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The Complex-type Pell p-Numbers in Finite Groups

Yeşim Aküzüm[a]

[a]*Department of Mathematics, Faculty of Science and Letters, Kafkas University 36100, Turkey*

Abstract. In this study, we study the complex-type Pell p-numbers modulo m and further we get the periods and the ranks of the complex-type Pell p-numbers modulo m. Additionally, we give some results on the periods and the ranks of the complex-type Pell p-numbers modulo m. Then, we consider the multiplicative orders of the complex-type Pell p-matrix when read modulo m. Also, we redefine the complex-type Pell p-numbers by means of the elements of groups. Finally, we produce the periods of the complex-type Pell 2-numbers in the semidihedral group SD_{2^m}, $(m \geq 4)$.

1. Introduction

The complex-type Pell p-numbers for any given p $(p = 2, 3, \ldots)$ is defined [2] by the following recurrence equation:

$$P_p^*(n+p+1) = 2i^{p+1} \cdot P_p^*(n+p) + i \cdot P_p^*(n) \tag{1}$$

for $n \geq 1$, where $P_p^*(1) = \cdots = P_p^*(p) = 0$, $P_p^*(p+1) = 1$ and $\sqrt{-1} = i$.

In [2], the complex-type Pell p-matrix K_p had been given as:

$$K_p = \begin{bmatrix} 2i^{p+1} & 0 & \cdots & 0 & i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}_{(p+1)\times(p+1)}.$$

Then, for $n \geq p$, they found that

$$\left(K_p\right)^n = \begin{bmatrix} P_p^*(n+p+1) & iP_p^*(n+1) & iP_p^*(n+2) & \cdots & iP_p^*(n+p) \\ P_p^*(n+p) & iP_p^*(n) & iP_p^*(n+1) & \cdots & iP_p^*(n+p-1) \\ \vdots & \vdots & \vdots & & \vdots \\ P_p^*(n+2) & iP_p^*(n-p+2) & iP_p^*(n-p+3) & \cdots & iP_p^*(n+1) \\ P_p^*(n+1) & iP_p^*(n-p+1) & iP_p^*(n-p+2) & \cdots & iP_p^*(n) \end{bmatrix}, \tag{2}$$

in addition, the determinant of the K_p matrix is $(-1)^p i$.

Corresponding author: YA mail address: yesim_036@hotmail.com ORCID:0000-0001-7168-8429.
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Definition 1.1. *A sequence is well known to be periodic if after a certain point it consists only of repeats of a fixed subsequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.*

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_u = a_{u+1}$, $0 \le u \le n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \ge 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted as $F_A(G)$ in [6].

A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is indicated by $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ in [15].

In [9], Deveci and Shannon showed that the following conditions apply for every elements x, y of the group G:

Definition 1.2. (i) *Suppose that $z = a + ib$ such that a and b are integers and suppose that e is the identity of G, then*
$* \; x^z \equiv x^{a(mod|x|)+ib(mod|x|)} = x^{a(mod|x|)} x^{ib(mod|x|)} = x^{ib(mod|x|)} x^{a(mod|x|)} = x^{ib(mod|x|)+a(mod|x|)}$,
$* \; x^{ia} = \left(x^i\right)^a = (x^a)^i$,
$* \; e^u = e$,
$* \; x^{0+i0} = e$.
(ii) *Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ such that a_1, b_1, a_2 and b_2 are integers , then $(x^{z_1} y^{z_2})^{-1} = y^{-z_2} x^{-z_1}$.*
(iii) *If $xy \ne yx$, then $x^i y^i \ne y^i x^i$.*
(iv) *$(xy)^i = y^i x^i$ and $\left(x^i y^i\right)^i = x^{-1} y^{-1}$.*
(v) *$xy^i = y^i x$ and so $\left(xy^i\right)^i = x^i y^{-1}$ and $\left(x^i y\right)^i = x^{-1} y^i$.*

In [1, 3, 4, 8, 11, 16], the authors have produced the cyclic groups and the semigroups through some special matrices and then, they have studied the orders of these algebraic structures. The study of the recurrence sequences in groups began with the earlier work of Wall [21]. Also, the theory extended to some special linear recurrence sequences by several authors; see for example, [5, 7, 10, 12–15, 17–20, 22]. In this study, we study the complex-type Pell p-numbers modulo m and then we get the periods and the ranks of the complex-type Pell p-numbers modulo m. Then, we consider the multiplicative orders of the complex-type Pell p-matrix when read modulo m. Also, we redefine the complex-type Pell p-numbers with the elements of groups and then we give the periods of the complex-type Pell 2-numbers in the semidihedral group.

