (1.10), first proposed by Woinowsky-Krieger [30] (1950), for the transverse motion of an extensible beam of the length $L$ whose ends are attached at a fixed distance

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} + \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\left(-\frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{1.10}$$

where $L$, $E$, $I$, $\rho$, $H$ and $A$ denote, respectively, the length of the beam in the rest position, the Young's modulus, the cross-sectional moment of inertia, the mass density, the tension in the rest position, and the cross-sectional area.

Antontsev et. al. [3], considered the nonlinear plate (or beam) Petrovsky equation with variable exponents as follows:

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \tag{1.11}$$

The authors, by using the Banach contraction mapping principle, obtained the local weak solutions. Also, they showed that the solution is global if $p(\cdot) \geq q(\cdot)$. Moreover, the authors proved that a solution with negative initial energy and $p(\cdot) < q(\cdot)$ blows up in finite time.

In [4], Antontsev et al., considered the Timoshenko-type equation with variable exponents as follows:

$$u_{tt} + \Delta^2 u - M\left(||\nabla u||^2\right)\Delta u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \tag{1.12}$$

The authors proved the local existence of the solution. Moreover, they investigated the nonexistence of solutions for negative initial energy.

In [5], Antontsev et al., studied the nonlinear $p(x)$-Laplacian equation with time delay and variable exponents as follows:

$$u_{tt} - \Delta_{p(x)} u + \mu_1 u_t(x, t)|u_t|^{m(x)-2}(x, t) + \mu_2 u_t(x, t-\tau)|u_t|^{m(x)-2}(x, t-\tau) = bu|u|^{q(x)-2}. \tag{1.13}$$

The authors proved the blow up of solutions. Then, by applying an integral inequality due to Komornik, they obtained the decay result. There are few results on Kirchhoff beam equation with delay. In [10] was considered the following nonlinear viscoelastic Kirchhoff beam equation with a time delay term in the internal feedback, given by

$$u_{tt} + \Delta^2 u - \mathrm{div} F(\nabla u) - \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)ds + \mu_1 u_t |u_t|^{m-1}(x, t) + \mu_2 u_t |u_t|^{m-1}(x, t-\tau) = 0,$$

where $\Omega \subset \mathbb{R}^n, (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The function $u = u(x, t)$ is the transverse displacement, and $\sigma(t)$ and $g(t)$ are positive functions defined on $\mathbb{R}^+$. $\mu_1, \mu_2$ are positive constants and $\tau > 0$ represents the time delay. Under suitable assumptions, the authors established the general rates of energy decay by using the energy perturbation method.

Kafini and Messaoudi [16], studied the equation with variable exponents and delay term as follows:

$$u_{tt} - \Delta u + \mu_1 u_t(x, t)|u_t|^{m(x)-2}(x, t) + \mu_2 u_t(x, t-\tau)|u_t|^{m(x)-2}(x, t-\tau) = bu|u|^{p(x)-2}. \tag{1.14}$$

They established the decay estimates and global nonexistence results for the equation (1.14).

Santos et al. [27], investigated the existence and the decay of the beam equation as follows:

$$u_{tt} + \Delta^2 u - M\left(||\nabla u||^2\right)\Delta u - \int_0^t h(t-s)\Delta u(s)ds + \alpha u_t = 0, \tag{1.15}$$

in a non-cylindrical domain. Recently, some other authors investigate hyperbolic type equations (see [11, 21–25, 28]).

Our aim in this work is to prove the stability of solutions for the Kirchhoff beam equation with the delay term ($\mu_2 u_t(x, t-\tau)$) and variable exponents which make the problem more different than from those considered in the literature. This manuscript extends the result of [16] to Kirchhoff beam equation.

The paper is organized as follows: In Section 2, the definition of the variable exponents Sobolev and Lebesgue spaces are stated. In Section 3, we obtain the stability result.

