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Research Article

Use of the Theory of Fischbein and the Theory of Shulman for the study of teachers' algorithmic knowledge concerning the concept of the altitude of a triangle

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Abstract

In this study I have used a theoretical framework integrating the theory of Fischbein and the theory of Shulman for examining the algorithmic knowledge of teachers. This framework has shaped the approach to both the formal and algorithmic aspects of knowledge, with reference to several components included in mathematical knowledge as well as pedagogical content knowledge. In this study it has been found that all teachers draw the altitudes of triangles as required. In addition, it has been found that most errors described by teachers have indeed been observed among students and that almost all errors observed among students were described by the teachers. Teachers are aware of the fact that concept imaging is a major cause of error but are not familiar with the term "concept image". Study participants included 32 teachers teaching math at junior high schools, and 25 eighth grade students learning the concept of altitude in a triangle. In order to check the mathematical-algorithmic knowledge and pedagogical-algorithmic knowledge of teachers, a questionnaire was developed including right-angled triangles and obtuse-angled triangles. The teachers were asked to draw in each of the triangles the altitude to the marked side, draw characteristic errors students make when drawing altitude, and possible sources of error. The students received a questionnaire that included the same triangles presented to the teachers and were asked only to draw the altitude to the marked side. When analyzing the findings I related to the drawings of the teachers, characteristic errors of students in drawing altitudes noted by the teachers, and sources of error in the drawings of the students.

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Introduction

Teaching is a dynamic profession with changes that involve, inter alia, technology, learning theory, students, study programs, teaching approaches, knowledge approaches, and more. These changes require teachers to pursue perpetual learning and enhancement of knowledge throughout their careers, in order to improve teaching quality (National Council of Teachers of Mathematics [NCTM], 2000).

Shulman (1986) relates to the types of knowledge required for teaching. In this study I shall focus on two of them:

Subject matter knowledge (SMK) of the studied material: This component includes knowledge of concepts, definitions, and claims, how the validity of claims is established, knowledge of the different ways in which concepts and claims are organized and interconnected, and how the ways of proving truths and refuting false claims are constructed and institutionalized (Shulman, 1986).

Pedagogical content knowledge (PCK): This component includes knowledge of the most beneficial ways of presenting ideas: analogies, examples, explanations, demonstrations, and illustrations. Understanding the degree of difficulty of a certain topic, conceptions and preconceptions students have, and knowledge of effective methods for dealing with students' misconceptions (Shulman, 1986).

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Studies focused on the teaching of geometry indicate a lack of knowledge among teachers on how to cope with students' difficulties, and the need to delve into and assess students' level of understanding (Gal, 1998; 2011; Gal & Vinner, 1997). Testing the SMK and PCK of teachers in this study was performed with reference to one of the components from the theoretical framework developed by Fishbein (1983) according to which three main aspects of knowledge can be distinguished in mathematical activity: the intuitive aspect, the algorithmic aspect, and the formal aspect. In this study I shall focus on the *algorithmic aspect*. The algorithmic aspect relates to the standard procedures a child learns during childhood, and later on, when he deals in higher mathematics. This knowledge includes solution methods, computational knowledge, and understanding the processes performed while solving. This is by nature procedural knowledge, i.e., knowledge that includes knowing the stages of the various procedures employed in performing the activity.

In their study, Tsamir and Tirosh (2008) described how integration of the two theories can contribute to testing teachers' knowledge. Table 1 describes the integration of Shulman's theory with Fischbein's theory. The structure in the table depicts integration of the dimensions of teachers' knowledge (SMK and PCK) according to Shulman with Fischbein's three aspects of mathematical knowledge (algorithmic, formal, and intuitive). We get six cells depicted in the table. In this study I shall treat two cells in connection with the concept of the altitude of a triangle.
Cell 1 – SMK algorithm: relates to the mathematical knowledge of teachers in drawing the altitude in various triangles
Cell 4 – PCK algorithm: Relates to the mathematical knowledge of teachers on characteristic errors of students in drawing the altitude in various triangles, and potential causes of error.

