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Review

HIROTA METHOD AND SOLITON SOLUTIONS

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Abstract: *Solitons are an important class of solutions to nonlinear differential equations which appear in different areas of physics and applied mathematics**. In this study, we provide** a general overview of the Hirota method which is one of the most powerful tools in finding the multi-soliton solutions of nonlinear wave and evaluation equations. Bright and dark soliton solutions of the nonlinear Schrödinger equation are discussed in detail.*

Keywords*: Nonlinear differential equations, integrable systems, Hirota method, solitons, nonlinear Schrödinger equation.*

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1. Introduction

Solitons are particular exact solutions of some nonlinear partial differential equations. Although there is no strict definition of solitons or solitary waves, they are characterized mainly by some common features: A solitary wave is a local disturbance or pulse which retains its shape during propagation. A soliton is a solitary wave that preserves its shape and velocity after interacting with other solitary waves. They are only affected by a phase shift after interactions and in this sense, they behave like particles. There are many nonlinear integrable differential equations that have soliton solutions such as the Korteweg-de Vries (KdV) equation, Boussinesq equation, nonlinear Schrödinger equation, sine-Gordon equation, et cetera.

Soliton theory begins with a phenomenon that the Scottish engineer J. Scott Russell observed by chance. Russell detected that a body of water set in motion by a canal boat, travels a long distance along the canal maintaining its shape and speed. As a result of later experiments done on this observation, he empirically derived a relation between the speed and the amplitude of the wave: $c^2 = g(h + a)$, where c is the speed, a is the maximum amplitude, h is the depth of the water and g is the acceleration due to gravity. This equation implies that the speed of the wave is related to its amplitude and a larger wave moves faster than a small one. Russell's work [1] triggered many debates on the subject, many of which were critical of his results. In the 1870s both Boussinesq [2] and Rayleigh [3] independently obtained similar results, which confirm Russell. They also showed that these long water waves have a $sech^2$ wave profile. Whereas the differential equation which is satisfied by this function remained unknown for about two more decades. Thus, the explanation of the phenomenon observed by Russell remained unsolved for more than 60 years. Finally, in 1895, a mathematical model proposed by Korteweg and de Vries achieved this task. This model is known as the KdV equation [4] and has been studied extensively in every aspect ever since. In its standard form, the equation is given by

$$u_t + \alpha u u_x + \beta u_{xxx} = 0. \tag{1}$$

Here the constants α and β are arbitrary and can be set to any values by scaling transformations of $u, x,$ and t. The conventional choice is given by $\alpha = 6$ and $\beta = 1$. This is one of the most important equations in the soliton theory and it is ubiquitous in physics problems, such as water waves, fluid mechanics, and plasma physics. There are two competing terms in this equation as in other nonlinear wave models. These terms ensure the coherence of the wave so that, it maintains the waveform and continues to propagate over a long period of time. The last linear term in the equation is the origin of the dispersion observed. On the other hand, the second term is a nonlinear term, and it steepens the wave and finally causes disintegration. When these competing effects are balanced, a stable waveform is formed.

The modern era of solitary waves began in 1955 with the studies of Fermi, Pasta, and Ulam (FPU) on a numerical model of a discrete nonlinear mass-spring system [5]. They tried to show that a smooth initial state would eventually relax to an equipartition of energy among all modes because of nonlinearity. Contrary to the expectations, results showed that the equipartition of energy among the modes did not occur. They put all the energy in a few lowest modes of the corresponding linear model at the beginning. In the linear problem, the energy in each mode would stay unchanged and no new mode would be excited. In the nonlinear problem, the energy is transferred from low modes to higher ones, and the expectation was a continuation of this process until the energy is completely distributed over all modes. Whereas, when the model starts to process, the energy is exchanged between various low-order modes, and it eventually returns to the lowest mode again. Hence, in the end, a series of recurring states show up. The next milestone is the work done by Zabusky and Kruskal on FPU results in 1965 [6]. In this study, they tried to understand why the recurrence phenomenon occurs and for this aim, they investigated a continuous model of the nonlinear mass-spring system. In fact, they analyzed the initial value problem of the KdV equation (1) in the form $q_t + 6qq_\xi + \delta^2 q_{\xi\xi\xi} = 0$ with a finite and small δ^2, i.e., for a weak nonlinear modulational term. What they get when starting with a smooth initial state $q(\xi, 0) \sim \cos(2\pi\xi)$, was summarized as follows: "*Initially the wave steepened in regions where it had a negative slope, a consequence of the dominant effects of nonlinearity over the dispersive term. As the wave steepens, the dispersive effect then becomes significant and balances the nonlinearity. At later times, the solutions develop a series of eight well-defined waves, each like $sech^2$ functions with the taller waves ever catching up and overtaking the shorter waves. These waves undergo nonlinear interaction according to the KdV equation and then emerge from the interaction without a change of form and amplitude, but with only a small change in their phases. Another surprising fact is that the initial profile reappears very similarly to the FPU recurrence phenomenon*" [6]. All these strange phenomena led the researchers to think that there are some conservation laws that operate in the background and somehow the KdV equation is integrable. After that, several conserved quantities were calculated by Zabusky-Kruskal, Whitham, and Miura. Miura also found one of the last pieces of the puzzle by introducing the famous Miura transformation [7]. He proved that another important integrable nonlinear differential equation, which is called the modified KdV (mKdV) equation, also has an infinite number of conserved quantities. Moreover, all these conserved quantities can be related to the corresponding counterparts in the KdV equation via the Miura transformation. The next step toward the integrability of the equation was the construction of an inverse scattering transformation method. Consequently, the complete integrability of the KdV equation was shown in a series of papers by Gardner et al. [8-10] and Zakharov and Faddeev [11].