## 2. Preliminaries

In this section, we present some material needed for the statement and proof of our results. In what follows, we present some properties related to $W^{1,p(\cdot)}(\Omega)$ Sobolev spaces with variable exponents, see [2, 9, 14]. The spaces $L^{p(\cdot)}(\Omega)$ are special cases of the generalized Orlicz Spaces originated by Nakano [19] and developed by Musielak [17] and Orlicz [18]. The study of these spaces have been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations.

Let $p : \Omega \to [1, \infty)$ be a measurable function. We define the variable exponent Lebesgue space with variable exponent $p(\cdot)$ by:

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to R;\ \text{measurable in } \Omega\ :\ \int_\Omega |u|^{p(\cdot)}\,dx < \infty \right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)}\,dx \leq 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space (see [8]).
The relation between the modular $\int_\Omega |f|^{p(x)}\,dx$ and the norm follows from

$$\min(\|f\|_{p(x)}^{p^-}, \|f\|_{p(x)}^{p^+}) \leq \int_\Omega |f|^{p(x)}\,dx \leq \max(\|f\|_{p(x)}^{p^-}, \|f\|_{p(x)}^{p^+}).$$

In the case $p(x) = const > 1$, these inequalities transform into equalities. For all $f \in L^{p(x)}(\Omega)$, $g \in L^{p'(x)}(\Omega)$ with

$$p(x) \in (1, \infty), \quad p'(x) = \frac{p(x)}{p(x) - 1}$$

the generalized Hölder inequality holds, that is,

**Lemma 2.1.** *[1] (Hölder's inequality) Let $p, q, s \geq 1$ be measurable functions defined on $\Omega$ and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}\ ,\ \text{for a.e. } y \in \Omega,$$

*satisfies. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then, $fg \in L^{s(\cdot)}(\Omega)$ and*

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ \nabla u\ \text{exists and}\ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

Variable exponent Sobolev space with respect to the norm:

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm:

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, similar to the usual Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.
We also assume that:

$$|m(x) - m(y)| \leq -\frac{B}{\log|x-y|}\ \text{for all } x, y \in \Omega, \tag{2.1}$$

$B > 0$ and $0 < \delta < 1$ with $|x - y| < \delta$. (log-Hölder condition)

**Lemma 2.2.** *[2] (Poincarè inequality) Suppose that $p(\cdot)$ satisfies (2.1) and let $\Omega$ be a bounded domain of $R^n$. Then,*

$$\|u\|_{p(\cdot)} \leq c\|\nabla u\|_{p(\cdot)}\ \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

*where $c = c\left(p^-, p^+, |\Omega|\right) > 0$.*

**Remark 2.3.** *We denote by $c$ various positive constants which may be different at different occurrences. Also, throughout this paper, we use the embedding*

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

*which implies*

$$\|u\|_p \leq C\|\nabla u\| \leq C\|\Delta u\|,$$

*where $2 \leq p < \infty$ ($n = 1, 2$), $2 \leq p \leq \frac{2n}{n-2}$ ($n \geq 3$).*

## 3. Stability of solutions

In this section, we get the stability results for the problem (3.1)-(3.7), with the exponent $m(\cdot)$.
Similar to the work of [20], we introduce a new variable

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \ x \in \Omega, \ \rho \in (0,1), \ t > 0;$$

hence, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ x \in \Omega, \ \rho \in (0,1), \ t > 0.$$

Consequently, problem (1.1)- (1.5) is transformed to:

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} + \mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2} = 0, \text{ in } \Omega \times (0,\infty), \tag{3.1}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } \Omega \times (0,1) \times (0,\infty), \tag{3.2}$$

with Dirichlet boundary condition

$$u(x,t) = 0 \ \text{ on } \ \partial\Omega \times [0,\infty), \tag{3.3}$$
$$z(x,\rho,t) = 0 \ \text{ on } \ \partial\Omega \times [0,1) \times [0,\infty), \tag{3.4}$$

and initial data

$$u(x,0) = u_0(x) \ \text{ in } \ \Omega, \tag{3.5}$$
$$u_t(x,0) = u_1(x) \ \text{ in } \ \Omega, \tag{3.6}$$
$$z(x,\rho,0) = f_0(x,-\rho\tau) \ \text{ in } \ \Omega \times (0,1). \tag{3.7}$$

