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Araştırma Makalesi / Research Article

A Note on 2-Normed Grand Sequence Spaces

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Abstract

In this paper, we define 2-normed grand sequence space by inspiration of (Gunawan, 2001) and (Rafeiro et. al., 2018). Also, we give some basic properties of these spaces.

Keywords: Grand sequence space, 2-normed space, Lebesgue sequence space.

2-Normlu Büyük Dizi Uzayları Üzerine Bir Not

Öz

Bu çalışmada, (Gunawan, 2001) ve (Rafeiro et. al., 2018) çalışmalarından esinlenerek 2-normlu büyük dizi uzaylarını tanımladık. Ayrıca, bu uzayların bazı temel özelliklerini verdik.

Anahtar Kelimeler: Büyük dizi uzayları, 2-normlu uzaylar, Lebesgue dizi uzayları.

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1. Introduction

Let X be a real vector space of dimension greater than one. If the real valued function $||.,.||$ on $X \times X$ satisfying the following conditions, then $||.,.||$ is called a 2-normed on X;

N1- $||x,y|| = 0$ if and only if x and y are linearly dependent,

N2- $||x,y|| = ||y,x||$,

N3- $||cx,y|| = |c|||x,y||$ for arbitrary $c \in \mathbb{R}$,

N4- $||x+z,y|| \leq ||x,y|| + ||z,y||$ for every $x,y,z \in X$.

The concept of 2-normed space was introduced by Gahler (Gahler, 1964). The 2-normed spaces and generalization to the n-normed spaces studied by many authors (Duyar et. al., 2016; Duyar et. al., 2017; Ogur, 2018). Later, Gunawan (Gunawan, 2001) defined, by using the standard 2-norm on ℓ^2, the natural 2-norm $||.,.||_p$ on $\ell^p \times \ell^p$,1 $\leq$ p $<\infty$ as follows;

$$||x,y||_p = \left[\frac{1}{2}\sum_j\sum_k \left|det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|^p\right]^{\frac{1}{p}}$$

and

$$||x,y||_\infty = sup_j sup_k \left|det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|$$

for p $=\infty$. Also, he gave the fixed point theorem for n −normed $\ell^p -$ spaces.

Iwaniec and Sbordone (Iwaniec and Sbordone, 1992) introduced the grand Lebesgue spaces $L^{p)}, 1 < p < \infty$. These spaces were studied by many authors (Jain 2010; Samko, 2017). Later, Raferio et. al., (Rafeiro et. al., 2018) defined the grand sequence space $\ell^{p),\theta}(X), \theta > 0$, by the norm

$$||x||_{\ell^{p),\theta}} = sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{p(1+\varepsilon)}}||x||_{\mathrm{p}(1+\varepsilon)}$$

where $||.||_{\mathrm{p}(1+\varepsilon)}$ is the standard norm on $\ell^{\mathrm{p}(1+\varepsilon)}$ and X is one of the sets $\mathbb{Z}^n, \mathbb{Z}, \mathbb{N}$ and $\mathbb{N}_0$. They studied some operators of harmonic analysis. Later, (Oğur, 2020) defined the grand Lorentz sequence spaces and studied some basic properties such as multiplication operators.

2. Materials and Methods

In this paper, we inspired by the above observations and defined 2-normed grand sequence spaces with 2-norm $||x,y||_{p),\theta}$ given as follows;

Let $\theta > 0$ and $1 \leq p < \infty$. Let define the function $||.,.||_{p),\theta}$ on $\ell^{p),\theta}\times\ell^{p),\theta}$ by

$$||x,y||_{p),\theta} := sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k \left|det\begin{pmatrix} x_j & x_k \\ y_j & y_k\end{pmatrix}\right|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} . \qquad (1)$$

Also, we studied some basic properties of these spaces.

3. Findings and Discussion

Firstly, we show that $||.,.||_{p),\theta}$ makes sense;

Lemma 1. Let $\theta > 0$ and $1 \leq p < \infty$. By Minkowski's inequality, we have

$$||x,y||_{p),\theta} = sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k |x_jy_k - x_ky_j|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left(|x_jy_k| + |x_ky_j|\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq sup_{\varepsilon>0}\left[\left\{\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left(|x_jy_k|\right)^{p(1+\varepsilon)}\right\}^{\frac{1}{p(1+\varepsilon)}}\right.$$

$$\left.+\left\{\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left(|x_ky_j|\right)^{p(1+\varepsilon)}\right\}^{\frac{1}{p(1+\varepsilon)}}\right]$$

$$\leq\left(sup_{\varepsilon>0}2^{\frac{-1}{p(1+\varepsilon)}}\right)\left(2||x||_{\ell^{p),\theta}}||y||_{\ell^{p),\theta}}\right)$$

$$=2||x||_{\ell^{p),\theta}}||y||_{\ell^{p),\theta}}$$

which shows that $||.,.||_{p),\theta}$ makes sense.