Ryogo Hirota introduced another powerful method to find the exact solutions of the KdV equation [12,13] in 1971. Hirota's method is the most suitable method for obtaining multi-soliton solutions of nonlinear differential equations. Soliton solutions can also be studied by using other methods like inverse scattering transformation, Bäcklund transformation, Darboux transformation, or Painleve expansion method. Especially, the inverse scattering method is a very powerful technique to obtain exact solutions to nonlinear equations; nevertheless, its application to practical problems requires a bit of cumbersome work. On the other hand, Hirota's method is much more manageable in this sense. After all these great achievements, soliton solutions of many nonlinear differential equations in 1-dimension, as well as higher dimensions, were studied extensively by using various methods [14-19].

In this study, we introduce the Hirota direct method to obtain the multi-soliton solutions of various differential equations. The plan of the paper is as follows. We will review the method in detail in the second section. In that section, the method is explained on a well-known example, the KdV equation. The soliton solutions up to the third order are constructed explicitly and then solutions are generalized to the N-soliton case. The third section is devoted to the nonlinear Schrödinger equation where focusing and defocusing nonlinear Schrödinger equations are presented. The soliton solutions of these equations are called bright and dark solitons respectively. Both, bright and dark soliton solutions of focusing and defocusing nonlinear Schrödinger equations, up to the two-solitons are calculated by the Hirota direct method. The fourth section includes a conclusion and discussions.

2. Hirota Direct Method

In this section, we will discuss the Hirota direct method to find the N-soliton solutions of any integrable nonlinear differential equation by following [13]. We will explain the method by reviewing its application to the KdV equation, which is also important for historical reasons in the sense that, it is the first introduced equation for explaining the previously observed solitary wave phenomenon. Having equipped with these tools our next goal will be to handle the nonlinear Schrodinger equation.

Multi-soliton solutions can be obtained by the inverse scattering transform [8-11], the dressing method [20-23] and the Hirota method [13]. The Hirota method is algebraic rather than analytic which can be treated as one of its advantages. The Hirota direct method also called bilinear method was first proposed by Hirota to obtain the N-soliton solutions of the KdV equation [12]. It is an efficient method for searching soliton solutions of the nonlinear evolution equations.

First, we will introduce the Hirota differential operator (from now on we will use D-operator in short) and then show how a nonlinear differential equation can be brought into the Hirota bilinear form by using those operators. The D-operator is a bilinear operator which acts on a pair of functions to produce a new function. We will work in 2-dimensional spacetime (t, x), but definitions can be extended to higher dimensions.

The Hirota D-operator is given by

$$D_{x_i}^m = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i{}'}\right)^m, \quad x_i = (t, x) \tag{2}$$

where m is a positive integer. It acts as a product of a pair of functions:

$$D_{x_i}^m(f \cdot g) = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i{}'}\right)^m f(x,t) \cdot g(x',t')|_{x'=x,t'=t} \,. \tag{3}$$

In what follows, we give some properties of the D-operator for later convenience. Equation (3) can be written for $x_i = x$ more explicitly as:

$$D_x^m(f \cdot g) = \sum_{k=0}^m (-1)^k \binom{m}{k} f_k g_{(m-k)} \,, \tag{4}$$

where f_k stands for $f_k \equiv \partial_x^k f$ and $\binom{m}{k}$ is the binomial coefficient. The anti-symmetrization property of the D-operator with respect to the second function is

$$D_{x_i}^m(g \cdot f) = (-1)^m D_{x_i}^m(f \cdot g). \tag{5}$$

Because of these properties, if we take the first or second function as a constant function, for example, if $g = 1$, we get

$$D_{x_i}^m(f \cdot 1) = \partial_{x_i}^m f \,. \tag{6}$$

On the other hand, if we take $f = g$ we then obtain

$$D_{x_i}^m(f \cdot f) = 0, \ \ if\ m = odd \tag{7}$$

and for even *m*, the first few equations are

$$\begin{aligned} m = 2: \quad & D_{x_i}^2(f \cdot f) = 2\left(f f_{x_i x_i} - f_{x_i}^2\right), \\ & D_x D_t(f \cdot f) = 2(f f_{xt} - f_x f_t), \\ m = 4: \quad & D_{x_i}^4(f \cdot f) = 2\left(f f_{x_i x_i x_i x_i} - 4 f_{x_i} f_{x_i x_i x_i} + 3 f_{x_i x_i}^2\right), \\ \vdots \end{aligned} \tag{8}$$

The following properties are especially useful for studying soliton solutions. Using the definition $\phi_i = k_i x + \omega_i t + \alpha_i$, where constant coefficients k_i, ω_i and α_i denote the wave number, angular momentum, and phase factor respectively, then

$$D_x^m D_t^n\left(e^{\phi_1} \cdot e^{\phi_2}\right) = (k_1 - k_2)^m (\omega_1 - \omega_2)^n e^{\phi_1 + \phi_2}\ , \tag{9}$$

and

$$D_{x_i}^m\left(e^{\phi_1} \cdot e^{\phi_1}\right) = 0\ . \tag{10}$$

The second part of the Hirota method includes the transformation of the dependent variable of the equation. The underlying motivation for such a transformation is to express the original equation as a quadratic equation of the dependent variable so that, the leading order derivative and the nonlinear term have the same degree and the same number of derivatives. Mainly three kinds of transformations are commonly used: logarithmic, rational, or arctan transformations. Once the equation is brought into a quadratic form it can be bilinearized via D-operators.