Similar to [16] we can define the strong solution as follows:

**Definition 3.1.** *Fix $T > 0$. We call $(u,z)$ a strong solution of* (3.1)-(3.7) *if*

$$u \in W^{2,\infty}([0,T];L^2(\Omega)) \cap W^{1,\infty}([0,T];H_0^2(\Omega)) \cap L^\infty([0,T];H^2(\Omega) \cap H_0^2(\Omega)),$$
$$u_t \in L^{m(\cdot)}(\Omega) \times [0,T)),$$
$$z \in W^{1,\infty}([0,1] \times [0,T];L^2(\Omega)) \cap L^\infty([0,1];L^{m(\cdot)(\Omega) \cap [0,T)}),$$

*and $(u,z)$ satisfies the initial data and* (3.1) *in the following sense*

$$\int_\Omega [u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} + \mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2}]v\,dx = 0,$$
$$\int_\Omega [\tau z_t(x,\rho,t) + z_\rho(x,\rho,t)]w\,dx = 0,$$

*for a.e. $t \in [0,t)$ and for $(v,w) \in H_0^1(\Omega) \cap L^2(\Omega)$.*

In order to state our main result, we define the "modified" energy functional of (3.1) is given by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho, \tag{3.8}$$

for $t \geq 0$, where $\xi$ is a continuous function yields

$$\tau|\mu_2|(m(x)-1) < \xi(x) < \tau(\mu_1 m(x) - |\mu_2|), x \in \overline{\Omega}. \tag{3.9}$$

The following lemma gives that, $E(t)$ is decreasing under the condition $\mu_1 > |\mu_2|$.

**Lemma 3.2.** *Let $(u,z)$ be a solution of* (3.1)-(3.7). *Then, there exists some $C_0 > 0$ such that*

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)}\right)dx \leq 0. \tag{3.10}$$

*Proof.* Multiplying (3.1) by $u_t$, integrating over $\Omega$, then, multiplying (3.2) by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2}z$ and integrating over $\Omega \times (0,1)$, summing up, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right]$$

$$= -\mu_1 \int_\Omega |u_t|^{m(x)}dx - \frac{1}{\tau}\int_\Omega \int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)d\rho dx - \mu_2 \int_\Omega u_t z(x,1,t)|z(x,1,t)|^{m(x)-2}dx. \tag{3.11}$$

The last two terms of the right-hand side of (3.11) can be estimated as follows:

$$-\frac{1}{\tau}\int_\Omega\int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2} zz_\rho(x,\rho,t)\,d\rho dx = -\frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\right)d\rho dx$$

$$= \frac{1}{\tau}\int_\Omega \frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)} - |z(x,1,t)|^{m(x)}\right)dx$$

$$= \int_\Omega \frac{\xi(x)}{\tau m(x)}|u_t|^{m(x)}\,dx - \int_\Omega \frac{\xi(x)}{\tau m(x)}|z(x,1,t)|^{m(x)}.$$

By using Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term, we get

$$|u_t||z(x,1,t)|^{m(x)-1} \leq \frac{1}{m(x)}|u_t|^{m(x)} + \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}.$$

Consequently, we obtain

$$-\mu_2\int_\Omega u_t z|z(x,1,t)|^{m(x)-2}\,dx \leq |\mu_2|\left(\int_\Omega \frac{1}{m(x)}|u_t(t)|^{m(x)}\,dx + \int_\Omega \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}\,dx\right).$$