2. The Complex-type Pell p-Numbers in Finite Groups

Reducing the complex-type Pell p-numbers by a modulus m, we obtain a repeating sequence, indicated by

$$\left\{P^*_{p,m}(n)\right\} = \left\{P^*_{p,m}(1), \; P^*_{p,m}(2), \; \ldots, \; P^*_{p,m}(j), \; \ldots\right\}$$

where $P^*_{p,m}(n) = P^*_p(n)\,(modm)$. This relation has the same recurrence relation as in (1)

Theorem 2.1. *For $p \ge 2$, the sequence $\left\{P^*_{p,m}(n)\right\}$ is simply periodic sequence.*

Proof. Consider the set

$$W = \left\{ \left(w_1, w_2, \ldots, w_{p+1}\right) \mid w_v\text{'s are complex numbers } a_v + ib_v \text{ where } a_v \text{ and } b_v \text{ are integers such that } 0 \le a_v, b_v \le m-1 \text{ and } 1 \le v \le p+1 \right\}.$$

Suppose that the notation $|W|$ is the order of the set W. Since the set W is finite, there are $|W|$ distinct $p+1$-tuples of the complex-type Pell p-numbers modulo m. So, at least one of the $p+1$-tuples appears twice in the sequence $\left\{P^*_{p,m}(n)\right\}$. Then, the subsequence following this $p+1$-tuple repeats; that is, $\left\{P^*_{p,m}(n)\right\}$ is a periodic sequence. Let $P^*_{p,m}(k) \equiv P^*_{p,m}(l)$, $P^*_{p,m}(k+1) \equiv P^*_{p,m}(l+1)$, $\ldots$, $P^*_{p,m}(k+p+1) \equiv P^*_{p,m}(l+p+1)$ and $k \ge l$, then $k \equiv l\,(mod p+1)$. It is obvious that

$$P^*_p(n) = (-i) \cdot P^*_p(n+p+1) + 2i^{p+2} \cdot P^*_p(n+p).$$

So we get $P^*_{p,m}(k-1) \equiv P^*_{p,m}(l-1)$, $P^*_{p,m}(k-2) \equiv P^*_{p,m}(l-2)$, $\ldots$, $P^*_{p,m}(1) \equiv P^*_{p,m}(k-l+1)$, which indicates that $\left\{P^*_{p,m}(n)\right\}$ is a simply periodic. □

We indicate the period of the sequence $\left\{P^*_{p,m}(n)\right\}$ by $t_p(m)$.

For given a matrix $B = \left[b_{ij}\right]$ with b_{ij}'s being integers, $B\,(mod m)$ means that each element of B are reduced modulo m, that is, $B\,(mod m) = \left(b_{ij}\,(mod m)\right)$. If $(\det B, m) = 1$, then the set $\langle B\rangle_m$ is a cyclic group; if $(\det B, m) \neq 1$, then the set $\langle B\rangle_m$ is a semigroup. Let the notation $|\langle B\rangle_m|$ indicates the order of the set $\langle B\rangle_m$.

Since $\det K_p = (-1)^p\, i$, the set $\left\langle K_p\right\rangle_m$ is a cyclic group for every positive integer $m \ge 2$. It is easy to see from (2) that it is $t_p(m) = \left|\left\langle K_p\right\rangle_m\right|$.

Theorem 2.2. *Let v be a prime. If r is the smallest positive integer such that $t_p\left(v^{r+1}\right) \neq t_p(v^r)$, then $t_p\left(v^{r+1}\right) = vt_p(v^r)$ for every integer $p \ge 2$*