Now, let us assume that a nonlinear differential equation is brought into the Hirota bilinear form by using one of the above transformations of the dependent variable and any combination of D-operators. This bilinear form is expressed by the equation $B(f \cdot f) = 0$ where *B* denotes a polynomial of D-operators. The N-soliton solution is obtained by taking a perturbative expansion for the function f such that

$$f = 1 + \sum_{i=1}^{\infty} \epsilon^i f_i, \tag{11}$$

where the parameter ϵ is a formal parameter, which can be set equal to 1 after getting all order solutions. When this expansion is plugged into the bilinear equation $B(f \cdot f) = 0$ and then grouped in order of the powers of ϵ one gets the following set of equations:

$$\epsilon^0: \quad B(1 \cdot 1) = 0 \tag{12.0}$$

$$\epsilon^1: \quad B(f_1 \cdot 1 + 1 \cdot f_1) = 0 \tag{12.1}$$

$$\epsilon^2: \quad B(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0 \tag{12.2}$$

$$\epsilon^3: \quad B(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0 \tag{12.3}$$

$$\vdots$$

$$\epsilon^n: \quad B\left(\sum_{j=0}^{m} f_{(m-j)} \cdot f_j\right) = 0\ . \tag{12.n}$$

This procedure is quite general and can be applied to any explicit bilinear operator expression. It can be proved that if the original equation admits an N-soliton solution, then the perturbative expansion (11) will truncate at the $n = N$ term, and hence, the convergence problem will be solved automatically.

2.1. KdV equation

In this section we will review the application of the Hirota method for the KdV equation as our first example, obtaining solutions up to 3-solitons explicitly. Then N-soliton solutions are given. In this section, we follow [13]. Let us recall Equation (1) in its standard form

$$u_{xxx} + 6uu_x + u_t = 0 \tag{13}$$

with the boundary condition $u \to 0$ as $|x| \to \infty$. To bring this equation into a quadratic form we will apply two successive transformations for the dependent variable u. We define $u = v_x$ and after integrating with respect to x we get the potential KdV equation

$$v_t + 3v_x^2 + v_{xxx} = c(t) \,, \tag{14}$$

where $c(t)$ is an arbitrary function of t and it can be set to 0 after applying boundary conditions v, $v_x, v_t, v_{xx}, v_{xxx} \to 0$ as $|x| \to \infty$. Now, if we choose a logarithmic transformation for the new variable such that

$$v(x,t) = 2\ln\,(f(x,t))_x \tag{15}$$

and insert this into Equation (14), we find

$$ff_{xxxx} - 4f_xf_{xxx} + 3f_{xx}^2 + ff_{xt} - f_xf_t = 0. \tag{16}$$

As it can be seen from Equation (16), this is a quadratic equation of the dependent variable f and it satisfies the previously mentioned property. Equation (16) can be written in terms of D-operators:

$$(D_xD_t + D_x^4)(f \cdot f) = 0. \tag{17}$$

Hence, the bilinearization operator *B* can be written as $B \equiv D_xD_t + D_x^4$.

2.2. Soliton solution:

The 1-soliton solution can be obtained by choosing the leading order term in the expansion (11) to contain only one exponential factor, and then solving the set of Equations (12) by orderwise iteration. For this aim, the following ansatz can be used:

$$f_1 = e^{\theta_1} \quad ; \quad \theta_1 = k_1x + \omega_1t + \alpha_1 \tag{18}$$

where k_1, ω_1 and α_1 are constants. Since Equation (12.0) is trivial, we start with Equation (12.1):

$$f_{1,xt} + f_{1,xxxx} = 0\,. \tag{19}$$

Ansatz (18) satisfies Equation (19) provided that the dispersion relation $\omega_1 = -k_1^3$, $(k_1 \neq 0)$ is satisfied. Equation (12.2) corresponds to

$$f_{2,xt} + f_{2,xxxx} = -(D_xD_t + D_x^4)(f_1 \cdot f_1) \tag{20}$$

and if we use the results of the previous order here, we see that the right-hand side of Equation (20) vanishes. Hence, we can choose $f_2(x,t) = 0$. Continuing to the next-order equations with these results, one can show that all the next-order solutions can be fixed to zero, $f_i(x,t) = 0$, for $i \geq 2$. Consequently, by setting $\epsilon = 1$ we end up with

$$f = 1 + e^{\theta_1} \tag{21}$$

for the 1-soliton solution. It is an easy task to obtain the original function $u(x,t) = 2(\ln f)_{xx}$ by going backward starting from the result (21)

$$u(x,t) = \frac{k_1^2}{2}\operatorname{sech}^2\frac{\theta_1}{2}. \tag{22}$$

2.3. Soliton solution:

In a similar way, the 2-soliton solution needs two exponential factors for the leading order term in the expansion of the function f. One can start with the ansatz

$$f_1 = e^{\theta_1} + e^{\theta_2} \,, \tag{23}$$

where

$$\theta_i = k_ix + \omega_it + \alpha_i \tag{24}$$

with constant coefficients k_i, ω_i, α_i. Equation (12.1) again gives the relations

$$\omega_i = -k_i^3 \ , \ (k_i \neq 0) \ , \tag{25}$$

but the right-hand side of Equation (12.2) is not zero anymore and we have

$$f_{2,xt} + f_{2,xxxx} = 3k_1k_2(k_1 - k_2)^2 e^{\theta_1+\theta_2} \ . \tag{26}$$

