A consistent statistical test based on bivariate random samples

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Abstract
We propose a consistent test for testing the distribution of bivariate random samples. The probability of type I, type II errors and probability of making no decisions under null and alternative hypotheses are obtained based on copula functions. The consistency of the proposed test is discussed under some null and alternative hypotheses. An unbiased, consistent estimator is proposed for probability of making no decision. Moreover, a simulation study is performed for showing the consistency of the proposed test for some well-known copulas such as independent, Clayton, Gumbel, Frank and Farlie-Gumbel-Morgenstern.

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1. Introduction
One of the classical and important problems of mathematical statistics is two sample problem which has many applications in different areas such as biology, medicine, economics, ecology and engineering. This problem aroused the interest of many researchers and there is a tremendous number of research works dealing with two sample problems in the univariate cases. With the developments of machine learning methods in classifications of statistical data, this problem becomes attractive to many researchers. The classical statement of the problem considers two training samples from two populations and a control sample which must be classified to one of these populations. The classical theory of hypothesis testing offers many interesting criteria based on distances between distribution functions, such as Kolmogorov-Smirnov, Cramer-Von Mises, Chi-square etc. Using these criteria the researchers accept or reject the hypothesis asserting that the control sample belongs to one of two populations and calculating the probability of miss classification expressed in probabilities of type I and type II errors. The consistency and efficiency of criteria are subject to the large sample sizes are also important.

In machine learning, classification based on a rejection option is widely preferred. In rejection option technique, if the patterns are most likely misclassified, then they are
rejected (i.e., they are not classified). Then these unclassified patterns can be treated by more advanced classification techniques or they can be left to human operators for classification. As a result the researcher is averted from cost of making unreliable decisions. Fumera et al. [12] proposed multiple rejection thresholds related to the data classes. Herbei and Wegkamp [13] studied on a binary classification method with a rejection option which is constructed on plug-in rules and empirical risk minimizers. For more studies about the rejection option we also refer to [17, 18, 21, 27].

In a univariate case Bairamov and Petunin [2] proposed a consistent nonparametric test based on univariate training samples which are relied on the order statistics (OSs) and use of different test procedure. The null hypothesis assumes that the control sample belongs to one of the populations. It may be accepted or rejected if the control sample belongs to one of two intervals determined by OSs of training samples. If the control sample doesn’t belong to the first or second interval, then no decision is taken and further investigations are required. The test is said to be consistent if probability of type I and type II errors both become infinitely small and if the sample size increases, the probability of making no decision is not one.

Computer-aided diagnostic systems are very important in medical sciences, when the diseases are diagnosed on the base of training samples of complex of symptoms of two groups of patients diagnosed with the disease A and B, respectively. The control sample is the complexes of symptoms of new patient who is determined to suffer with one of the diseases A or B and must be diagnosed to A or B by computer-aided diagnosis system. For example, A may be diabetes with normal blood sugar level (Diabetes Insipidus) and B may be diabetes with high blood sugar level (Diabetes Mellitus). However, in some cases it can be difficult to diagnose the case by a computer-aided diagnostic system. In such cases, the cases are examined by the expert as an advanced stage. Thus, the workload of the experts is reduced and the loss of time is minimized. If the probability of type I and type II errors are small and the probability of making no decision is large, the patient should be investigated in more detail until a correct diagnosis is made. For studies about the applications of rejection option in medicine, we refer to [7, 14, 19].

There are many interesting recent papers in statistical literature related to equality of two probability distributions, see, e.g., [1, 23–25]. Jiménez-Gamero et al. [15] proposed a test for testing equality of distribution of two samples based on empirical characteristic functions. Similarly, Alba-Fernández et al. [1] studied on a test statistic which relies on probability generating function. For more recent studies about univariate two-sample problems we also refer to [20].

In the case of bivariate observations, nonparametric two sample problem has not been studied much. These type of tests naturally involve the bivariate OSs and concomitants of order statistics. The theory of bivariate OSs is closely related to bivariate binomial distributions. Eryilmaz and Bairamov [10] studied the distribution of the bivariate training sample ranks of OSs and its concomitants. For some recent work on this topic we can refer to [4, 16]. There are some works dealing with the bivariate OSs and concomitants. Kemalbay and Bayramoglu [16] considered the joint distribution of ranks of OSs based on bivariate random samples. Stoimenova and Balakrishnan [24] also dealt with the two sample problem constructing a consistent test using precedence and exceedance statistics. Afterwards, Erem and Bayramoglu [9] obtained the exact and asymptotic distributions of exceedance statistics based on bivariate random sequences. Recently, Erem [8] proposed bivariate two sample test based on exceedance statistics for testing equality of two copula functions. For more results on the studies investigating the equality of two copula functions we refer to [6, 22]. Also, Susam and Ucer [26] studied the independency of Archimedean copulas based on Bernstein estimate of Kendall distribution function.

In this paper we consider two sample problem for bivariate observations. Similar to Bairamov and Petunin [2] considering univariate case, we construct statistical tests based
on bivariate OSs for bivariate random samples. We divide a possible set of training samples into three mutually exclusive sets. If the control sample belongs to the first set we accept the null hypothesis. If it belongs to the second set we reject it and if it belongs to third set we do not make any decision and further investigations are made. The probabilities of type I and type II errors and also the probability of making no decision are calculated. The unbiased and consistent estimators for probability of not making a decision are constructed.

The test proposed in this work can be used in many applications, in particular may be successful in the case of small sample observations.

In this paper we propose a test statistic for testing the equality of two copula functions based on bivariate OSs of training samples. In Section 2 problem statement and test procedure are introduced. Under some cases, the consistency of the test is discussed based on probability of type I, type II errors and probability of making no decision. In Section 3 an unbiased and consistent estimator for probability of making no decision is proposed. Finally in Section 4, a simulation study is performed for probability of type I, type II errors and probability of making no decision under some well-known copulas such as independent, Clayton, Frank, Gumbel and Farlie-Gumbel-Morgenstern (FGM) copulas.

2. Problem statement and the test procedure

Let $Z_1 = \{ (X^{(1)}_k, Y^{(1)}_k) , k = 1, 2, \ldots, n \}$ be a sequence of independent random variables with joint cumulative distribution function (cdf) $F(x, y) = C_1(F_X(x), F_Y(y))$, where $C_1(u, v), (u, v) \in [0, 1]^2$ is a connecting copula and $F_X(x), F_Y(y)$ are the marginal cdfs of $X$ and $Y$, respectively. Furthermore, let $Z_2 = \{ (X^{(2)}_k, Y^{(2)}_k) , k = 1, 2, \ldots, n \}$ be another sequence of independent random variables with joint cdf $G(x, y) = C_2(F_X(x), F_Y(y))$, where $C_2(u, v), (u, v) \in [0, 1]^2$ is a connecting copula and $F_X(x), F_Y(y)$ are the marginal cdfs of $X$ and $Y$, respectively. Let $f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$, $g(x, y) = \frac{\partial^2 G(x, y)}{\partial x \partial y}$, and $f_Y(y) = \frac{dF_Y(y)}{dy}$. We assume that $Z_1$ and $Z_2$ are independent and we call them the training samples from populations with joint cdfs $F$ and $G$, respectively.