Thus,

$$\frac{dE(t)}{dt} \leq -\int_\Omega\left[\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)}\right)\right]|u_t(t)|^{m(x)}\,dx - \int_\Omega\left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)}\right)|z(x,1,t)|^{m(x)}\,dx.$$

As a result, for all $x \in \overline{\Omega}$, the relation (3.9) satisfies

$$f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)}\right) > 0, \text{ and } f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. Hence, if we define

$$C_0(x) = \min\{f_1(x),\, f_2(x)\} > 0 \text{ for any } x \in \overline{\Omega},$$

and take $C_0(x) = \inf_{\overline{\Omega}}C_0(x)$, so $C_0(x) \geq C_0 > 0$. Therefore,

$$E'(t) \leq -C_0\left[\int_\Omega |u_t(t)|^{m(x)}\,dx + \int_\Omega |z(x,1,t)|^{m(x)}\,dx\right] \leq 0.$$

$$\square$$

We need the following lemmas before obtain our stability results.

**Lemma 3.3.** *(Komornik, [15]) Let $E : R^+ \to R^+$ be a nonincreasing function and suppose that there are constants $\sigma$, $\omega > 0$ such that*

$$\int_s^\infty E^{1+\sigma}(t)\,dt \leq \frac{1}{\Omega}E^\sigma(0)E(s) = cE(s), \forall s > 0.$$

*Then, we have*

$$\begin{cases} E(t) \leq cE(0)(1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ E(t) \leq cE(0)e^{-\omega t} & \text{if } \sigma = 0. \end{cases}$$

*for all $t \geq 0$.*

**Lemma 3.4.** *[16] The functional*

$$F(t) = \tau\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)|z(x,\rho,t)|^{m(x)}\,dxd\rho$$

*satisfies*

$$F'(t) \leq \int_\Omega \xi(x)|u_t|^{m(x)}\,dx - \tau e^{-\tau}\int_0^1\int_\Omega \xi(x)|z(x,\rho,t)|^{m(x)}\,dxd\rho$$

*along the solution of (3.1)-(3.7).*

**Theorem 3.5.** *Assume that conditions (1.6) and (2.1) are satisfied. Then, there exist two constants $c$, $\alpha > 0$ independent of $t$ such that any global solution of (3.1)-(3.7) satisfies,*

$$\begin{cases} E(t) \leq ce^{-\alpha t} & \text{if } m(\cdot) = 2, \\ E(t) \leq cE(0)(1+t)^{2/(m^+-2)} & \text{if } m^+ > 2. \end{cases}$$

*Proof.* We multiply the equation (3.1) by $uE^q(t)$, for $q > 0$ to be specified later, and integrate over $\Omega \times (s, T)$, $s < T$, to have

$$\int_s^T E^q(t) \int_\Omega \left[ uu_{tt} + u\Delta^2 u - u\Delta u - \|\nabla u\|^{2\gamma} u\Delta u + \mu_1 uu_t |u_t|^{m(x)-2} + \mu_2 uz(x,1,t) |z(x,1,t)|^{m(x)-2} \right] dxdt = 0,$$

which implies that

$$\int_s^T E^q(t) \int_\Omega \left( \frac{d}{dt}(uu_t) - u_t^2 + |\Delta u|^2 + |\nabla u|^2 + \|\nabla u\|^{2\gamma} |\nabla u|^2 + \mu_1 uu_t(x,t) |u_t(x,t)|^{m(x)-2} + \mu_2 uz(x,1,t)|z(x,1,t)|^{m(x)-2} \right) dxdt = 0. \tag{3.12}$$