It can be shown that the solution to this equation is given by

$$f_2 = e^{A_{12}} e^{\theta_1+\theta_2} \ , \tag{27}$$

with the constant coefficient

$$e^{A_{12}} = (\frac{k_1 - k_2}{k_1 + k_2})^2 \ . \tag{28}$$

The next order Equation (12.3) is

$$f_{3,xt} + f_{3,xxxx} = -B(f_1 \cdot f_2 + f_2 \cdot f_1) \ , \tag{29}$$

and the right-hand side of this equation is zero, since $B(f_1 \cdot f_2) = B(f_2 \cdot f_1) = 0$. In that case one can choose $f_3 = 0$ and similarly all higher order terms can be set to zero as well, i.e. $f_i = 0, (i \geq 3)$. Finally, we get

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_1+\theta_2+A_{12}} \ , \tag{30}$$

and hence the 2-soliton solution is given by

$$u(x,t) = -2 \frac{k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + \left(k_1^2 e^{\theta_2} + k_2^2 e^{\theta_1}\right) e^{\theta_1+\theta_2+A_{12}} + 2(k_1 - k_2)^2 e^{\theta_1+\theta_2}}{(1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_1+\theta_2+A_{12}})^2} \tag{31}$$

This result can also be rearranged in terms of hyperbolic trigonometric functions however, for later convenience we keep it in this form. Now, let us continue with the 3-soliton solution.

2.4. Soliton solution and generalization:

The 3-soliton solution goes in a similar way, except that the ansatz for the first term of the expansion (11) has one more exponential factor**:**

$$f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \tag{32}$$

When this form is inserted into Equation (12.1), dispersion Equations (25) are obtained. After a few algebraic calculations, the next-order term in the expansion can be written as

$$f_2 = e^{A_{12}} e^{\theta_1+\theta_2} + e^{A_{13}} e^{\theta_1+\theta_3} + e^{A_{23}} e^{\theta_2+\theta_3} \tag{33}$$

from Equation (12.2). Here, the coefficients are

$$e^{A_{ij}} = (\frac{k_i - k_j}{k_i + k_j})^2, \quad i,j = 1,2,3 \ and \ i < j \ . \tag{34}$$

If we proceed with Equation (12.3), we see that the right-hand side of Equation (29) does not vanish anymore, and the solution to Equation (12.3) can be given by

$$f_3 = e^{A_{123}} e^{\theta_1+\theta_2+\theta_3} \tag{35}$$

with the constant coefficient

$$e^{A_{123}} = e^{A_{12}} e^{A_{13}} e^{A_{23}}. \tag{36}$$

The perturbative expansion will cease at this order as stated before. It can be checked that since

$$B(f_1 \cdot f_3 + f_2 \cdot f_2 + f_1 \cdot f_3) = 0 \tag{37}$$

the next order equation (12.4)

$$f_{4,xt} + f_{4,xxxx} = 0 \tag{38}$$

can be solved by setting $f_4 = 0$ indeed. Finally, all following orders can be set to zero in the same fashion. Hence, we reach the 3-soliton solution by setting the expansion parameter $\epsilon = 1$:

$$\begin{aligned} f &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{A_{12}} e^{\theta_1+\theta_2} + e^{A_{13}} e^{\theta_1+\theta_3} + e^{A_{23}} e^{\theta_2+\theta_3} \\ &\quad + e^{A_{123}} e^{\theta_1+\theta_2+\theta_3} \ . \end{aligned} \tag{39}$$

We see that this solution includes no additional freedom and is totally obtained from the preceding parameters. We will not get into details but, unlike the previous orders, the 3-soliton solution is quite restrictive and closely related to the integrability of the equation. The 3-soliton solution guides us to obtain the N-soliton solution which contains a finite polynomial of exponential factors given below

$$f(x,t) = \sum_{\substack{\mu_i=0,1 \\ 1\le i\le N}} \exp\left(\sum_{1\le i<j\le N} A_{ij}\,\mu_i\mu_j + \sum_{j=1}^{N}\mu_j\theta_j\right). \tag{40}$$

Proof of this can be found in [13]. Here, it should be noted that the combination of lower-order solutions in the form given above to get higher-order solutions is possible only for integrable equations. For a detailed discussion see [24], [25]. This is called the Hirota integrability condition. Hirota integrability can be used equivalently instead of the usual integrability because no counterexamples have been found so far.

3. Nonlinear Schrödinger equation

In this section, we will present another important class of nonlinear differential equations, the nonlinear Schrödinger (NLS) equation. The NLS equation is an extension of the well-known linear Schrodinger equation, and it can be defined as an approximation to a wide class of nonlinear wave equations [26] that arise in many branches of physics such as plasma physics, nonlinear optics, and fluid dynamics. The most common applications of the NLS equation include self-focusing of beams in nonlinear optics, modeling of the propagation of electromagnetic pulses in nonlinear optical fibers which act as waveguides, and stability of Stokes waves in water. In hydrodynamics, the NLS equation describes the dynamics of surface gravity waves in finite or infinite depth, depending on the ratio between the water depth and wavelength. It is also shown that the nonlinear modulation of a quasi-monochromatic wave is described by the NLS equation. For a further discussion see [27], [28] and references therein. The integrability of the NLS equation has been shown by Zakharov and Shabat by using the inverse scattering method [20].