Let $Z = \{ (X_1, Y_1), \ldots, (X_n, Y_n) \}$ be a control sample with joint distribution function $H(x, y) = C(F_X(x), F_Y(y))$, where $C(u, v), (u, v) \in [0, 1]^2$ is a connecting copula. Recall $Z_1$, $Z_2$ are training samples and $Z$ is a control sample. Consider the null hypothesis

$H_0 : H(x, y) = F(x, y)$ and the alternative hypothesis $H_1 : H(x, y) = G(x, y)$. To test $H_0$ against alternative $H_1$ we construct a test based on bivariate OSs of the samples $Z_1$ and $Z_2$.

The $(i,j)$th bivariate OSs of $Z_1$ are defined as $(X^{(i)}_{1:n}, Y^{(i)}_{j:n})$, where $1 \leq i < j \leq n, X^{(i)}_{1:n} \leq X^{(i)}_{2:n} \leq \cdots \leq X^{(i)}_{n:n}$ and $Y^{(i)}_{1:n} \leq Y^{(i)}_{2:n} \leq \cdots \leq Y^{(i)}_{n:n}$ are the OSs of $\{X^{(i)}_{k}, k = 1, 2, \ldots, n \}$ and $\{Y^{(i)}_{k}, k = 1, 2, \ldots, n \}$, respectively. Similarly, the bivariate $(i,j)$th order statistics of $Z_2$ is $(X^{(2)}_{i:n}, Y^{(2)}_{j:n})$, where $X^{(2)}_{1:n} \leq X^{(2)}_{2:n} \leq \cdots \leq X^{(2)}_{n:n}$ and $Y^{(2)}_{1:n} \leq Y^{(2)}_{2:n} \leq \cdots \leq Y^{(2)}_{n:n}$ are the OSs of $\{X^{(2)}_{k}, k = 1, 2, \ldots, n \}$ and $\{Y^{(2)}_{k}, k = 1, 2, \ldots, n \}$, respectively.

We assume that $X^{(1)}_{1:n} \leq X^{(2)}_{1:n}$ and $X^{(1)}_{n:n} \leq X^{(2)}_{n:n}$. Similarly $Y^{(1)}_{1:n} \leq Y^{(2)}_{1:n}$ and $Y^{(1)}_{n:n} \leq Y^{(2)}_{n:n}$.

2.1. The test

Let $E_1 = E_1(X, Y) = (-\infty, X^{(2)}_{1:n}) \times (-\infty, Y^{(2)}_{1:n})$ and $E_2 = E_2(X, Y) = (X^{(1)}_{n:n}, \infty) \times (Y^{(1)}_{n:n}, \infty)$. The test $T(n, r) \ (1 < r < n)$ for testing $H_0$ against $H_1$ is defined as follows:

(1) Accept the hypothesis $H_0$ if at least $r$ of the sample values of $Z$ are in $E_1$.
(2) Accept $H_1$ if at least $r$ of the sample values of $Z$ are in $E_2$.
(3) In all other cases no decision is made about $H_0$ and $H_1$. 

Since the sets $E_1$ and $E_2$ are disjoint, the alternatives 1 and 2 cannot be observed at the same time. The graphical illustration of $E_1$ and $E_2$ is given in Figure 1.

**Figure 1.** Graphical illustration of critical set.

**Remark.** To apply this test we need the following conditions:

$$X^{(2)}_{1:n} \leq X^{(1)}_{1:n} \text{ and } X^{(1)}_{n:n} \leq X^{(2)}_{n:n}, \text{ similarly } Y^{(1)}_{1:n} \leq Y^{(2)}_{1:n} \text{ and } Y^{(1)}_{n:n} \leq Y^{(2)}_{n:n}. \quad (2.1)$$

One of the referees rightly observes that the condition in Equation (2.1) is restrictive and will put strong relationship between two distribution functions $F$ and $G$. This condition is satisfied in some situations if the training samples are not separated from each other, they are mixed. The verification of this condition is not difficult, because it is put on the data and required to check the training samples. The test can be applied and will be consistent if this condition is satisfied. In many practical applications the data from different samples are mixed, not separated. Alternatively, if

$$X^{(1)}_{1:n} \leq X^{(2)}_{1:n} \text{ and } X^{(2)}_{n:n} \leq X^{(1)}_{n:n}, \text{ similarly } Y^{(2)}_{1:n} \leq Y^{(1)}_{1:n} \text{ and } Y^{(1)}_{n:n} \leq Y^{(2)}_{n:n},$$

then the test procedure will be applied by changing places of the training samples $Z_1$ and $Z_2$. In the case when the training samples are separated, i.e.

$$X^{(1)}_{n:n} \leq X^{(2)}_{1:n} \text{ and } Y^{(1)}_{n:n} \leq Y^{(2)}_{1:n},$$

and in other cases not satisfying (2.1) different test procedures can be investigated and applied.

**Definition 2.1.** The test $T(n, r)$ based on bivariate training samples $Z_1$, $Z_2$ and control sample $Z$, is called consistent if probability of making no decision (ND) given that the hypothesis $H_i$, $i = 0, 1$ ($P(ND \mid H_i)$) is true, is less than 1, i.e. $P(ND \mid H_i) < 1$, $i = 0, 1$ and the probability of type I and type II errors tend to zero when the sample size $n$ increases to infinity.
2.2. The probability of type I error

The probability of type I error $\alpha_n$ of the test $T(n, r)$ can be calculated as follows:

$$\alpha_n = P \{ H_1 \mid H_0 \}$$

$$= P \left\{ \text{at least} \ r \ \text{of the sample values in} \ Z \ \text{are in} \ (X_{n1:n}^{(1)}, \infty) \times (Y_{n1:n}^{(1)}, \infty) \mid H(x, y) = F(x, y) \right\}$$

$$= P \left\{ X_{n-r+1:n} > X_{n1:n}^{(1)}, Y_{n-r+1:n} > Y_{n1:n}^{(1)} \mid H(x, y) = F(x, y) \right\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(t, s)]^{n} f_{X_{n-r+1:n}, Y_{n-r+1:n}}(t, s) dt ds,$$  \hspace{1cm} (2.2)

where $f_{X_{n-r+1:n}, Y_{n-r+1:n}}$ is probability density function (pdf) of $X_{n-r+1:n}^{(1)}$, $Y_{n-r+1:n}^{(1)}$ in training sample $Z_t$. The joint pdf $f_{X_{n-r+1:n}, Y_{n-r+1:n}}$ of bivariate OSs $X_{n-r+1:n}^{(1)}$ and $Y_{n-r+1:n}^{(1)}$ is studied in [3] and [16] has the following form:

$$f_{X_{n-r+1:n}, Y_{n-r+1:n}}(t, s) = \sum_{t_1=a_1}^{a_2} l_1 [F(t, s)]^{t_1} (F_X(t) - F_1(t, s)) (F_Y(s) - F_1(t, s))^n r-t_1$$