Table 1.
Integration of the Theories of Shulman and Fischbein

Shulman's theory / **Fishbein's theory**	**SMK**	**PCK**
Algorithmic aspect	Cell 1 – SMK algorithm	Cell 4 – PCK algorithm
Formal aspect	Cell 2 – SMK formal	Cell 5 – PCK formal
Intuitive aspect	Cell 3 – SMK intuitive	Cell 6 – PCK intuitive

In Hershkowitz's (1987) study students of grades 5-8 were asked to draw the altitude to a marked side of a triangle. The test included a collection of triangles, and the students were asked to draw the altitude to the marked side of each triangle. It was found that students had more difficulty drawing the altitude of a right-angled triangle and an obtuse-angled triangle than an acute-angled triangle. Several incorrect solutions were gotten for the task (drawing altitude to side a): Methodical drawing of a median to side a, thus violating the requirement of "perpendicular"; drawing a certain segment within the triangle that exits one of the vertices; drawing a perpendicular bisector to side a, thus violating the requirement of exiting the vertex; drawing the altitude to a side other than a, while making sure the altitude is within the triangle.

According to the Fischbein and Nachlieli (1998) study, students of grades 9-11 were asked to define the altitude of a triangle and draw the altitude to a certain side in three triangles. In the study it was found that students who correctly define the altitude of a triangle often fail to correctly draw the heights of triangles that are not acute angled.

Gutierrez and Jaime (1999) studied images of the concept "altitude of a triangle" among 190 teachers. The common errors in drawing altitudes were: Drawing a median instead of the altitude; drawing a perpendicular bisector, thus violating the requirement of exiting the vertex; partial concept image that does not include altitudes outside of a triangle; partial concept image without reference to the length of the altitude. Here we have a violation of the requirement of "being a segment of the triangle", drawing a line outside of the triangle in all of the triangles (over inclusion from an obtuse-angled triangle in which one of the altitudes is outside of the triangle).

In Linchevsky's study (1985), elementary school teachers were asked to draw altitudes in triangles. In one of the questions, the teachers were asked to draw all the altitudes in five triangles. Four different types of answers were gotten**:** Altitude to the horizontal side only – the concept image in this group of responders includes perpendicularity to the lower margins of the page; hence, they draw only one altitude, and this only in triangles in which the triangle's position includes a side parallel to the lower margins of the page; median – the altitude is replaced by a median. The difficulty is revealed only in some cases, because in some of the triangles the median and the altitude of one of the sides converge; a vertical exiting the end of a side (sometimes within and sometimes outside of the triangle) – the concept of altitude in this group of responders contains elements of perpendicularity, and the vertical exiting the end of a side or continuation of a side, altitude to a "non-problematic" side – since each triangle has at least one side to which the altitude is less problematic, in the sense that it is drawn within the triangle, the responders belonging to this group did indeed choose this altitude in each triangle.

Vinner (1991) refers to the difference between "concept definition" and "concept image". While the definition of a concept is the formal representation of the concept as at appears in the definition, the concept image is the representation of the concept for the student, and usually includes an example that constitutes a prototype, a visual representation, an assemblage of properties, and connections with other concepts and associations.

Vinner and Hershkowitz (1983) note that the prototype of a concept often constitutes the basis for judgment. These scholars identified an inclination to use an example that constitutes a prototype as a framework of reference for judging the other examples, instead of using the concept definition. For example: The "content" of the altitude of a triangle is an example of a prototype among examples of altitudes in a triangle, created apparently due to exposure mainly to acute-angled triangles. As a result, in acute-angled triangles, there is a common inclination to draw an internal segment that is not the altitude as the altitude.

In a study by Hilf and Abu Naja (2019), the students were presented with the pictures of 48 different obtuse-angled triangles, each with a segment drawn in it. The students were asked to determine whether the segment is the altitude of the triangle. It was found that students are more successful at identifying altitudes found within triangles.

No studies were found examining teachers' knowledge of characteristic errors of students being asked to draw altitudes. The present study treats this issue.

As studies show, most errors in drawing and identifying altitudes occur with obtuse-angled triangles and right-angled triangles. The main erroneous inclination is to draw a line within the triangle. I shall treat each characteristic error and indicate possible causes for it based on the literature.

Drawing altitude only within the triangle: A potential cause of this error is a high concept image, based apparently on experience with acute-angled triangles (Hilf and Abu Naja, 2019; Linchevsky, 1985; Gutierrez & Jaime, 1999).

Drawing an altitude to a horizontal side only: A possible cause of this error is instructions presenting mainly examples of acute-angled triangles in a characteristic position in which one of the sides is parallel to the lower margins of the page (Linchevsky, 1985; Gutierrez & Jaime, 1999).

Drawing a median instead of an altitude: In some curricula, the two segments are studied consecutively, and this is a potential reason for the inclination to draw a median (within the triangle) instead of an altitude (Ministry of Education, 2014).