In this section, we mainly follow the references [13] and [28]. The derivation of the NLS equation and the related definitions can be found in detail in [28]. Soliton solution technic can be found in [13] and [29].

The NLS equation is given by

$$iq_t + \tfrac{1}{2}q_{xx} + p|q^2|q = 0\ , \tag{41}$$

where p is a parameter that takes values $p = \mp 1$. The equation for $p = 1$ is called the focusing NLS (fNLS) equation and it has an N-envelop solution [20, 30]. The case for $p = -1$ is called defocusing NLS (dNLS) equation and it is shown that it has a dark pulse solution [30]. We will keep this parameter undetermined and at the end, possible values of the parameter will be considered separately. Bilinearization of the NLS equation goes as follows: a transformation of the dependent variable is defined as follows

$$q(x,t) = \frac{u(x,t)}{v(x,t)}\ , \tag{42}$$

where $u(x,t)$ is a complex function and $v(x,t)$ is a real function. Exploiting Equation (42) and D-operators, Equation (41) can be split into two distinct equations:

$$\left(iD_t + \frac{1}{2}D_x^2\right)(u\cdot v) = 0 \tag{43}$$

and

$$\frac{1}{2}D_x^2(v\cdot v) - p|u|^2 = 0\ . \tag{44}$$

Now let us expand new functions u and v into series

$$u(x,t) = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots \ ,$$
$$v(x,t) = 1 + \epsilon v_1 + \epsilon^2 v_2 + \cdots \qquad , \tag{45}$$

with the assumption that all the functions $u_1, u_2, \cdots, v_1, v_2, \cdots$ go to zero as $|x| \to \infty$, which is the boundary condition also satisfied by the original function q. Keeping in mind that, $B = iD_t + \frac{1}{2}D_x^2$ for Equation (43) and $B = \frac{1}{2}D_x^2$ for Equation (44), one can expand the equations in powers of ϵ as before. This gives us

$$\epsilon^n\!: B\left(\sum_{k=1}^{n} u_k \cdot v_{n-k}\right) = 0$$
$$\Rightarrow\ iu_{n,t} + \frac{1}{2}u_{n,xx} = -B\left(\sum_{k=1}^{n-1} u_k \cdot v_{n-k}\right), \tag{46}$$

and

$$\epsilon^n\!: B\left(\sum_{k=0}^{n} v_k \cdot v_{n-k}\right) = p\sum_{k=1}^{n-1} u_k \cdot u_{n-k}^{*}$$
$$\Rightarrow\ v_{n,xx} = p\left(\sum_{k=1}^{n-1} u_k \cdot u_{n-k}^{*}\right) - \frac{1}{2}D_x^2\left(\sum_{k=1}^{n-1} v_k \cdot v_{n-k}\right). \tag{47}$$

respectively.

3.1. Soliton solutions

3.1.1 Soliton solution:

After getting the bilinearization of the equation, one can now start seeking the soliton solutions. As we saw in the previous section the N-soliton solution is given by defining the first order function of the expansion (45) as a sum of N exponential term: $u_1 = \sum_{i=1}^{N} e^{\theta_i}$ where θ_i is defined as in Equation (24). Let us start by taking $N = 1$ for the 1-soliton solution. If one takes $n = 1$ in Equation (46) it gives the following linear equation

$$iu_{1,t} + \tfrac{1}{2}u_{1,xx} = 0\ . \tag{48}$$

Inserting the ansatz for the 1-soliton solution into Equation (48) gives the condition $\omega_1 = \frac{i}{2}k_1^2$ which defines the dispersion relation of the wave. The equation for the function v at the same order of ϵ, i.e., taking $n = 1$ in Equation (47),

$$v_{1,xx} = 0 \tag{49}$$

allows us to choose $v_1 = 0$. Hence, we have

$$u_1 = e^{\theta_1}\ ,$$
$$v_1 = 0. \tag{50}$$

For the next order, taking $n = 2$, Equations (46) and (47) can be read as

$$iu_{2,t} + \tfrac{1}{2}u_{2,xx} = -B(u_1 \cdot v_1),\ v_{2,xx} = p|u_1|^2 - B(v_1 \cdot v_1) \tag{51}$$

respectively. It can be shown that, since $B(u_1 \cdot v_1) = 0 = B(v_1 \cdot v_1)$, one can choose $u_2 = 0$ from the first of Equation (51). On the other hand, the second equation gives the solution

$$v_2 = e^{\theta_1+\theta_1^*+A_{11}} \tag{52}$$

with

$$e^{A_{11}} = \frac{p}{(k_1+k_1^*)^2}. \tag{53}$$

Therefore, one can see that the function v is real as assumed. It can be verified that all the next order terms in the expansion (45) can be set to zero, $u_i = v_i = 0,\ (i \geq 3)$. In this way, the 1-soliton solution is obtained by setting $\epsilon = 1$ as

$$q = \frac{u}{v} = \frac{e^{\theta_1}}{1+e^{\theta_1+\theta_1^*+A_{11}}}. \tag{54}$$

Now, let $k_1 = a + ib$ as a general complex number. Plugging this definition into Equation (54) and setting $p = +1$ to deal with the fNLS solution gives

$$q(x,t) = ae^{i\left[bx+\frac{a^2-b^2}{2}t\right]}\operatorname{sech}[a(x-bt)+\alpha_2] \tag{55}$$

where α_2 is a constant. If one chooses $p = -1$, i.e., if the dNLS case is considered, one can see that the coefficient (53) becomes negative, and one gets

$$q(x,t) = -a\frac{e^{i\left[bx+\frac{a^2-b^2}{2}t\right]}}{\sinh[a(x-bt)+\alpha_2]}. \tag{56}$$

This solution has a singularity and therefore does not yield a soliton solution for the dNLS equation. This situation will be considered in detail in section 3.2, before that let us continue investigating the 2-soliton solution.