$$\times \left[ F(t, s) \right]^{2r-n-1+t_1} f(t, s) + \sum_{t_4=a_1}^{a_2} \sum_{t_2=a_1}^{a_2} \sum_{t_1=a_1}^{a_2} l_2 [F(t, s)]^{t_1}$$

$$\times [(F_X(t) - F(t, s))]^{n-r-1-t_2} [(F_Y(s) - F(t, s))]^{n-r-1-t_4}$$

$$\times \left[ F(t, s) \right]^{2r-n+t_1+t_2+t_4-2} [F_{-1}(t, s)]^{t_2} [F_{-1}(s) - F_{-1}(t, s)]^{1-t_2}$$

$$\times \left[ F_{-1}(t, s) \right]^{t_4} \left[ f_X(t) - F_{-1}(t, s) \right]^{1-t_4},$$  \hspace{1cm} (2.3)

where $a_1 = \max(0, n-2r+1)$, $a_2 = n-r$, $b_1 = \max(0, n-2r+2-t_2-t_4)$, $b_2 = \min(n-r-t_2, n-r-t_4)$, $c_1 = \max(0, 2-r)$, $c_2 = \min(1, n-r)$, $d_1 = \max(0, 2-r)$, $d_2 = \min(1, n-r)$,

$$F_{-1}(t, s) = \frac{\partial F_1(t, s)}{\partial t},$$

$$F_{-1}(t, s) = \frac{\partial F_1(t, s)}{\partial s},$$

and the constants $l_1$ and $l_2$ are

$$l_1 = \frac{n!}{t_1! \left[ (n-r-t_1)! \right]^2 (2r-n+t_1-1)!},$$

$$l_2 = \frac{n!}{t_1! \left( n-r-t_1-t_2 \right)! \left( n-r-t_1-t_4 \right)! (2r-n-2+t_1+t_2+t_4)!}.$$  \hspace{1cm} (2.4)

2.3. The probability of type II error

The probability of Type II error $\beta_n$ of the test $T(n, r)$ can be calculated as follows:

$$\beta_n = P \{ H_0 \mid H_1 \}$$

$$= P \left\{ \text{at least} \ r \ \text{of the sample values in} \ Z \ \text{are in} \ (-\infty, X_{1:n}^{(2)}) \times (-\infty, Y_{1:n}^{(2)}) \mid H(x, y) = G(x, y) \right\}$$

$$= P \left\{ X_{r:n} < X_{1:n}^{(2)}, Y_{r:n} < Y_{1:n}^{(2)} \mid H(x, y) = G(x, y) \right\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [G(t, s)]^{n} g_{X_{r:n}, Y_{r:n}}(t, s) dt ds,$$  \hspace{1cm} (2.4)
where \( g_{X_{r_{n},Y_{r_{n}}}}(t,s) \) is pdf of \( X_{r_{n}}^{(2)} \) and \( Y_{r_{n}}^{(2)} \) in training sample \( Z_2 \) and \( G(t,s) = 1 - F_X(t) - F_Y(s) + G(t,s) \).

### 2.4. Consistency of the test

By using Equations (2.2) and (2.4) we have the following theorems.

**Theorem 2.2.** The test \( T(n, n-1) \) is consistent under the following hypotheses

\[
\begin{align*}
H_0 : C(u, v) &= C_1(u, v) = uv \\
H_1 : C(u, v) &= C_2(u, v).
\end{align*}
\]

**Proof.** See Appendix A.

In Figure 2, the plot of \( \alpha_n \) is provided for \( C_1(u, v) = uv \).

![Figure 2](image)

**Figure 2.** The plot of \( \alpha_n \) for \( C_1(u, v) = uv \).

**Theorem 2.3.** The test \( T(n, n-1) \) is consistent under \( H_0 : C(u, v) = C_1(u, v) \) against \( H_1 : C(u, v) = C_2(u, v) = uv \).

**Proof.** See Appendix B.

In Figure 3, the plot of \( \beta_n \) is provided for \( C_2(u, v) = uv \).

![Figure 3](image)

**Figure 3.** The plot of \( \beta_n \) for \( C_2(u, v) = uv \).
3. An unbiased and consistent estimator of probability of making no decision

In this section, an unbiased and consistent estimator of probability of making no decisions under the null and alternative hypotheses \( P \{ ND \mid H_0 \} \) and \( P \{ ND \mid H_1 \} \) are constructed. Here, ND denotes the event making no decision. By constructing the unbiased and consistent estimator, the investigator can make inference on probability of making no decision under the null hypothesis before using \( T(n, n - 1) \) test.

Firstly consider the unbiased and consistent estimator of \( P \{ ND \mid H_0 \} \) under the following hypotheses:

\[
H_0 : C(u, v) = C_1(u, v) = uv \\
H_1 : C(u, v) = C_2(u, v).
\]

Let \( A \) denotes the event that at least \( r \) of the sample values of \( Z \) are in \( (-\infty, X_{1:n}^{(2)}) \times (-\infty, Y_{1:n}^{(2)}) \) and \( B \) denotes the event at least \( r \) of the sample values of \( Z \) are in \( (X_{1:n}^{(1)}, \infty) \times (Y_{n:n}^{(1)}, \infty) \). The events \( A \) and \( B \) are mutually exclusive events. Then by using Equation (A.4) in Appendix A, we have

\[
P \{ ND \mid H_0 \} = 1 - \alpha_n - P \{ A \mid H_0 \},
\]

where

\[
\alpha_n = \frac{n (n - 1)2 (n + 1) (n - 2)! + (n^2 - 1) (n - 1)! (n + 1)!}{4^n A(n)^2}.
\]

Since \( P \{ A \mid H_0 \} \) consists of a double integral which depends on unknown copula function \( \hat{C}_2 \), an estimator for estimating the integral based on training samples \( Z_1 \) and \( Z_2 \) are constructed. The following sampling scheme is considered for bivariate training sample \( Z_1 \). We divide \( Z_1 \) sequentially into \( l = \left[ \frac{n}{k} \right] \) parts, such that

\[
z_i = \{(x_1^{(i)}, y_1^{(i)}), \ldots, (x_k^{(i)}, y_k^{(i)}), (x_{k+1}^{(i)}, y_{k+1}^{(i)}), \ldots, (x_{2k}^{(i)}, y_{2k}^{(i)}), \ldots, (x_{k(n-1)}^{(i)}, y_{k(n-1)}^{(i)}), \ldots, (x_{kn}^{(i)}, y_{kn}^{(i)})\},
\]

and each sequential group is called as a subgroup of bivariate training sample of \( Z_1 \). Then, the OSs of \( i \)-th subgroup of training sample \( Z_1 \) with a sample size of \( k \) is

\[
X_{i_1:n}^{(1)} \leq X_{i_2:n}^{(1)} \leq \cdots \leq X_{i_k:n}^{(1)} \\
Y_{i_1:n}^{(1)} \leq Y_{i_2:n}^{(1)} \leq \cdots \leq Y_{i_k:n}^{(1)}.
\]

Define the event \( A_i \) as at least \( k - 1 \) of \( i \)-th subgroup of \( Z_1 \) is in \( (-\infty, X_{1:n}^{(2)}) \times (-\infty, Y_{1:n}^{(2)}) \).