Proof. Suppose that r is the smallest positive integer such that $t_p\left(v^{r+1}\right) \neq t_p(v^r)$ and suppose that z is a positive integer. If $\left(K_p\right)^{t_p\left(v^{z+1}\right)} \equiv I\left(mod v^{z+1}\right)$, then $\left(K_p\right)^{t_p\left(v^{z+1}\right)} \equiv I\,(mod v^z)$. Thus we obtain that $t_p(v^z)$ divides $t_p\left(v^{z+1}\right)$. Also, writing $\left(K_p\right)^{t_p(v^z)} = I + \left(m^{(z)}_{i,j} \cdot v^z\right)$, by the binomial theorem, we obtain

$$\left(K_p\right)^{vt_p(v^z)} = \left(I + \left(m^{(z)}_{i,j} \cdot v^z\right)\right)^v = \sum_{i=0}^{v} \binom{v}{i} \left(m^{(z)}_{i,j} \cdot v^z\right)^i \equiv I\left(\text{mod } v^{z+1}\right).$$

and so it appears that $t_p\left(v^{z+1}\right)$ divides $v\,t_p(v^z)$. Therefore, $t_p\left(v^{z+1}\right) = t_p(v^z)$ or $t_p\left(v^{z+1}\right) = v\,t_p(v^z)$, and the latter holds if and only if there is a $m^{(z)}_{i,j}$ which is not divisible by v. Since we assume that r is the smallest positive integer such that $t_p\left(v^{r+1}\right) \neq t_p(v^r)$, there is an $m^{(z)}_{i,j}$ that is not divisible by v. This shows that ${}_p\left(v^{r+1}\right) = vt_p(v^r)$. So, the proof is complete. □

Definition 2.3. *The rank of the sequence $\left\{P^*_{p,m}(n)\right\}$ is the least positive integer α such that $P^*_{p,m}(\alpha) \equiv P^*_{p,m}(\alpha+1) \equiv \cdots \equiv P^*_{p,m}(\alpha+p-1) \equiv 0\,(mod m)$, and we indicate the rank of $\left\{P^*_{p,m}(n)\right\}$ by $r_p(m)$.*

If $P^*_{p,m}(\alpha+p-1) \equiv 0\,(mod m)$, then the terms of the sequence $\left\{P^*_{p,m}(n)\right\}$ starting with index $r_p(m)$, namely $\underbrace{0, 0, \ldots, 0}_{p}, \theta, \theta, \ldots$, are exactly the initial terms of $\left\{P^*_{p,m}(n)\right\}$ multiplied by a factor θ.

The exponents ϖ for which $\left(K_p\right)^{\varpi} \equiv I\,(modm)$ form a simple aritmetic progression. So we give

$$\left(K_p\right)^{\varpi} \equiv I\,(modm) \Longleftrightarrow t_p\,(m) \mid \varpi.$$

Similarly, the exponents ϖ for which $\left(K_p\right)^{\varpi} \equiv \theta I\,(modm)$ for some $\theta \in \mathbb{C}$ form a simple aritmetic progression, and so

$$\left(K_p\right)^{\varpi} \equiv \theta I\,(modm) \Longleftrightarrow r_p\,(m) \mid \varpi.$$

Thus, it is simple to show that $r_p\,(m)$ divides $t_p\,(m)$.

The order of the sequence $\left\{P^{*}_{p,m}\,(n)\right\}$ is defined by $\frac{t_p(m)}{r_p(m)}$ and we indicate it by $Q_p\,(m)$. Let $\left(K_p\right)^{r_p(m)} \equiv \theta I\,(modm)$, then $ord_m\,(\theta)$ is the least positive value of δ such that $\left(K_p\right)^{\delta r_p(m)} \equiv I\,(modm)$. So it is confirm that $ord_m\,(\theta)$ is the least positive integer δ with $t_p\,(m) \mid \delta r_p\,(m)$. Thus, we obtain $ord_m\,(\theta) = \delta$. As a result, we may easily conclude that $Q_p\,(m)$ is always a positve integer, and that $Q_p\,(m) = ord_m\left(P^{*}_p\left(r_p\,(m) + p\right)\right)$, the multiplicative order of $P^{*}_{p,m}\left(r_p\,(m) + p\right)$.

Example 2.4. *Since*

$$\left\{P^{*}_{5,2}\,(n)\right\} = \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, i, 0, 0, 0, 0, 0, 1, 0, \ldots, \},$$

we have $t_5\,(2) = 12$, $r_5\,(2) = 6$ *and* $Q_5\,(2) = 2$.

Theorem 2.5. *suppose that* m_1 *and* m_2 *are positive integers with* $m_1, m_2 \geq 2$, *then* $r_p\,(lcm\,[m_1, m_2]) = lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]$. *In the same way,* $t_p\,(lcm\,[m_1, m_2]) = lcm\left[t_p\,(m_1)\,, t_p\,(m_2)\right]$.