3.1.2 Soliton solution

As one can see above, since the 1-soliton solution contains ϵ^2 terms, it is reasonable to expect that for the 2-soliton solution we should go up to the ϵ^4 orders. For $N = 2$ one starts with the ansatz

$$u_1 = e^{\theta_1} + e^{\theta_2}\ , \tag{57}$$

which satisfies Equation (48) with the dispersion relation $\omega_i = \frac{i}{2}k_i^2,\ (i = 1,2)$. Equation (49) gives $v_1 = 0$ as before. Since the right-hand side of the first equation and the second term in the right-hand side of the second equation in (51) vanish, these equations give us

$$\begin{gathered} u_2 = 0, \\ v_2 = e^{\theta_1+\theta_1^*+A_{11}} + e^{\theta_1+\theta_2^*+A_{12}} + e^{\theta_2+\theta_1^*+A_{21}} + e^{\theta_2+\theta_2^*+A_{22}} \end{gathered} \tag{58}$$

respectively. The constant factors are given by

$$e^{A_{ij}} = \frac{p}{(k_i+k_j^*)^2}. \tag{59}$$

If we proceed to the third-order equations by taking $n = 3$ in Equations (46) and (47) we get

$$\begin{aligned} iu_{3,t} + \frac{1}{2}u_{3,xx} &= -B(u_1\cdot v_2 + u_2\cdot v_1), v_{3,xx} \\ &= p(u_1u_2^* + u_2u_1^*) - D_x^2(v_1\cdot v_2) \end{aligned} \tag{60}$$

respectively. One can show that the right-hand side of the second equation in (60) vanishes. On the other hand, the non-vanishing part on the right-hand side of the first equation can be written as

$$B(u_1\cdot v_2) = p\frac{(k_1-k_2)^2}{(k_1+k_1^*)(k_2+k_1^*)}e^{\theta_1+\theta_2+\theta_1^*} + p\frac{(k_1-k_2)^2}{(k_1+k_2^*)(k_2+k_2^*)}e^{\theta_1+\theta_2+\theta_2^*}. \tag{61}$$

One can solve the set of equations (60) to obtain the following results:

$$\begin{aligned} u_3 &= e^{\theta_1+\theta_1^*+\theta_2+B_{121}} + e^{\theta_1+\theta_2+\theta_2^*+B_{122}}, \\ v_3 &= 0, \end{aligned} \tag{62}$$

where

$$e^{B_{ijk}} = \frac{p(k_i - k_j)^2}{(k_i + k_k^*)^2 (k_j + k_k^*)^2}. \tag{63}$$

For $n = 4$ one can write the fourth-order equations as

$$iu_{4,t} + \frac{1}{2} u_{4,xx} = 0, v_{4,xx} = p(u_1 u_3^* + u_3 u_1^*) - \frac{1}{2} D_x^2 (v_2 \cdot v_2). \tag{64}$$

The solution of the first equation is obvious and after a bit of long but straightforward calculations the solution of the second equation can be obtained as

$$u_4 = 0,$$
$$v_4 = e^{\theta_1 + \theta_1^* + \theta_2 + \theta_2^* + C_{1212}}, \tag{65}$$

where

$$e^{C_{ijkl}} = \frac{p^2 \left(k_i - k_j\right)^2 (k_k^* - k_l^*)^2}{(k_i + k_k^*)^2 (k_i + k_l^*)^2 \left(k_j + k_k^*\right)^2 \left(k_j + k_l^*\right)^2}. \tag{66}$$

We should emphasize that because of the symmetry of constant factors in the solutions, both v_2 and v_4 are real functions as it is stated at the beginning. On the other hand, it can easily be shown that the coefficient $e^{C_{1212}}$ is positive. The perturbative expansion is truncated at this order and all the higher order terms can be chosen as zero; indeed, $u_n = v_n = 0, for\ n \geq 5$. Therefore, we end up with

$$q(x,t) = \frac{u_1 + u_3}{1 + v_2 + v_4} \tag{67}$$

for 2-soliton solution. After taking $p = 1$ and rearranging the terms one can write the solution for the fNLS equation as

$$q = \frac{\Lambda_{122}\, e^{i\xi_1} \cosh \zeta_2 + \Lambda_{121}\, e^{i\xi_2} \cosh \zeta_1}{\Lambda_{1212} \cosh(\zeta_1 + \zeta_2) + \Lambda_{11}\Lambda_{22} \cosh(\zeta_1 - \zeta_2) + \Lambda_{12}\Lambda_{21} \cos(\xi_1 - \xi_2)}, \tag{68}$$

where coefficients Λ_{ijkl} stand for the square root of the exponential coefficients with the same index structure in Equations (59), (63) and (66), $\zeta_i = \frac{\theta_i + \theta_i^*}{2}$ and $\xi_i = \frac{\theta_i - \theta_i^*}{2}$ are the real and the imaginary parts of θ parameters respectively. This solution is nonsingular since the condition $\Lambda_{1212} + \Lambda_{11}\Lambda_{22} > \Lambda_{12}\Lambda_{21}$ is satisfied by definitions of the coefficients. If we look at the dNLS solution, we encounter the same problem as in the 1-soliton case. Namely, for the case of $p = -1$, the second term in the denominator takes a minus sign and therefore the solution again includes a singularity.