Then

\[
P(A_i) = P \left\{ \text{at least } k - 1 \text{ of } i \text{-th subgroup of } Z_1 \text{ is in } (-\infty, X_{1:n}^{(2)}) \times (-\infty, Y_{1:n}^{(2)}) \right\}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \hat{G}(t, s) \right]^{k-1} f_{X_{k-1:n}, Y_{k-1:n}}(t, s) dt ds.
\]

Let define the binary random variable \( \xi_i \) as follows:

\[
\xi_i = \begin{cases} 
1 & \text{if event } A_i \text{ occurs,} \\
0 & \text{if event } A_i \text{ does not occur,}
\end{cases}
\]

then \( S_i = \xi_1 + \xi_2 + \ldots + \xi_l \) denotes the number of events \( A_i \)'s within \( l \) outcomes. It is clear that \( P \{ \xi_i = 1 \} = P(A_i) \) and \( P \{ \xi_i = 0 \} = 1 - P(A_i) \). From Equation (3.2) we can observe that \( P(A_i) \) is equiprobable for \( i = 1, 2, \ldots, l \). Let \( h = \frac{S_i}{l} \) and \( P(A_i) = p \). Then

\[
E(h) = \frac{1}{l} E(S_i) = E(\xi_i) = p.
\]
Therefore, \( h = \bar{S}_T \) is an unbiased estimator for \( P \{ A \mid H_0 \} \) in Equation (3.1). Now consider the variance of statistic \( h \):

\[
V(h) = \frac{1}{l^2} V(S_T)
\]

\[
= \frac{1}{l^2} \left( \sum_{i=1}^{l} V(\xi_i) + 2 \sum_{i=1}^{l} \sum_{j=1}^{l} \text{Cov}(\xi_i, \xi_j) \right)
\]

\[
= \frac{1}{l^2} \left( lp(1-p) + 2 \left( \frac{l}{2} \right) \text{Cov}(\xi_i, \xi_j) \right)
\]

\[
= \frac{p(1-p)}{l} + \frac{l-1}{l} \left[ P(A_iA_j) - E(\xi_i)^2 \right]
\]

\[
= \frac{p(1-p)}{l} + \left[ \frac{[n]}{[n]} - \frac{1}{[n]} \right] \left[ P(A_iA_j) - p^2 \right],
\]

where \( P(A_iA_j) = \{ \xi_i = 1, \xi_j = 1 \} \).

\[
P(A_iA_j) = P \left\{ X_{1:n}^{(1)} < X_{2:n}^{(2)}, Y_{1:n}^{(1)} < Y_{2:n}^{(2)}, X_{1:k-1}^{(1)} < X_{2:k-1}^{(1)}, Y_{1:k-1}^{(1)} < Y_{2:k-1}^{(1)} \right\}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left\{ X_{1:k-1}^{(1)} < t, Y_{1:k-1}^{(1)} < s, X_{2:k-1}^{(1)} < t, Y_{2:k-1}^{(1)} < s \right\} g_{X_{1:n},Y_{1:n}}(t,s) \, dt \, ds
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P \left\{ X_{1:k-1}^{(1)} < t, Y_{1:k-1}^{(1)} < s \right\} P \left\{ X_{2:k-1}^{(1)} < t, Y_{2:k-1}^{(1)} < s \right\} g_{X_{1:n},Y_{1:n}}(t,s) \, dt \, ds
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{X_{1:n},Y_{1:n}}(t,s)]^2 g_{X_{1:n},Y_{1:n}}(t,s) \, dt \, ds,
\]

where \( F_{X_{1:n},Y_{1:n}}(t,s) \) is the joint cdf of bivariate OSs \( X_{1:n}^{(1)} \) and \( Y_{1:n}^{(1)} \) in \( i \)-th subgroup of training sample \( Z_1 \) with a sample size of \( k \). Also \( g_{X_{1:n},Y_{1:n}}(t,s) \) is pdf of \( X_{1:n}^{(2)} \) and \( Y_{1:n}^{(2)} \) in training sample \( Z_2 \). By probability integral transformation, Equation (3.3) can be written for \( C_1(u,v) = uv \) as follows:

\[
P(A_iA_j) = \int_0^1 \int_0^1 \left[ C_{k,k}(u,v) \right]^2 \left\{ n \left[ \hat{C}_2 (1 - u, 1 - v) \right]^{n-1} c_2(u,v) + n(n-1) \left[ \hat{C}_2 (1 - u, 1 - v) \right]^{n-2} \left( 1 - C_2^{-1} (u,v) \right) \left( 1 - C_2^{-1} (u,v) \right) \right\} \, du \, dv,
\]

where \( \hat{C}_2(1-u,1-v) = 1 - u - v + C_2(u,v) \), \( c_2(u,v) = \frac{\partial^2 C_2(u,v)}{\partial u \, \partial v} \) and

\[
C_{k,k}(u,v) = \sum_{i=k-1}^{k} \sum_{j=k-1}^{k} \min(i,j) \sum_{l=\max(0,i+j-k)}^{k} \frac{k!}{i! \, (i-l)! \, (j-l)! \, (k-i-j+l)!} [C_1(u,v)]^l [u - C_1(u,v)]^{i-l} \times [v - C_1(u,v)]^{j-l} [1 - u - v + C_1(u,v)]^{k-i-j+l}.
\]

Then it is obvious that \( \lim_{n \to \infty} P(A_iA_j) = 0 \). Therefore, \( \lim_{n \to \infty} V(h) = 0 \).

So an unbiased and consistent estimator of \( P \{ ND \mid H_0 \} \) under the following null and alternative hypotheses

\[
H_0 : C(u,v) = C_1(u,v) = uv
\]

\[
H_1 : C(u,v) = C_2(u,v)
\]

is given by

\[
t_0(n) = 1 - \left[ \frac{n(n-1)^2(n+1)(n-2)!}{4^n A(n)} + \frac{(n^2-1)(n-1)(n+1)!}{4^n (A(n))^2} + h \right].
\]

In a similar way, one can also construct an unbiased and consistent estimator for \( P \{ ND \mid H_1 \} \). Define the event \( B_i \) as at least \( k-1 \) of \( i \)-th subgroup of \( Z_2 \) is in \( \left( X_{1:n}^{(1)}, \infty \right) \times \)
Then an unbiased and consistent estimator of \( P \{ ND \mid H_1 \} \) under the following null and alternative hypotheses

\[
H_0 : C(u, v) = C_1(u, v) \\
H_1 : C(u, v) = C_2(u, v) = uv
\]

is given by

\[
t_1(n) = 1 - \left[ \frac{n(n - 1)^2 [(n - 2)!]^2 [(n + 1)!]^2 + (n - 1)(n - 1)! n![(n + 1)!]^2 + h^*}{[(2n)!]^2} \right],
\]

where \( h^* \) is proportion of total number of events \( B_i \)'s in \( l \) subgroup of \( Z_2 \). In this section a simulation study is performed to show the consistency of the test under some well-known copulas (Independent, Clayton, Frank, Gumbel and FGM) in R programme for different values of \( n, r \) and dependence parameters. The number of repetition is 5000.