Recalling the definition of $E(t)$, given in (3.8) adding and subtracting some terms and using the relation

$$\frac{d}{dt}\left( E^q(t) \int_\Omega uu_t dx \right) = qE^{q-1}(t) E'(t) \int_\Omega uu_t dx + E^q(t) \frac{d}{dt} \int_\Omega uu_t dx,$$

the equation (3.12) satisfies

$$2\int_s^T E^{q+1}(t) dt = -\int_s^T \frac{d}{dt}\left( E^q(t) \int_\Omega uu_t dx \right) dt + q\int_s^T E^{q-1}(t) E'(t) \int_\Omega uu_t dxdt$$
$$- \frac{\gamma}{\gamma+1} \int_s^T E^q \int_\Omega \|\nabla u\|^{2\gamma} |\nabla u^2| dxdt + 2\int_s^T E^q(t) \int_\Omega u_t^2 dxdt - \mu_1 \int_s^T E^q(t) \int_\Omega uu_t |u_t|^{m(x)-2} dxdt$$
$$- \mu_2 \int_s^T E^q(t) \int_\Omega uz(x,1,t) |z(x,1,t)|^{m(x)-2} dxdt + 2\int_s^T E^q(t) \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dxd\rho dt. \tag{3.13}$$

Next, we estimate the parts of the right side in inequality (3.13), respectively.
The first term is estimated as follows:

$$\left| -\int_s^T \frac{d}{dt}\left( E^q(t) \int_\Omega uu_t dx \right) dt \right| = \left| E^q(s) \int_\Omega uu_t(x,s) dx - E^q(T) \int_\Omega uu_t(x,T) dx \right|$$
$$\leq \frac{1}{2} E^q(s) \left[ \int_\Omega u^2(x,s) dx + \int_\Omega u_t^2(x,s) dx \right] + \frac{1}{2} E^q(T) \left[ \int_\Omega u^2(x,T) dx + \int_\Omega u_t^2(x,T) dx \right]$$
$$\leq \frac{1}{2} E^q(s) \left[ C_p \|\Delta u(s)\|_2^2 + 2E(s) \right] + \frac{1}{2} E^q(T) \left[ C_p \|\Delta u(T)\|_2^2 + 2E(T) \right]$$
$$\leq E^q(s) \left[ C_p E(s) + E(s) \right] + E^q(T) \left[ C_p E(T) + E(T) \right],$$

where $C_p$ is the Poincare's constant. Because of $E(t)$ is nonincreasing, we infer that

$$\left| -\int_s^T \frac{d}{dt}\left( E^q(t) \int_\Omega uu_t dx \right) dt \right| \leq cE^{q+1}(s) \leq cE^q(0) E(s) \leq cE(s). \tag{3.14}$$

In similar way, we handle the term

$$\left| q\int_s^T E^{q-1}(t) E'(t) \int_\Omega uu_t dxdt \right| \leq -q\int_s^T E^{q-1}(t) E'(t) \left[ C_p E(T) + E(T) \right] dt$$
$$\leq -c\int_s^T E^q(t) E'(t) \leq cE^{q+1}(s) \leq cE(s). \tag{3.15}$$

We estimate the other term as follows:

$$\left| -\frac{\gamma}{\gamma+1} \int_s^T E^q \int_\Omega \|\nabla u\|^{2\gamma} |\nabla u^2| dxdt \right| = \left| -2\gamma \int_s^T E^q \left( \frac{\|\nabla u\|^{2\gamma}}{2(\gamma+1)} \int_\Omega |\nabla u^2| dx \right) dt \right|$$
$$= \left| -2\gamma \int_s^T E^q \left( \frac{\|\nabla u\|^{2(\gamma+1)}}{2(\gamma+1)} \right) dt \right|$$
$$\leq \left| -2\gamma \int_s^T E^q(E(t)) dt \right|$$
$$\leq C^* \int_s^T E^{q+1}(t) dt$$
$$\leq C^* E(s) \tag{3.16}$$

where $C^*$ is a generic constant.
To treat the other term, we set

$$\Omega_+ = \{x \in \Omega, \ |u_t(x,t)| \geq 1\} \ \text{and} \ \Omega_- = \{x \in \Omega, \ |u_t(x,t)| < 1\}.$$