Theorem 1. $\ell^{p),\theta}, 1\leq p<\infty$, is a 2-normed space with the function $||.,.||_{p),\theta}$.

Proof. It is easy to see N2) and N3) by the definition of the 2-norm. For N1), let $||x,y||_{p),\theta}=0$, then we have

$det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}=0$ if and only if x and y are linearly dependent.

For N4), let $x,y,z\in\ell^{p),\theta}$. Then, by Minkowski inequality and property of the determinant, we get

$$||x+y,z||_{p),\theta}=sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left|det\begin{pmatrix}x_j+y_j & x_k+y_k\\ z_j & z_k\end{pmatrix}\right|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left(\left|det\begin{pmatrix}x_j & x_k\\ z_j & z_k\end{pmatrix}\right|+\left|det\begin{pmatrix}y_j & y_k\\ z_j & z_k\end{pmatrix}\right|\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$\leq sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left|det\begin{pmatrix}x_j & x_k\\ z_j & z_k\end{pmatrix}\right|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$+sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_j\sum_k\left|det\begin{pmatrix}y_j & y_k\\ z_j & z_k\end{pmatrix}\right|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$=||x,z||_{p),\theta}+||y,z||_{p),\theta}.$$

Remark 1. By Lemma 2.4 in (Gunawan, 2001) we have that a sequence in ℓ^p is convergent (Cauchy sequence) in the 2-norm $||.,.||_p$ if and only if it is convergent (Cauchy sequence) in the usual norm $||.||_p$. Also, by 2.7. Theorem in (Swe, 2019), we have that the function $||x||^*_{\ell^{p),\theta}}$ defined by

$$||x||^*_{\ell^{p),\theta}}:=||x,z||_{p),\theta}+||x,w||_{p),\theta} \qquad (2)$$

, where z and w are linearly independent, is a norm on $\ell^{p),\theta}$.

Similarly, we get that a sequence in $\ell^{p),\theta}$ is convergent (Cauchy sequence) in the 2-norm $||.,.||_{p),\theta}$ if and only if it is convergent (Cauchy sequence) in the usual norm $||.||_{\ell^{p),\theta}}$. By using similar way as in (Gunawan, 2001), we have

Lemma 2. The derived norm $||.||^*_{\ell^{p),\theta}}$ is equivalent to the $||.||_{\ell^{p),\theta}}$ on $\ell^{p),\theta}$ and the inequality

$$2^{\frac{-1}{p}}||x||_{\ell^{p),\theta}}\leq||x||^*_{\ell^{p),\theta}}\leq 2||x||_{\ell^{p),\theta}} \qquad (3)$$

holds for all $x\in\ell^{p),\theta}$.

Proof. Let choose $e_1 = (1,0,0,\dots)$ and $e_2 = (0,1,0,\dots)$ $and\ define$ $||x||^*_{\ell^{p)},\theta}$ with respect to $\{e_1, e_2\}$. Thus, we have

$$\begin{aligned}||x||^*_{\ell^{p)},\theta} &= ||x, e_1||_{p),\theta} + ||x, e_2||_{p),\theta}\\ &= sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_{k\neq 1}|x_k|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &+ sup_{\varepsilon>0}\left[\frac{\varepsilon^\theta}{2}\sum_{k\neq 2}|x_k|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\leq 2\left(sup_{\varepsilon>0}2^{\frac{-1}{p(1+\varepsilon)}}\right) sup_{\varepsilon>0}\left[\varepsilon^\theta\sum_k|x_k|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\leq 2||x||_{\ell^{p)},\theta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}||x||_{\ell^{p)},\theta} &= sup_{\varepsilon>0}\left[\varepsilon^\theta\sum_k|x_k|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\leq sup_{\varepsilon>0}\left[\varepsilon^\theta|x_1|^{p(1+\varepsilon)} + \varepsilon^\theta|x_2|^{p(1+\varepsilon)} + 2\varepsilon^\theta\sum_{k\geq 3}|x_k|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &= sup_{\varepsilon>0}\left[2\frac{\varepsilon^\theta}{2}\sum_{k\neq 1}|x_k|^{p(1+\varepsilon)} + 2\frac{\varepsilon^\theta}{2}\sum_{k\neq 2}|x_k|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\leq sup_{\varepsilon>0}2^{\frac{1}{p(1+\varepsilon)}}\left\{\left(\frac{\varepsilon^\theta}{2}\sum_{k\neq 1}|x_k|^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} + \left(\frac{\varepsilon^\theta}{2}\sum_{k\neq 2}|x_k|^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\right\}\\ &\leq 2^{\frac{1}{p}}\left(||x, e_1||_{p),\theta} + ||x, e_2||_{p),\theta}\right)\\ &= 2^{\frac{1}{p}}||x||^*_{\ell^{p)},\theta}\end{aligned}$$