3.2. Soliton solution to the dNLS equation

The soliton solutions obtained in the previous subsection are called bright solitons and they are characterized by the vanishing boundary values at infinity. Although the dNLS equation does not admit bright soliton solutions, it has been shown that by changing the boundary conditions it can support other kinds of soliton solutions which are called dark and gray solitons. They are typically in the form $q \sim e^{ikx} tanh(\omega t)$ and $q \sim e^{ikx}(\cos\alpha + i \sin\alpha \tanh\theta)$ respectively and in this sense, dark solitons are a special case of the gray solitons in the limit $\cos\alpha \to 0$. Such solitons satisfy the boundary conditions $|q|^2 \to constant$ as $|x| \to \infty$ and appear as localized dips on the finite background [28]. These kinds of solutions have been detected in various experiments [31-35].

To bring the equation into bilinear form, an appropriate redefinition of the dependent variable with the above-mentioned boundary condition is given by

$$q(x,t) = \rho e^{i\theta} \frac{u(x,t)}{v(x,t)}, \quad \theta = \alpha x - \beta t \tag{69}$$

where u and v are real functions, α and ρ are real constants, and $\beta = \alpha^2 - p\rho^2$. One can choose $u/v \to 1$ as $|x| \to \infty$ without loss of generality. Substituting this definition into Equation (41) leads to two distinct equations as follows

$$\begin{gathered}\left(iD_t + i\alpha D_x + \frac{1}{2}D_x^2\right)(u \cdot v) = 0, \\ \left(\frac{1}{2}D_x^2 + p\rho^2\right)(v \cdot v) = p\rho^2|u|^2 \ .\end{gathered} \tag{70}$$

Choosing the following ansatzes for the functions

$$\begin{gathered}u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \\ v = 1 + \epsilon v_1 + \epsilon^2 v_2 + \cdots\end{gathered} \tag{71}$$

one obtains,

$$\epsilon^n: (i\partial_t + i\alpha\partial_x)(u_n - v_n) + \frac{1}{2}\partial_x^2(u_n + v_n) = -(iD_t + i\alpha D_x + D_x^2)\sum_{k=1}^{n-1} u_k \cdot v_{n-k}, \tag{72}$$

and

$$\epsilon^n: v_{n,xx} + 2p\rho^2 v_n = -\frac{1}{2}D_x^2\left(\sum_{k=1}^{n-1} v_k \cdot v_{n-k}\right) - p\rho^2\left(\sum_{k=1}^{n-1} v_k \cdot v_{n-k} - \sum_{k=1}^{n-1} u_k \cdot u_{n-k}^*\right) \tag{73}$$

3.2.1 Soliton solution

Compared to section 3.1, we reached a different set of equations that require different ansatzes for solutions. We see that the leading order equation in Equation (72) contains both functions u and v on the left-hand side. Thus, one can assume

$$\begin{gathered}u_1 = e^{\eta_1 + 2i\phi_1}, \\ v_1 = e^{\eta_1},\end{gathered} \tag{74}$$

where $\eta_1 = \kappa_1 x + \omega_1 t + \tau_1$ and all the coefficients $\kappa_1, \omega_1, \tau_1$ and ϕ_1 are real constants. If one solves the first order ϵ equations, which assume $n = 1$ in Equations (72) and (73), one obtains

$$\begin{gathered}\omega_1 = \frac{\kappa_1^2}{2}\cot\phi_1 - \alpha\kappa_1, \\ \kappa_1^2 = -4p\rho^2\sin^2\phi_1.\end{gathered} \tag{75}$$

As it can be seen from Equations (75), these coefficients and hence the related functions are real only for $p = -1$. Therefore, we proceed with this p-value. One can easily show that solving the second-order equations, which are given by $n = 2$ in Equations (72) and (73), gives $u_2 = 0 = v_2$. All the higher-order terms in the expansion (71) can be set to zero by the same reasoning. Hence, after a few easy calculations one obtains the 1-soliton solution as

$$q(x,t) = \rho e^{i(\theta+\phi_1)}\left(\cos\phi_1 + i\sin\phi_1\tanh\frac{\eta_1}{2}\right), \tag{76}$$

which defines a gray soliton.