Proof. Let $lcm\,[m_1, m_2] = m$. Then

$$P^{*}_p\left(r_p\,(m)\right) \equiv P^{*}_p\left(r_p\,(m) + 1\right) \equiv \cdots \equiv P^{*}_p\left(r_p\,(m) + p - 1\right) \equiv 0\,(modm)$$

and

$$P^{*}_p\left(r_p\,(m_w)\right) \equiv P^{*}_p\left(r_p\,(m_w) + 1\right) \equiv \cdots \equiv P^{*}_p\left(r_p\,(m_w) + p - 1\right) \equiv 0\,(modm)$$

for $w = 1, 2$. Using the least common multiple operation implies that $P^{*}_p\left(r_p\,(m)\right) \equiv P^{*}_p\left(r_p\,(m) + 1\right) \equiv \cdots \equiv P^{*}_p\left(r_p\,(m) + p - 1\right) \equiv 0 modm_w$ for $w = 1, 2$. Hence we get $r_p\,(m_1) \mid r_p\,(m)$ and $r_p\,(m_2) \mid r_p\,(m)$, which signifies that $lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]$ divides $r_p\,(lcm\,[m_1, m_2])$. We also know that

$$P^{*}_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]\right) \equiv P^{*}_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + 1\right) \equiv \cdots \equiv P^{*}_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + p - 1\right) \equiv 0\,(modm_w)$$

for $w = 1, 2$. Then we can write

$$P^{*}_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]\right) \equiv P^{*}_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + 1\right) \equiv \cdots \equiv P^{*}_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + p - 1\right) \equiv 0\,(modm)\,,$$

and it follows that $r_p\,(lcm\,[m_1, m_2])$ divides $lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]$. Thus, the proof is complete.

The period $t_p\,(m)$ is proved with a similar proof method. □

Now we take into account the complex-type Pell p-numbers in groups.

Suppose that G be a finite j-generator group and let $X = \{(x_1, x_2, \ldots, x_j) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid\ <\left\{x_1, x_2, \ldots, x_j\right\}>= G\}$. We call $\left(x_1, x_2, \ldots, x_j\right)$ a generating j-tuple for G.

Definition 2.6. *Suppose that G is a j-generator group and suppose that* $\left(x_1, x_2, \ldots, x_j\right)$ *is a generating j-tuple for G. So we define the complex-type Pell p-orbit* $P_p^*\left(G; x_1, x_2, \ldots, x_j\right) = \left\{a_p(n)\right\}$ *as shown:*

$$a_p(n+p) = a_p(n-1)^i \, a_p(n+p-1)^{2i^{p+1}} \quad (n > 1)$$

where

$$\begin{cases} a_p(1) = x_1,\ a_p(2) = x_2, \ldots,\ a_p(j) = x_j, a_p(j+1) = e, \ldots, a_p(p+1) = e & \text{if } j < p+1, \\ a_p(1) = x_1,\ a_p(2) = x_2, \ldots, a_p(p+1) = x_{p+1} & \text{if } j = p+1. \end{cases}$$

Theorem 2.7. *Suppose that G is a j-generator group. If G is finite, then the complex-type Pell p-orbit of G is periodic.*

Proof. We think of the set

$$\begin{aligned} H = & \left\{\left((h_1)^{a_1(mod|h_1|)+ib_1(mod|h_1|)},\right.\right. \\ & (h_2)^{a_2(mod|h_2|)+ib_2(mod|h_2|)}, \ldots, \\ & \left.\left(h_j\right)^{a_j\left(mod\left|h_j\right|\right)+ib_j\left(mod\left|h_j\right|\right)}\right): \\ & \left. h_1, h_2, \ldots, h_j \in G \text{ and } a_n, b_n \in Z \text{ such that } 1 \leq n \leq j\right\}. \end{aligned}$$

If G is finite, the H is a finite set. For any $c \geq 0$, there exists $k \geq c + j$ such that $a_p(c+1) = a_p(k+1)$, $a_p(c+2) = a_p(k+2), \ldots, a_p(c+j) = a_p(k+j)$. Due to repeating, for all generating j-tuples, the sequence $P_p^*\left(G; x_1, x_2, \ldots, x_j\right)$ is periodic. □

We indicate the length of the period of the complex-type Pell p-orbit $P_p^*\left(G; x_1, x_2, \ldots, x_j\right)$ by $hP_p^*\left(G; x_1, x_2, \ldots, x_j\right)$. Now we give the lengths of the periods of the complex-type Pell 2-orbit of the semidihedral group SD_{2^m}. The semidihedral group SD_{2^m} of order 2^m is defined by the presentation

$$SD_{2^m} = \langle x, y \mid x^{2^{m-1}} = y^2 = e, \ y^{-1}xy = x^{-1+2^{m-2}} \rangle$$

for every $m \geq 4$. Note that the orders x and y are 2^{m-1} and 2, respectively.