In this study I have checked whether students' errors in drawing altitude and the errors noted by teachers in this context are similar to the errors noted in the studies.

Method

Research Model

This research is a descriptive study in the form of a field survey, but it has a qualitative character due to the examination of the documents obtained from the students in order to reveal the intellectual skills in a special field of mathematics in detail and in-depth.

Participant

Thirty-two junior high school mathematics teachers in the north and in the south of Israel participated in this study. The participants had different experiences (at least 3 years) in teaching mathematics, and taught mathematics in different grades. All the participants taught the geometric concept "altitude", and twenty-five eighth grade students learned the concept of altitude of a triangle.

The Research Tool

In this study, several research tools were used for the following purposes:

a. Testing mathematical-algorithmic knowledge of teachers concerning the concept of altitude in various types of triangles.
b. Testing pedagogical-algorithmic knowledge of teachers concerning the concept of altitude in various types of triangles, and potential causes of error.

In order to check the mathematical-algorithmic knowledge and pedagogical-algorithmic knowledge of teachers, a questionnaire was developed, designed to check the algorithmic SMK and algorithmic PCK of teachers with regard to this concept.

The questionnaire included two types of triangles: Right-angled (five triangles) and obtuse-angled (four triangles). The triangles were presented in different positions (see tables 3 and 4). The teachers were asked to:

a. Draw the altitude to a certain side.
b. Draw characteristic errors of students and refer to potential causes of error.

In addition to this questionnaire, relating directly to the research questions, a students' questionnaire was developed including the same triangles presented to the teachers, in which they were asked to draw only the altitude to the marked side.

The questionnaires were submitted for review by two mathematic education scholars, and the questionnaires were altered according to their comments.

Data Analysis

In the study I checked the mathematical-algorithmic SMK and algorithmic PCK of the teachers. The mathematical SMK of the teachers was checked through several teachers, who correctly drew the altitude to the requested side of the triangle. The algorithmic PCK of the teachers was checked through the characteristic errors of students when drawing an altitude noted by the teachers, and through errors made by students in drawing the altitude to the marked side. The characteristic errors noted by the teachers and errors in drawing altitudes observed among the students were divided into categories. For each category, I marked whether it was noted by teachers and whether it was observed among the students [(+) noted or observed; (-) not noted or not observed]. I then checked for each type of triangle (right-angled and obtuse angled) while relating to each triangle of both types, how many teachers noted the category and among how many students it was observed. The teachers were also asked to note potential causes of error among students. I classified the sources of error in categories and noted Number of teachers who noted the category at least once, the number of teachers that noted each category for each type of triangle, the overall number of categories for each type of triangle, and the number of times each category was noted for all triangles as well as examples for each category.

Results

Mathematical and Pedagogical Content Knowledge

In this chapter I shall describe the findings connected with the SMK and the mathematical-algorithmic PCK with regard to the concept of altitude. First, the findings connected with algorithmic SMK will be presented, and then those connected with algorithmic PCK.

Algorithmic SMK

As part of the study, the teachers were presented with nine triangles (five right-angled and four obtuse-angled). The teachers were asked to draw the altitude to a certain side of each triangle.

It was found that all the teachers correctly drew altitudes in all of the triangles.

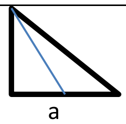
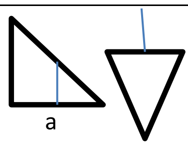
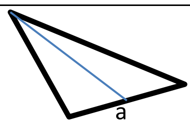
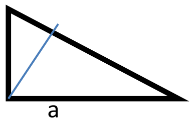
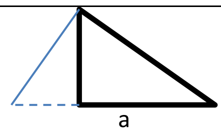
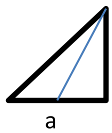
Algorithmic PCK

As part of the study, in addition to drawing altitudes, the teachers were asked to note, with regard to each drawing, characteristic errors by students and potential causes of them. I shall first refer to the drawings, and then to the causes of error.

Treatment of the pedagogical-algorithmic knowledge in this part of the findings' description is based on the errors observed among these students.

Table 2 details all types of errors observed among the students and described by the teachers, including examples of each error (it should be noted that sorting and naming of the errors was performed and validated by two experts in mathematics education). I shall first present the findings that apply to right-angled triangles, and then as apply to obtuse-angled triangles.