3.2.2 Soliton solution

In order to derive the 2-soliton solution we take two exponential functions in the ansatzes for the leading order functions

$$\begin{gathered}u_1 = e^{\eta_1 + 2i\phi_1} + e^{\eta_2 + 2i\phi_2}, \\ v_1 = e^{\eta_1} + e^{\eta_2},\end{gathered} \tag{77}$$

where $\eta_i = \kappa_i x + \omega_i t + \tau_i$. When these ansatzes plugged into Equations (72) and (73) with $n = 1$, it gives the dispersion relations

$$\omega_i = \frac{\kappa_i^2}{2}\cot\phi_i - \alpha\kappa_i\,,$$
$$\kappa_i^2 = -4p\rho^2\sin^2\phi_i\,. \tag{78}$$

The next order terms in the expansion (71) can be obtained with $n = 2$ in equations (72) and (73) and with the help of Equation (77), as

$$u_2 = e^{\eta_1+\eta_2+2i(\phi_1+\phi_2)},$$
$$v_2 = e^{\eta_1+\eta_2+A_{12}}\,. \tag{79}$$

Also, one can show that the coefficient in v_2 is given by

$$e^{A_{12}} = \left[\frac{\sin\left(\frac{1}{2}(\phi_1-\phi_2)\right)}{\sin\left(\frac{1}{2}(\phi_1+\phi_2)\right)}\right]^2. \tag{80}$$

The perturbative expansion is truncated at this level and all the higher-order terms can be set to zero as before, $u_n = 0 = v_n,\ n \geq 3$. Here we see that the coefficient given by the equation (80) is positive for real ϕ's and by using that, one can write the solution as

$$q = \frac{\rho e^{i(\theta+2\phi_+)}}{\Delta_{12}\cosh\eta_+ + \cos\eta_1}\{\Delta_{12}\cosh\eta_+\cos(2\phi_+) + \cosh\eta_-\cos(2\phi_-)$$
$$+ i\Delta_{12}\sinh\eta_+\sin(2\phi_+) + i\sinh\eta_-\sin(2\phi_-)\}\,, \tag{81}$$

where the coefficient Δ_{12} stands for the square root of the coefficient given by equation (80), $\eta_{\mp} = \frac{\eta_1 \mp \eta_2}{2}$ and $\phi_{\mp} = \frac{\phi_1 \mp \phi_2}{2}$. Consequently, one obtains the 2-soliton solution of the dNLS equation as in Equation (81), and this procedure can be repeated in each following order to obtain the N-soliton solution. Although the calculations become more cumbersome and complex as more terms are added at each higher level, the application of the method is quite systematic and clear. In this sense, the Hirota direct method is one of the most powerful methods to obtain N-soliton solutions of any integrable nonlinear differential equation.

4. Conclusion

Solitary wave solutions of nonlinear differential equations are an active research topic. These solutions appear in a variety of types, such as solitons, kinks, peakons, cuspons, and others. They play a significant role in almost every branch of physics from fluid dynamics [36] and oceanography [37] to Bose-Einstein condensation [38] and cosmology [39]. Solitons are a special type of solitary wave solutions due to their particle-like properties and because of that, they attract a great deal of interest. It is now quite well understood that solitons appear as a result of a balance between the competing properties, weak nonlinearity, and dispersion. Soliton solutions can be obtained by various methods. Although the inverse scattering technic is the most powerful one, its applicability to practical problems is a bit troublesome. Bäcklund transformation, Darboux transformation, Painleve expansion method can be mentioned as other solution technics. However, the Hirota method is the most efficient way to obtain the multi-soliton solutions of nonlinear differential equations. Many soliton equations, such as the nonlinear Schrödinger equation [18], the 2-dimensional Toda lattice [40], the AKNS hierarchy [41] and some equations constrained from the high-dimensional KP hierarchy [42] admit solutions in Hirota forms. Recent researches have shown that the Hirota method can also be used to construct soliton solutions with rogue-like phenomena [43]. These are localized waves both in time and in space and Peregrine solution [44] was the first example of this kind of solution. They represent an unexpected wave event on an otherwise flat background and are observed in water waves [45] and in optical fibers [46]. Because of that property, they are called “waves that appear from nowhere and disappear without a trace". Recently, for high-dimensional soliton equations, there are a lot of work on lump solutions by the Hirota method [47]. Furthermore, the Hirota method can also be used to solve the nonlinearization systems of Lax pairs [48, 49]. The advantage of the Hirota method is that it does not depend on Lax pairs.

Recently, the supersymmetric (susy) extensions of integrable systems are another hot topic that has been studied a lot. *N=1* susy extension of the KdV equation is defined by Manin and Radul [50], and Mathieu [51]. *N=2* susy extension of the KdV equation is defined in [52] and later other susy extensions of the KdV equation have also defined [53]. In a similar fashion, susy extensions of the other integrable systems are defined as well [54, 55]. The integrability of these susy extended models have been proved by similar methods to the original bosonic counterparts: infinite number of conservation laws, a bihamiltonian structure, the Lax operator etc. Another method of integrability is the existence of soliton solutions and for this aim, Hirota method has also been adapted to bilinearize supersymmetric systems [56-58].

In this study, we give an overview of the Hirota direct method. We construct the bilinear forms and study the multi-soliton solutions of the KdV and the nonlinear Schrödinger equations by using Hirota's method. We explicitly demonstrate how both bright/dark one and two-soliton solutions of the nonlinear focusing/defocusing Schrödinger equations can be obtained. We showed that fNLS equation admits bright soliton solutions for the vanishing boundary value at infinity. An appropriate redefinition of the dependent variable splits up the equation into a set of bilinear equations which are to be solved to obtain the term of the solution by term. On the other hand, dNLS equation admits dark soliton solutions by changing both the boundary condition and the definition of the dependent variable. The powerful property of Hirota's method is that it gives solutions in terms of a series of exponential functions and this series expansion truncates at a certain finite order for any soliton degree. Hence, one can obtain soliton solutions of any degree directly without dealing with the initial value problem of the related differential equation.

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