Then, by using the Hölder's and Young's inequalities, we get

$$\left| \int_s^T E^q(t) \int_\Omega u_t^2 \, dx dt \right| = \left| \int_s^T E^q(t) \left[ \int_{\Omega_+} u_t^2 \, dx + \int_{\Omega_-} u_t^2 \, dx \right] dt \right|$$

$$\leq c \int_s^T E^q(t) \left[ \left( \int_{\Omega_+} |u_t|^{m^-} \, dx \right)^{2/m^-} + \left( \int_{\Omega_-} |u_t|^{m^+} \, dx \right)^{2/m^+} \right] dt$$

$$\leq c \int_s^T E^q(t) \left[ \left( \int_\Omega |u_t|^{m(x)} \, dx \right)^{2/m^-} + \left( \int_\Omega |u_t|^{m(x)} \, dx \right)^{2/m^+} \right] dt$$

$$\leq c \int_s^T E^q(t) \left[ \left( -E'(t) \right)^{2/m^-} + \left( -E'(t) \right)^{2/m^+} \right] dt$$

$$\leq c\varepsilon \int_s^T [E(t)]^{qm^-/(m^- - 2)} \, dt + c(\varepsilon) \int_s^T \left( -E'(t) \right) dt + c\varepsilon \int_s^T E(t)^{q+1} \, dt + c(\varepsilon) \int_s^T \left( -E'(t) \right)^{2(q+1)/m^+} \, dt.$$

For $m^- > 2$ and the choice of $q = m^+/2 - 1$ will give $\frac{qm^-}{m^- - 2} = q + 1 + \frac{m^+ - m^-}{m^- - 2}$.

Therefore,

$$\left| \int_s^T E^q(t) \int_\Omega u_t^2 \, dx dt \right| \leq c\varepsilon \int_s^T E(t)^{q+1} \, dt + c\varepsilon [E(0)]^{\frac{m^+ - m^-}{m^- - 2}} \int_s^T [E(t)]^{q+1} \, dt + c(\varepsilon) E(s)$$

$$\leq c\varepsilon \int_s^T E(t)^{q+1} \, dt + c(\varepsilon) E(s). \tag{3.17}$$

For the case $m^- = 2$ and the choice of $q = m^+/2 - 1$ will give the similar result.
For the other term, utilizing Young's inequality we conclude

$$\left| -\mu_1 \int_s^T E^q(t) \int_\Omega u |u_t|^{m(x)-1} \, dx dt \right| \leq \varepsilon \int_s^T E^q(t) \int_\Omega |u(t)|^{m(x)} \, dx dt + c \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |u_t(t)|^{m(x)} \, dx dt$$

$$\leq \varepsilon \int_s^T E^q(t) \left[ \int_{\Omega_+} |u(t)|^{m^-} \, dx + \int_{\Omega_-} |u(t)|^{m^+} \, dx \right] dt + c \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |u_t(t)|^{m(x)} \, dx dt,$$

where we have used Young's inequality with

$$p(x) = \frac{m(x)}{m(x) - 1}, \, p'(x) = m(x)$$

and hence

$$c_\varepsilon(x) = (m(x) - 1) m(x)^{m(x)/(1-m(x))} \varepsilon^{1/(1-m(x))}.$$

By using the embeddings $H_0^2(\Omega) \hookrightarrow L^{m^-}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we obtain

$$\left| -\mu_1 \int_s^T E^q(t) \int_\Omega u |u_t|^{m(x)-1} \, dx dt \right| \leq \varepsilon \int_s^T E^q(t) \left[ c \|\Delta u(s)\|_2^{m^-} + c \|\Delta u(s)\|_2^{m^+} \right] dt + c \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |u_t(t)|^{m(x)} \, dx dt$$

$$\leq \varepsilon \int_s^T E^q(t) \left[ cE^{(m^- - 2)/2}(0) E(t) + cE^{(m^+ - 2)/2}(0) E(t) \right] dt + c \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |u_t(t)|^{m(x)} \, dx dt$$

$$\leq c\varepsilon \int_s^T E^{q+1}(t) \, dt + \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |u_t(t)|^{m(x)} \, dx dt. \tag{3.18}$$