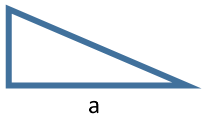
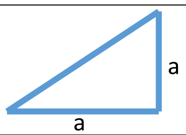
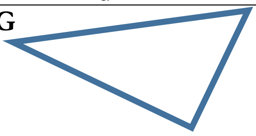
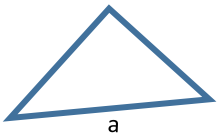
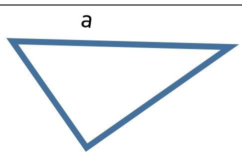
Table 2.
Categories of Characteristic Errors and Examples for Each Category

	The category	Example	The category Noted by Teachers	Category observed among students
a	Ignoring the "perpendicular" property Abbreviated: Ignoring perpendicular		+	+
b	Ignoring the "exit from vertex" property. Abbreviated: Ignoring vertex.		+	+
c	Ignoring the two properties "perpendicular" and "exit from vertex" Abbreviated: Ignoring perpendicular and vertex.		+	+
d	Drawing altitude to side other than required side. Abbreviated: Ignoring required side.		+	+
e	Drawing to extend the side. Abbreviated: Extended side.		+	+
f	Ignoring critical property "perpendicular" and required side. Abbreviated: Ignoring perpendicular and required side.		+	-

Right-angled Triangles

Of the five right-angled triangles (E, F, G, H, J) presented to students and teachers, two were presented in characteristic positions (triangles F and E) in which one of the perpendiculars was parallel to the lower margins of the page. The remaining triangles were presented in different positions (drawings of triangles as shown in the questionnaire are depicted in table 3). Table 3 details the types of errors noted by the teachers and observed among the students for each triangle, as well as the number of teachers who noted each error and the number of students who made each error.

Table 3.
Types of Errors Noted by Teachers and Types of Students' Errors in Right-Angled Triangles

Type of error / The triangle	"Ignoring perpendicular" Teachers (Students)	"Ignoring vertex" Teachers (Students)	"Ignoring required side" Teachers (Students)	"Ignoring perpendicular and required side" Teachers (Students)	"Extended side" Teachers (Students)	Errors Total Teachers (Students)
E 	18 (2)	7 (6)	5 (5)	0 (0)	0 (2)	30 (15)
F 	19 (6)	8 (6)	11 (3)	3 (0)	2 (5)	43 (20)
G 	16 (5)	9 (1)	9 (1)	9 (0)	3 (2)	46 (9)
H 	21 (1)	9 (6)	2 (0)	2 (0)	2 (0)	36 (7)
J 	20 (3)	18 (6)	3 (0)	3 (0)	0 (1)	44 (10)

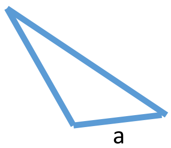
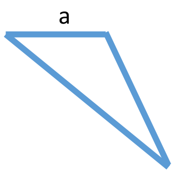
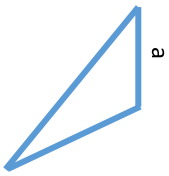
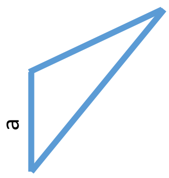
Number of errors noted by teachers as apply to three of the five triangles (triangles G, F, J) is similar (43-46 errors). The number of errors noted by the teachers in the two remaining triangles (triangles E and H) is smaller (30-36 errors). It should be noted that there is no correlation between the number of errors noted by the teachers with regard to each triangle and the number of errors observed among the students for each triangle. With regard to triangle G, for example, the frequency of the errors noted by the teachers is the highest, and the number of errors observed among the students is low in relation to the number of errors observed in the other triangles. Number of errors observed among students for the two triangles (triangles E, F) is similar (15-20 errors). Number of errors observed among students for the three other triangles (triangles G, H, J) is smaller (7-10 errors).

Obtuse-angled Triangles

Out of four obtuse-angled triangles presented to students and teachers, one was presented in typical position (L triangle) in which the required side is parallel to the lower margins of the page. The rest of the triangles were presented in different positions (The drawings of the triangles as presented in the questionnaire are depicted in table 4).

Table 4 details the types of errors noted by teachers and observed among students, with regard to each of the four triangles, as well as the number of teachers who noted each error and the number of students who made each error.