Theorem 2.8. *For generating pairs* (x, y)*, the length of the period of the complex-type Pell 2-orbit in the semidihedral group* SD_{2^m} *is* $2^{m-3} \cdot t_2(2)$.

Proof. For the complex-type Pell 2-orbit, we consider $t_2(2) = 6$. The orbit $P_2^*(SD_{2^m}; x, y)$ is

$$\begin{array}{l} x, y, e, x^i, y^i x^2, x^{-4i}, x^{-9}, yx^{20i}, x^{44}, \\ x^{-97i}, y^i x^{42}, x^{-40i}, x^{17}, yx^{8i}, x^{56}, \ldots, \end{array}$$

and so the orbit becomes:

$$\begin{aligned} a_2(1) &= x,\ a_2(2) = y, a_2(3) = e, \ldots \\ a_2(2 \cdot t_2(2)\alpha + 1) &= x^{8\alpha\lambda_1 + 1}, a_2(2 \cdot t_2(2)\alpha + 2) = yx^{4\alpha\lambda_2 \cdot i}, a_2(2 \cdot t_2(2)\alpha + 3) = x^{4\alpha\lambda 3}, \ldots. \end{aligned}$$

where λ_1, λ_2 and λ_3 are positive integers such that $gcd(\lambda_1, \lambda_2, \lambda_3) = 1$. Thus, for $\beta \in \mathbb{N}$, we need the smallest integer α such that $8\alpha = 2^{m-1} \cdot \beta$. If we choose $\alpha = 2^{m-4}$, we get

$$a_2\left(2^{m-3} \cdot t_2(2) + 1\right) = x, a_2\left(2^{m-3} \cdot t_2(2) + 2\right) = y, a_2\left(2^{m-3} \cdot t_2(2) + 3\right) = e \ldots.$$

Since the elements succeeding $a_2\left(2^{m-3} \cdot t_2(2) + 1\right)$, $a_2\left(2^{m-3} \cdot t_2(2) + 2\right)$ and $a_2\left(2^{m-3} \cdot t_2(2) + 3\right)$ depend on x, y, e for their values, the cycle begins again with the $a_2\left(2^{m-3} \cdot t_2(2) + 1\right)$ nd element. Thus it is verified that the length of the period of the complex-type Pell 2-orbit in SD_{2^m} is $2^{m-3} \cdot t_2(2)$. □

Example 2.9. *The sequence* $P_2^*(SD_{64}; x, y)$ *is*

$$x, y, e, x^i, y^i x^2, x^{-4i}, x^{-9}, yx^{20i}, x^{12}, x^{-i}, y^i x^{10},$$
$$x^{-8i}, x^{17}, yx^{8i}, x^{24}, x^i, yx^{26}, x^{4i}, x^7, yx^{12i}, x^{20},$$
$$x^{-i}, y^i x^{18}, x^{16i}, x, yx^{16i}, x^{16}, x^i, y^i x^{18}, x^{12i}, x^{23},$$
$$yx^{4i}, x^{28}, x^{-i}, y^i x^{26}, x^{8i}, x^{17}, yx^{24i}, x^8, x^i, y^i x^{10},$$
$$x^{20i}, x^7, yx^{28i}, x^4, x^{31i}, y^i x^2, e, x, y, e, \ldots.$$

which implies that $hP_2^*(SD_{32}; x, y) = 48.$

3. Conclusion

In this study, we have considered the complex-type Pell p-numbers modulo m and then we have obtained the periods and the ranks of the complex-type Pell p-numbers modulo m. Also, we have studied the multiplicative orders of the complex-type Pell p-matrix when read modulo m. Finally, we have redefined the complex-type Pell p-numbers with the elements of groups and then we have obtained the periods of the complex-type Pell 2-numbers in the semidihedral group SD_{2^m}, $(m \geq 4)$.

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