which gives the proof.
Now, we can give the following theorem.

Theorem 2. The space $\ell^{p),\theta}, 1 \leq p < \infty$, is a complete 2-normed space with its 2-norm $||.,.||_{p),\theta}$.

Proof. Let $(x(m))$ be a Cauchy sequence in $\ell^{p),\theta}$ with respect to $||.,.||_{p),\theta}$. By the Lemma 2 $(x(m))$ is a Cauchy sequence in $\ell^{p),\theta}$ with respect to $||.||_{\ell^{p)},\theta}$. Also, since the space $\ell^{p),\theta}$ is a complete space with respect to $||.||_{\ell^{p)},\theta}$, then there is $x \in \ell^{p),\theta}$ such that $lim_{m\to\infty}||x(m) - x||_{\ell^{p)},\theta} = 0$. By the inequality (3), $x(m)$ converges to x in $\ell^{p),\theta}$ with respect to $||.,.||_{p),\theta}$. This shows $\ell^{p),\theta}$ is a complete 2-normed space with respect to $||.,.||_{p),\theta}$.

Theorem 3. Let, F be a self-mapping on $\ell^{p),\theta}$ and contractive with respect to $||.,.||_{p),\theta}$. Then, F has a unique fixed point with respect to derived norm $||x||^*_{\ell^{p)},\theta}$.

Proof. Using similar way as in (Gunawan, 2001) and by the inequality (3), the proof can be obtained.

4. Conclusions and Recommendations

Here, we give the definition of 2-normed grand sequence space and show that $\ell^{p),\theta}$ is a complete 2-normed space with respect to its 2-norm $||.,.||_{p),\theta}$. Also, we get an inequality for derived norm $||x||^*_{\ell^{p)},\theta}$. The results in this paper can be generalized to the n-normed concept as in (Gunawan, 2001).

Statement of Conflicts of Interest

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

References

Duyar, C., Kanber, O. and Sağır, B., (2016). On n-normed Cesaro sequence space *Ces*n,p. *Int. Electron. J. Pure Appl. Math.,* 10(2): 151-159.

Duyar, C., Sağır, B. and Oğur, O., (2017). On a class of n-normed double sequences related to p-summable double sequence space $\ell_p^{(2)}$. *E. J. Mathematical Anal. App.*, 5(1): 106-111.

Gahler, S., (1964). Lineare 2-normierte Raume, *Math. Nachr.*, 28: 1-43.

Gunawan, H., (2001). The space of p-summable sequences and its natural n-norm. *Bull. Austral. Math. Soc.,* 64, 137-147.

Iwaniec, T. and Sbordone, C., (1992). On the integrability of the Jacobian under minimal hypotheses, *Arch. Ration. Mech. Anal.*, 119(2), 129-143.

Jain, P. and Kumari, S., 2010. On grand Lorentz spaces and the maximal operator, *Math. Student*, 79.

Oğur, O., (2018). Superposition operator on some 2-normed sequence spaces. *Karaelmas Science and Engineering journal*, 8(1), 288-291.

Oğur, O., (2020). Grand Lorentz sequence space and its multiplication operator. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics* , 69 (1) , 771-781.

Rafeiro, H., Samko, S. and Umarkhadzhiev, S., (2018). Grand Lebesgue sequence spaces. *Georgian Math. J.*, 19(2), 235-246.

Samko, S. and Umarkhadzhiev, S., (2017), On grand Lebesgue spaces on sets of infinite measure, *Math. Nachr.*, 290, 913-919.

Swe, T. T., (2019). Bounded linear 2-functionals in linear 2-normed space. *Dagon University Commemoration of 25. Anniversary Silver Jubilee Research Journal, 9, 203.*