Table 4.
Types of Errors Noted by Teachers and Types of Students' Errors in Obtuse-Angled Triangles

Type of error / Triangle	"Ignoring perpendicular" Teachers (Students)	"Ignoring vertex" Teachers (Students)	"Ignoring required side" Teachers (Students)	"Ignoring perpendicular and vertex" Teachers (Students)	"Ignoring perpendicular and required side" Teachers (Students)	Errors Total Teachers (Students)
L 	18 (0)	6 (6)	13 (6)	0 (1)	0 (0)	37 (13)
M 	20 (2)	4 (5)	14 (6)	0 (0)	0 (0)	38 (13)
N 	19 (4)	4 (3)	13 (11)	3 (0)	3 (0)	42 (18)
P 	24 (3)	2 (4)	13 (7)	3 (2)	0 (0)	42 (16)

With regard to each of the triangles, in all of them it can be seen that most of the errors noted by the teachers are "ignoring perpendicular" and "ignoring required side". In triangles M, L and P, most of the errors observed among students are "ignoring vertex" and "ignoring required side". In the N triangle they are "ignoring required side".

The number of errors noted by teachers is similar for all triangles (37-42). It should be noted that there is no correlation between the number of errors noted by the teachers with regard to each triangle and the number of errors observed among the students for each triangle. With regard to triangle P, for example, the frequency of the errors noted by the teachers is the highest, and the number of errors observed among the students is low in relation to the number of errors observed in the other triangles.

Causes of Error in Drawing Altitudes

Table 5 describes the causes of errors according to the defined categories, the number of teachers noting each category for each type of triangle, the overall number of categories in each type of triangle, and the number of times each category was noted in each type of triangle. The category noted by the largest number of teachers (column 2 of the table) is the first, and the category noted least is the last (the categories are arranged in descending order of the number of teachers who noted them).

Table 5.

Categories of Causes of Error in Altitude Drawings (distribution in numbers)

The category	Number of teachers who noted the category at least once	Number of times the category was noted for right-angled triangles	Number of times the category was noted for obtuse-angled triangles	Number of times the category was noted (total for all triangles)
Concept image	30	19	67	86
Ignoring a critical property	28	33	21	54
Forgot the definition	26	68	15	83
Confusion between terms	22	39	4	43
Difficulty drawing	22	11	6	17
Drawing to non-required side	20	17	19	36
Inaccuracy in drawing	18	6	0	6
Confusion between types of triangles	17	21	13	34
Total		214	145	

From the table it can be seen that teachers noted several categories. I shall present examples for each category:
A large number of teachers (30) noted causes attributed to the "concept image" category (the teachers did not use that term). In this category we included statements that included reference to prototypical properties of altitudes, noted in the literature. For example: "Altitude is always from top down", "he will think the altitude must be within the triangle", "he doesn't know the altitude may be external". This category was noted a great many times (67) in obtuse-angled triangles. It should be noted that the teachers did not use the term "concept image" (this is the name I gave this category). Most of the teachers (28) cited causes attributed to the "ignoring a critical property" category, for example: "The student is unaware that the altitude is perpendicular to the side", and thus "ignores the requirement that it must exit the vertex". This category was noted a great many times (33) in right-angled triangles. It should be noted that the teachers did not use the term "concept image" (this is the name that I gave this category).

A large number of teachers (26) noted the category "forgot the definition". This category was noted a fairly large number of times (68) for right-angled triangles.
Twenty-two teachers noted the category "confusion between terms", for example: "Confused with median" or "confused with perpendicular bisector". This category was noted mainly in connection with right-angled triangles (39).

Twenty-two teachers noted causes attributed to the "difficulty drawing" category, for example: "The student shall not draw at all since he does not know how to form a right angle with side a"; "the student will errand think that side a is the altitude and will therefore not draw". This category was noted a great many times in right-angled triangles.

About two thirds of the teachers (20) noted causes attributed to the "drawing to non-required side" category, for example: "He will get confused and draw to another side"; "he will draw to another side for the sake of ease in drawing". This category was noted mainly for right-angled triangles and obtuse-angled triangles.

Eighteen teachers noted causes attributed to the "inaccuracy in drawing" category, for example: "Difficulty in drawing accurately". This category was noted mainly in connection with acute-angled triangles (20) and was not noted for obtuse-angled triangles.

With regard to all types of triangles, the categories "failure to understand the definition", "concept image", and "ignoring a critical property" were noted a great many times. The largest number of causes of error were noted concerning right-angled triangles (214) and after that with regard to obtuse-angled triangles (145). In right-angled triangles, the category "forgot the definition" is the most common for causes of error. In obtuse-angled triangles, the most common category of reasons for error is "concept image". This applies to the conception, according to which the altitude of a triangle is a segment within a triangle; hence, the difficulty with obtuse-angled triangles, in which the altitude from the vertex of the acute angle lies outside of the triangle.