**Table 1.** Probability of type I Error for \( C_1(u, v) = uv \) and different values of \( n \) and \( r \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( r )</th>
<th>( \alpha )</th>
<th>( n )</th>
<th>( r )</th>
<th>( \alpha )</th>
<th>( n )</th>
<th>( r )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>0.0712</td>
<td>20</td>
<td>1</td>
<td>0.0462</td>
<td>40</td>
<td>1</td>
<td>0.0192</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>0.0078</td>
<td>20</td>
<td>2</td>
<td>0.0016</td>
<td>40</td>
<td>2</td>
<td>0.0002</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.0015</td>
<td>20</td>
<td>3</td>
<td>0.0004</td>
<td>50</td>
<td>1</td>
<td>0.0188</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>0.0002</td>
<td>30</td>
<td>1</td>
<td>0.029</td>
<td>50</td>
<td>2</td>
<td>0.0006</td>
</tr>
<tr>
<td>30</td>
<td>2</td>
<td>0.0002</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.** Probability of type I Error for \( C_1(u, v) \) is FGM copula and different values of \( n, r \) and \( \theta_2 \).

<table>
<thead>
<tr>
<th>( (n, r) )</th>
<th>( \theta_2 )</th>
<th>( \alpha )</th>
<th>( (n, r) )</th>
<th>( \theta_2 )</th>
<th>( \alpha )</th>
<th>( (n, r) )</th>
<th>( \theta_2 )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 1)</td>
<td>-1</td>
<td>0.0224</td>
<td>(20, 1)</td>
<td>-1</td>
<td>0.0084</td>
<td>(30, 2)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>0.5</td>
<td>0.0116</td>
<td>(20, 1)</td>
<td>0.5</td>
<td>0.061</td>
<td>(30, 2)</td>
<td>0.5</td>
<td>0.002</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>1</td>
<td>0.1266</td>
<td>(20, 1)</td>
<td>1</td>
<td>0.094</td>
<td>(30, 2)</td>
<td>1</td>
<td>0.0062</td>
</tr>
<tr>
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<td>-1</td>
<td>0.0002</td>
<td>(20, 2)</td>
<td>-1</td>
<td>0.0002</td>
<td>(30, 3)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(10, 2)</td>
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<td>0.0134</td>
<td>(20, 2)</td>
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<td>0.0052</td>
<td>(30, 3)</td>
<td>0.5</td>
<td>0.0002</td>
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<td>(20, 2)</td>
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<td>(30, 3)</td>
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<td>(10, 3)</td>
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<td>-1</td>
<td>0.001</td>
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<td>0.0028</td>
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<td>0.0004</td>
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<td>0.5</td>
<td>0.027</td>
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<td>0.005</td>
<td>(20, 3)</td>
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<td>0.002</td>
<td>(50, 1)</td>
<td>1</td>
<td>0.0388</td>
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<tr>
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<td>(30, 1)</td>
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<td>0.0028</td>
<td>(50, 2)</td>
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<td>0</td>
</tr>
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<td>(10, 4)</td>
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<td>0.004</td>
<td>(30, 1)</td>
<td>0.5</td>
<td>0.0412</td>
<td>(50, 2)</td>
<td>0.5</td>
<td>0.0012</td>
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<tr>
<td>(10, 4)</td>
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<td>0.006</td>
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<td>0.052</td>
<td>(50, 2)</td>
<td>1</td>
<td>0.0022</td>
</tr>
</tbody>
</table>

**Table 3.** Probability of type I Error for \( C_1(u, v) \) is Clayton copula and different values of \( n, r \) and \( \theta_1 \).

<table>
<thead>
<tr>
<th>( (n, r) )</th>
<th>( \theta_1 )</th>
<th>( \alpha )</th>
<th>( (n, r) )</th>
<th>( \theta_1 )</th>
<th>( \alpha )</th>
<th>( (n, r) )</th>
<th>( \theta_1 )</th>
<th>( \alpha )</th>
<th>( (n, r) )</th>
<th>( \theta_1 )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 1)</td>
<td>-1</td>
<td>0</td>
<td>(20, 1)</td>
<td>-1</td>
<td>0</td>
<td>(30, 1)</td>
<td>-1</td>
<td>0</td>
<td>(50, 1)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>1</td>
<td>0.124</td>
<td>(20, 1)</td>
<td>1</td>
<td>0.0782</td>
<td>(30, 1)</td>
<td>1</td>
<td>0.0618</td>
<td>(50, 1)</td>
<td>1</td>
<td>0.034</td>
</tr>
<tr>
<td>(10, 1)</td>
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<td>0.166</td>
<td>(20, 1)</td>
<td>2</td>
<td>0.1064</td>
<td>(30, 1)</td>
<td>2</td>
<td>0.0774</td>
<td>(50, 1)</td>
<td>2</td>
<td>0.0578</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>-1</td>
<td>0</td>
<td>(20, 2)</td>
<td>-1</td>
<td>0</td>
<td>(30, 2)</td>
<td>-1</td>
<td>0</td>
<td>(50, 2)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>1</td>
<td>0.023</td>
<td>(20, 2)</td>
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<td>0.0104</td>
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<td>0.0032</td>
<td>(50, 2)</td>
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<tr>
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<td>0.0352</td>
<td>(20, 2)</td>
<td>2</td>
<td>0.018</td>
<td>(30, 2)</td>
<td>2</td>
<td>0.0116</td>
<td>(50, 2)</td>
<td>2</td>
<td>0.006</td>
</tr>
<tr>
<td>(10, 5)</td>
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<td>0</td>
<td>(20, 3)</td>
<td>-1</td>
<td>0</td>
<td>(30, 3)</td>
<td>-1</td>
<td>0</td>
<td>(50, 3)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(10, 5)</td>
<td>1</td>
<td>0</td>
<td>(20, 3)</td>
<td>1</td>
<td>0.0016</td>
<td>(30, 3)</td>
<td>1</td>
<td>0.0012</td>
<td>(50, 3)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(10, 5)</td>
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<td>0.0012</td>
<td>(20, 3)</td>
<td>2</td>
<td>0.0063</td>
<td>(30, 3)</td>
<td>2</td>
<td>0.0016</td>
<td>(50, 3)</td>
<td>2</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

In Tables 1-3, probability of type I error are provided under the following hypotheses:

\[
H_0 : C(u, v) = C_1(u, v) = uv \\
H_1 : C(u, v) = C_2(u, v),
\]
increases, probability of type I error also increases. Similar conclusions can be done from
respectively. From Tables 1-3, it is clear that probability of type I error decreases when
while $r$ increases probability of type I error decreases. Furthermore for fixed values of $n$ and $r$, when the dependence parameters ($\theta_1$ and $\theta_2$) increases, probability of type I error also increases. Similar conclusions can be done from
Tables 4-6 for probability of type II error.