The next term of (3.13) can be estimated in a similar attitude to get

$$\left| -\mu_2 \int_s^T E^q(t) \int_\Omega u |z(x,1,t)|^{m(x)-1} \, dx dt \right| \leq \varepsilon \int_s^T E^q(t) \left[ c \|\Delta u(s)\|_2^{m^-} + c \|\Delta u(s)\|_2^{m^+} \right] dt + c \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |z(x,1,t)|^{m(x)} \, dx dt$$

$$\leq c\varepsilon \int_s^T E^{q+1}(t) \, dt + \int_s^T E^q(t) \int_\Omega c_\varepsilon(x) |z(x,1,t)|^{m(x)} \, dx dt. \tag{3.19}$$

For the last term of (3.13), from Lemma 3.4, we get

$$2 \int_s^T E^q(t) \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} \, dx d\rho dt \leq \frac{2}{m^-} \int_s^T E^q(t) \int_0^1 \int_\Omega \xi(x) |z(x,\rho,t)|^{m(x)} \, dx d\rho dt$$

$$\leq -\frac{2\tau}{m^-} \int_s^T E^q(t) \frac{d}{dt} \left( \int_0^1 \int_\Omega e^{-\rho\tau} \xi(x) |z|^{m(x)} \, dx d\rho \right) dt$$

$$+ \frac{2}{m^-} \int_s^T E^q(t) \int_\Omega \xi(x) |u_t|^{m(x)} \, dx dt$$

$$\leq -\frac{2\tau}{m^-} \left[ E^q(t) \int_0^1 \int_\Omega e^{-\rho\tau} \xi(x) |z|^{m(x)} \, dx d\rho \right]_{t=s}^{t=T} + \frac{2}{m^-} \int_s^T E^q(t) \int_\Omega \xi(x) |u_t|^{m(x)} \, dx dt.$$

As $\xi(x)$ is bounded, by (3.8), we obtain

$$2\int_s^T E^q(t)\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt \leq \frac{2\tau e^{-\tau}}{m^-}E^q(s)E(s)+\frac{2c}{m^-}E^{q+1}(T)$$

$$\leq \frac{2\tau e^{-\tau}}{m^-}E^q(0)E(s)+\frac{2c}{m^-}E^q(T)E(s)\leq cE(s), \qquad (3.20)$$

for some $c>0$.

By combining (3.13)-(3.20), we conclude that

$$\int_s^T E^{q+1}(t)dt \leq \varepsilon\int_s^T E^{q+1}(t)dt+cE(s)+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}dxdt. \qquad (3.21)$$

Choosing $\varepsilon$ so small such that

$$\int_s^T E^{q+1}(t)dt \leq cE(s)+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}dxdt.$$

Once $\varepsilon$ is fixed, then $c_\varepsilon(x)\leq M$, since $m(x)$ is bounded. Therefore, we infer that

$$\int_s^T E^{q+1}(t)dt \leq cE(s)+cM\int_s^T E^q(t)\int_\Omega |z(x,1,t)|^{m(x)}dxdt$$

$$\leq cE(s)-C_0M\int_s^T E^q(t)E'(t)dt$$

$$\leq cE(s)+\frac{C_0M}{q+1}\left[E^{q+1}(s)-E^{q+1}(T)\right]\leq cE(s). \qquad (3.22)$$

By taking $T\to\infty$, we obtain

$$\int_s^\infty E^{q+1}(t)dt \leq cE(s).$$

Thus, Komornik's Lemma (with $\sigma=q=m^+/2-1$) implies the desired result.          $\square$

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## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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