Summary of Findings

A main finding of the present study, in connection with the algorithmic aspect, is that all teachers who participated in the study accurately drew the altitudes, in different types of triangles, in different positions.
to the best of my knowledge, no test has hitherto been made of teachers' knowledge of students' characteristic errors with regard to the algorithmic aspect (drawing) of altitudes in triangles and potential causes of these errors. Another

main finding is that the teachers who participated in the study noted most of the errors actually observed among students of the population in which the study was conducted. In addition, the teachers attributed the errors observed among the students to causes of characteristic student errors in various subjects described in the professional literature. For example, "ignoring a critical property" and "confusion between terms".

Discussion

The main purpose of the present study is to examine teachers' knowledge of two main components of mathematical knowledge required for teaching: Mathematical-algorithmic knowledge (algorithmic SMK) and algorithmic pedagogical-content knowledge (algorithmic PCK).

The teachers were asked to draw an altitude to marked sides of nine triangles (right-angled, obtuse-angled). All the teachers correctly drew the altitudes in all of the triangles. These findings do not correspond to those of Linchevsky (1985), who reported that some participants in the study conducted by her did not correctly draw altitudes in the triangles. This discrepancy is explained by the fact that while Linchevsky's study was conducted among elementary school teachers, the population of the present study is junior high school teachers, i.e., among teachers teaching math who specialized in teaching this subject. Elementary school teachers specialize in teaching a number of various subjects, and in many cases, mathematics is not one of them.

In analysis of the findings connected with algorithmic PCK concerning the concept of altitude, I shall refer to the answers given by teachers to the questionnaire, in which they were asked to draw the altitude of a triangle and note potential causes of error, and the degree of correspondence between the errors noted by the teachers and the errors made by the students as observed in this study and described in the literature.

It should be noted that almost all the errors described by teachers were indeed observed among students, and almost all the errors observed among students were described by the teachers.

The common errors observed among students with regard to right-angled triangles and obtuse-angled triangles were as follows: "Ignoring perpendicular", i.e., drawing a segment in a triangle that is not perpendicular to the opposite segment; "ignoring vertex", i.e., drawing a segment in a triangle that does not exit the vertex; and "ignoring required side", i.e., drawing an altitude to another side.

Most of the errors observed in this study are described in the research literature in mathematics education. The "ignoring perpendicular", "ignoring vertex", "ignoring perpendicular and vertex" and "ignoring required side" errors have been observed in the present study and described by several scholars (Fischbein & Nachlieli, 1998; Hershkowitz, 1987; 1989a; Vinner, 1991).

The "ignoring perpendicular" error (i.e., drawing a segment within a triangle instead of outside the triangle in the case of an obtuse-angled triangle or a segment merging with one of the sides in the case of a right-angled triangle) is discussed at length in the literature, in connection with the impact of the concept image on the way in which examinees respond to various problems in mathematical areas (Vinner, 1991).

The "extended side" error has been observed in the present study but has not been described in the research literature.

The teachers were asked to note potential causes of characteristic errors in drawing altitudes. To the best of my knowledge, this is the first study in which the teachers have been asked to note potential causes of students' errors in connection with drawing altitudes in triangles; therefore, in this case I cannot refer to the research literature.

The causes noted by the teachers have been classified into categories, and I shall briefly relate to each one (the order in which the causes are presented is according to descending frequency): *Concept image.* We shall clarify that prior to the teachers' training course none used the term "concept image". To this category were classified statements such as: "He'll think the altitude must be within the triangle"; "he'll think that the altitude is always from top down", *Ignoring a critical property*. To this category were classified statements such as: "Ignoring the requirements to start from the vertex"; "doesn't know that the altitude must be perpendicular to the side", *Forgot the definition*. To this category were classified statements such as: "Forgot the definition", *Difficulty drawing.* To this category were classified statements such as: "The student will not draw at all, because he won't know how to create a right angle with side a", *Confusion between terms*. To this category were classified statements such as: "Confused with median" or "confused with perpendicular bisector", *Drawing to non-required side*. To this category were classified statements such as: "He will err and draw to another side", *Confusion between types of triangles*. To this category were classified statements such as: "Gets confused with an acute-angled triangle".

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