Table 4. Probability of type II Error for $C_2(u,v) = uv$ and different values of $n$ and $r$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>$\beta$</th>
<th>$n$</th>
<th>$r$</th>
<th>$\beta$</th>
<th>$n$</th>
<th>$r$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.081</td>
<td>1</td>
<td>0</td>
<td>0.04</td>
<td>1</td>
<td>0</td>
<td>0.022</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.0086</td>
<td>20</td>
<td>0</td>
<td>0.0034</td>
<td>40</td>
<td>2</td>
<td>0.0012</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.0001</td>
<td>20</td>
<td>3</td>
<td>0</td>
<td>50</td>
<td>1</td>
<td>0.0172</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>1</td>
<td>0.0282</td>
<td>50</td>
<td>2</td>
<td>0.0014</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Probability of type II Error for $C_2(u,v)$ is FGM copula and different values of $n$, $r$ and $\theta_2$.

<table>
<thead>
<tr>
<th>$(n,r)$</th>
<th>$\theta_2$</th>
<th>$\beta$</th>
<th>$(n,r)$</th>
<th>$\theta_2$</th>
<th>$\beta$</th>
<th>$(n,r)$</th>
<th>$\theta_2$</th>
<th>$\beta$</th>
<th>$(n,r)$</th>
<th>$\theta_2$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(10,1)$</td>
<td>$-1$</td>
<td>0.021</td>
<td>$(20,1)$</td>
<td>$-1$</td>
<td>0.009</td>
<td>$(30,1)$</td>
<td>$-1$</td>
<td>0.004</td>
<td>$(50,1)$</td>
<td>$-1$</td>
<td>0.01</td>
</tr>
<tr>
<td>$(10,1)$</td>
<td>0.5</td>
<td>0.0984</td>
<td>$(20,1)$</td>
<td>0.5</td>
<td>0.0504</td>
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<td>0.4</td>
<td>$(50,1)$</td>
<td>0.5</td>
<td>0.037</td>
</tr>
<tr>
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<td>0.132</td>
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<td>$(30,1)$</td>
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<td>0.06</td>
<td>$(50,1)$</td>
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<td>0.045</td>
</tr>
<tr>
<td>$(10,2)$</td>
<td>$-1$</td>
<td>0.001</td>
<td>$(20,2)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(30,2)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(50,2)$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
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<td>0.015</td>
<td>$(20,2)$</td>
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<td>$(30,2)$</td>
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<td>0.007</td>
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<td>0</td>
<td>$(20,3)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(30,3)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(50,4)$</td>
<td>$-1$</td>
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</tr>
<tr>
<td>$(10,3)$</td>
<td>0.5</td>
<td>0.003</td>
<td>$(20,3)$</td>
<td>0.5</td>
<td>0</td>
<td>$(30,3)$</td>
<td>0.5</td>
<td>0</td>
<td>$(10,4)$</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>$(10,3)$</td>
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<td>0.008</td>
<td>$(20,3)$</td>
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<td>0.002</td>
<td>$(30,3)$</td>
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<td>0.001</td>
<td>$(10,4)$</td>
<td>1</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table 6. Probability of type II Error for $C_2(u,v)$ is Clayton Copula and different values of $n$, $r$ and $\theta_1$.

<table>
<thead>
<tr>
<th>$(n,r)$</th>
<th>$\theta_1$</th>
<th>$\beta$</th>
<th>$(n,r)$</th>
<th>$\theta_1$</th>
<th>$\beta$</th>
<th>$(n,r)$</th>
<th>$\theta_1$</th>
<th>$\beta$</th>
<th>$(n,r)$</th>
<th>$\theta_1$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(10,1)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(20,1)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(30,1)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(50,1)$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$(10,1)$</td>
<td>0.317</td>
<td>$(20,1)$</td>
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<td>0.305</td>
<td>$(30,1)$</td>
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<td>$(50,1)$</td>
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<td>0.274</td>
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<td>0.392</td>
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<td>2</td>
<td>0.383</td>
<td>$(50,1)$</td>
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<td>0.4</td>
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<tr>
<td>$(10,2)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(20,2)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(30,2)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(50,2)$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>$(20,2)$</td>
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<td>0.096</td>
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<td>$(20,2)$</td>
<td>2</td>
<td>0.142</td>
<td>$(30,2)$</td>
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<td>0.155</td>
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<td>0.144</td>
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</tr>
<tr>
<td>$(10,3)$</td>
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<td>0</td>
<td>$(20,3)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(30,3)$</td>
<td>$-1$</td>
<td>0</td>
<td>$(50,3)$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$(10,3)$</td>
<td>1</td>
<td>$(20,3)$</td>
<td>1</td>
<td>0.001</td>
<td>$(30,3)$</td>
<td>1</td>
<td>0.024</td>
<td>$(50,3)$</td>
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<td>0.0878</td>
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</tr>
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<td>$(10,3)$</td>
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<td>$(20,3)$</td>
<td>2</td>
<td>0.006</td>
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<td>$(50,3)$</td>
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<td>0.1458</td>
<td></td>
</tr>
</tbody>
</table>

In Tables 4-6 the simulation study is performed for probability of type II error under the following hypotheses, respectively:

$H_0 : C(u,v) = C_1(u,v)$

$H_1 : C(u,v) = C_2(u,v) = uv$, 

(3.7)
\( H_0 : C(u,v) = C_1(u,v) \)
\( H_1 : C(u,v) = C_2(u,v) = uv(1 + \theta_2(1 - u)(1 - v)), \ -1 \leq \theta_2 \leq 1, \ \text{(FGM copula)}, \)
and
\( H_0 : C(u,v) = C_1(u,v) \)
\( H_1 : C(u,v) = C_2(u,v) = \max \left( \left[ u^{-\theta_1} + v^{-\theta_1} - 1 \right]^{1/\theta_1}, 0 \right), \ \theta_1 \in [-1,\infty) \setminus \{0\} , \ \text{(Clayton copula)}, \)

**Table 7.** \( P(ND \mid H_0) \) under hypothesis (3.8) for some values of \( n, r \) and \( \theta_1. \)

<table>
<thead>
<tr>
<th>( (n,r) )</th>
<th>( \theta_1 )</th>
<th>( P(ND \mid H_0) )</th>
<th>( (n,r) )</th>
<th>( \theta_1 )</th>
<th>( P(ND \mid H_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10,1)</td>
<td>4</td>
<td>0.457</td>
<td>(15,1)</td>
<td>4</td>
<td>0.4978</td>
</tr>
<tr>
<td>(10,1)</td>
<td>5</td>
<td>0.437</td>
<td>(15,1)</td>
<td>5</td>
<td>0.4854</td>
</tr>
<tr>
<td>(10,1)</td>
<td>6</td>
<td>0.4144</td>
<td>(15,1)</td>
<td>6</td>
<td>0.4728</td>
</tr>
<tr>
<td>(10,2)</td>
<td>4</td>
<td>0.8282</td>
<td>(15,2)</td>
<td>4</td>
<td>0.8498</td>
</tr>
<tr>
<td>(10,2)</td>
<td>5</td>
<td>0.812</td>
<td>(15,2)</td>
<td>5</td>
<td>0.8414</td>
</tr>
<tr>
<td>(10,2)</td>
<td>6</td>
<td>0.8016</td>
<td>(15,2)</td>
<td>6</td>
<td>0.8306</td>
</tr>
</tbody>
</table>

**Table 8.** \( P(ND \mid H_0) \) under hypothesis (3.9) for some values of \( n, r, \theta_1 \) and \( \theta_2. \)

<table>
<thead>
<tr>
<th>( (n,r) )</th>
<th>( (\theta_1, \theta_2) )</th>
<th>( P(ND \mid H_0) )</th>
<th>( (n,r) )</th>
<th>( (\theta_1, \theta_2) )</th>
<th>( P(ND \mid H_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10,1)</td>
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<td>0.56</td>
<td>(15,1)</td>
<td>(1,5)</td>
<td>0.5894</td>
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<tr>
<td>(10,1)</td>
<td>(2,5)</td>
<td>0.4526</td>
<td>(15,1)</td>
<td>(2,5)</td>
<td>0.5034</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(3,5)</td>
<td>0.4026</td>
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<td>(3,5)</td>
<td>0.4452</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(4,5)</td>
<td>0.387</td>
<td>(15,1)</td>
<td>(4,5)</td>
<td>0.425</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(5,5)</td>
<td>0.3458</td>
<td>(15,1)</td>
<td>(5,5)</td>
<td>0.4092</td>
</tr>
<tr>
<td>(10,1)</td>
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<td>0.3998</td>
<td>(15,1)</td>
<td>(5,2)</td>
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</tr>
<tr>
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<td>(15,1)</td>
<td>(5,4)</td>
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</tr>
<tr>
<td>(10,1)</td>
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<td>0.3452</td>
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<td>(5,6)</td>
<td>0.3936</td>
</tr>
<tr>
<td>(10,2)</td>
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<td>0.8638</td>
<td>(15,2)</td>
<td>(1,5)</td>
<td>0.8852</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(2,5)</td>
<td>0.8042</td>
<td>(15,2)</td>
<td>(2,5)</td>
<td>0.8824</td>
</tr>
<tr>
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<td>0.7886</td>
<td>(15,2)</td>
<td>(3,5)</td>
<td>0.8014</td>
</tr>
<tr>
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<td>0.7474</td>
<td>(15,2)</td>
<td>(4,5)</td>
<td>0.7822</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,5)</td>
<td>0.7398</td>
<td>(15,2)</td>
<td>(5,5)</td>
<td>0.7696</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,2)</td>
<td>0.7816</td>
<td>(15,2)</td>
<td>(5,2)</td>
<td>0.8188</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,3)</td>
<td>0.7676</td>
<td>(15,2)</td>
<td>(5,3)</td>
<td>0.7918</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,4)</td>
<td>0.7462</td>
<td>(15,2)</td>
<td>(5,4)</td>
<td>0.7766</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,6)</td>
<td>0.7212</td>
<td>(15,2)</td>
<td>(5,6)</td>
<td>0.7588</td>
</tr>
</tbody>
</table>

**Table 9.** \( P(ND \mid H_0) \) under hypothesis (3.10) for some values of \( n, r \) and \( \theta_3. \)

<table>
<thead>
<tr>
<th>( (n,r) )</th>
<th>( \theta_3 )</th>
<th>( P(ND \mid H_0) )</th>
<th>( (n,r) )</th>
<th>( \theta_3 )</th>
<th>( P(ND \mid H_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10,1)</td>
<td>2</td>
<td>0.458</td>
<td>(15,1)</td>
<td>2</td>
<td>0.4774</td>
</tr>
<tr>
<td>(10,1)</td>
<td>3</td>
<td>0.3458</td>
<td>(15,1)</td>
<td>3</td>
<td>0.3542</td>
</tr>
<tr>
<td>(10,1)</td>
<td>4</td>
<td>0.2822</td>
<td>(15,1)</td>
<td>4</td>
<td>0.3072</td>
</tr>
<tr>
<td>(10,1)</td>
<td>5</td>
<td>0.2652</td>
<td>(15,1)</td>
<td>5</td>
<td>0.2552</td>
</tr>
<tr>
<td>(10,2)</td>
<td>2</td>
<td>0.8362</td>
<td>(15,2)</td>
<td>2</td>
<td>0.8412</td>
</tr>
<tr>
<td>(10,2)</td>
<td>3</td>
<td>0.7674</td>
<td>(15,2)</td>
<td>3</td>
<td>0.772</td>
</tr>
<tr>
<td>(10,2)</td>
<td>4</td>
<td>0.728</td>
<td>(15,2)</td>
<td>4</td>
<td>0.738</td>
</tr>
<tr>
<td>(10,2)</td>
<td>5</td>
<td>0.6876</td>
<td>(15,2)</td>
<td>5</td>
<td>0.7116</td>
</tr>
</tbody>
</table>

In Tables 7-10, the simulation study is performed for probability of making no decision when the null hypothesis is true \( (P(ND \mid H_0)) \). We consider the following hypotheses, respectively:

\( H_0 : C(u,v) = C_1(u,v) = \max \left( \left[ u^{-\theta_1} + v^{-\theta_1} - 1 \right]^{1/\theta_1}, 0 \right), \ \theta_1 \in [-1,\infty) \setminus \{0\} , \ \text{(Clayton copula)}, \)
\( H_1 : C(u,v) = C_2(u,v) = uv, \) \hspace{1cm} (3.8)
\( H_0 : C(u,v) = C_1(u,v) = \max \left( \left[u^{-\theta_1} + v^{-\theta_1} - 1\right]^{-\frac{1}{\theta_1}}, 0 \right), \theta_1 \in [-1, \infty) \setminus \{0\} \), (Clayton copula),
\( H_1 : C(u,v) = C_2(u,v) = \exp \left\{ - \left[(- \ln u)^{\theta_3} + (- \ln v)^{\theta_3}\right]^{1/\theta_3} \right\}, \theta_3 \in [1, \infty) \), (Gumbel copula),
(3.9)

\( H_0 : C(u,v) = C_1(u,v) = \exp \left\{ - \left[(- \ln u)^{\theta_3} + (- \ln v)^{\theta_3}\right]^{1/\theta_3} \right\}, \theta_3 \in [1, \infty) \), (Gumbel copula),
\( H_1 : C(u,v) = C_2(u,v) = uv, \) 
(3.10)

\( H_0 : C(u,v) = C_1(u,v) = \exp \left\{ - \left[(- \ln u)^{\theta_3} + (- \ln v)^{\theta_3}\right]^{1/\theta_3} \right\}, \theta_3 \in [1, \infty) \), (Gumbel copula),
\( H_1 : C(u,v) = C_2(u,v) = -\frac{1}{\theta_4} \ln \left(1 + \frac{(e^{-\theta_4 u} - 1)(e^{-\theta_4 v} - 1)}{(e^{-\theta_4 u} - 1)}\right), \theta_4 \in (-\infty, \infty) \setminus \{0\}, \)
(Frank copula). 
(3.11)

### Table 10. \( P(ND \mid H_0) \) under hypothesis (3.11) for some values of \( n, r, \theta_3 \) and \( \theta_4 \).

<table>
<thead>
<tr>
<th>( (n,r) )</th>
<th>( (\theta_3, \theta_4) )</th>
<th>( P(ND \mid H_0) )</th>
<th>( (n,r) )</th>
<th>( (\theta_3, \theta_4) )</th>
<th>( P(ND \mid H_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10,1)</td>
<td>(3,5)</td>
<td>0.3106</td>
<td>(15,1)</td>
<td>(3,5)</td>
<td>0.3332</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(4,5)</td>
<td>0.2738</td>
<td>(15,1)</td>
<td>(4,5)</td>
<td>0.2892</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(5,5)</td>
<td>0.2126</td>
<td>(15,1)</td>
<td>(5,5)</td>
<td>0.2382</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(5,2)</td>
<td>0.2452</td>
<td>(15,1)</td>
<td>(5,2)</td>
<td>0.2482</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(5,3)</td>
<td>0.2454</td>
<td>(15,1)</td>
<td>(5,3)</td>
<td>0.2572</td>
</tr>
<tr>
<td>(10,1)</td>
<td>(5,4)</td>
<td>0.2318</td>
<td>(15,1)</td>
<td>(5,4)</td>
<td>0.2638</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(3,5)</td>
<td>0.7444</td>
<td>(15,2)</td>
<td>(3,5)</td>
<td>0.754</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(4,5)</td>
<td>0.7008</td>
<td>(15,2)</td>
<td>(4,5)</td>
<td>0.7258</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,5)</td>
<td>0.684</td>
<td>(15,2)</td>
<td>(5,5)</td>
<td>0.6834</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,2)</td>
<td>0.6936</td>
<td>(15,2)</td>
<td>(5,2)</td>
<td>0.7032</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,3)</td>
<td>0.6918</td>
<td>(15,2)</td>
<td>(5,3)</td>
<td>0.7008</td>
</tr>
<tr>
<td>(10,2)</td>
<td>(5,4)</td>
<td>0.6742</td>
<td>(15,2)</td>
<td>(5,4)</td>
<td>0.6838</td>
</tr>
</tbody>
</table>

From Tables 7-10, it can be observed that when \( n \) increases, \( P(ND \mid H_0) \) increases. For fixed values of \( n \) and dependence parameters, when \( r \) increases \( P(ND \mid H_0) \) increases. In Tables 7 and 9, when \( \theta_3 (\theta_4) \) increases \( P(ND \mid H_0) \) decreases. In Table 8 for fixed values of \( n, r \) and \( \theta_3 (\theta_4) \), as \( \theta_1 (\theta_3) \) increases \( P(ND \mid H_0) \) decreases. Furthermore, from Table 10, one can observe that for fixed values of \( n, r \) and \( \theta_3 (\theta_4) \), as \( \theta_1 (\theta_3) \) increases \( P(ND \mid H_0) \) decreases.

In Tables 11-15 the results of simulation study are provided for probability of making no decision, when the alternative hypothesis is true \( (P(ND \mid H_1)) \). We consider the following hypotheses:

\( H_0 : C(u,v) = C_1(u,v) = uv, \)
\( H_1 : C(u,v) = C_2(u,v) = \max \left( \left[u^{-\theta_1} + v^{-\theta_1} - 1\right]^{-\frac{1}{\theta_1}}, 0 \right), \theta_1 \in [-1, \infty) \setminus \{0\} \), (Clayton copula),
(3.12)

\( H_0 : C(u,v) = C_1(u,v) = uv (1 + \theta_2 (1 - u) (1 - v)), -1 \leq \theta_2 \leq 1 \) (FGM copula),
\( H_1 : C(u,v) = C_2(u,v) = \max \left( \left[u^{-\theta_1} + v^{-\theta_1} - 1\right]^{-\frac{1}{\theta_1}}, 0 \right), \theta_1 \in [-1, \infty) \setminus \{0\} \), (Clayton copula),
(3.13)

\( H_0 : C(u,v) = C_1(u,v) = uv, \)
\( H_1 : C(u,v) = C_2(u,v) = \exp \left\{ - \left[(- \ln u)^{\theta_3} + (- \ln v)^{\theta_3}\right]^{1/\theta_3} \right\}, \theta_3 \in [1, \infty) \), (Gumbel copula),
(3.14)
A consistent statistical test based on bivariate random samples

\[ H_0 : C(u, v) = C_1(u, v) = \frac{1}{\theta_4} \ln \left( 1 + \frac{e^{-\theta_4 u} - 1}{(e^{-\theta_4} - 1)} \right), \quad \theta_4 \in (-\infty, \infty) \setminus \{0\}, \]

(Frank copula),

\[ H_1 : C(u, v) = C_2(u, v) = \exp \left\{ -\left( -\ln u \right)^{\theta_3} + (- \ln v)^{\theta_3} \right\}^{1/\theta_3}, \quad \theta_3 \in [1, \infty), \quad \text{(Gumbel copula).} \]

(3.15)

**Table 11.** \( P(ND \mid H_1) \) under hypothesis (3.12) for \( n, r \) and \( \theta_1 \).

<table>
<thead>
<tr>
<th>( n, r )</th>
<th>( \theta_1 )</th>
<th>( P(ND \mid H_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 1)</td>
<td>3</td>
<td>0.3214</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>4</td>
<td>0.3626</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>5</td>
<td>0.3498</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>6</td>
<td>0.342</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>3</td>
<td>0.7766</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>4</td>
<td>0.7604</td>
</tr>
<tr>
<td>(10, 2)</td>
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<td>0.7416</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>6</td>
<td>0.7152</td>
</tr>
</tbody>
</table>

**Table 12.** \( P(ND \mid H_1) \) under hypothesis (3.13) for some values of \( n, r, \theta_1 \) and \( \theta_2 \).

<table>
<thead>
<tr>
<th>( n, r )</th>
<th>( \theta_1, \theta_2 )</th>
<th>( P(ND \mid H_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 1)</td>
<td>(4, -1)</td>
<td>0.3724</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(4, 1)</td>
<td>0.3522</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(5, -1)</td>
<td>0.3484</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(5, 1)</td>
<td>0.3346</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(4, -1)</td>
<td>0.7602</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(4, 1)</td>
<td>0.7338</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(5, -1)</td>
<td>0.746</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(5, 1)</td>
<td>0.721</td>
</tr>
</tbody>
</table>

**Table 13.** \( P(ND \mid H_1) \) under hypothesis (3.9) for some values of \( n, r, \theta_1 \) and \( \theta_2 \).

<table>
<thead>
<tr>
<th>( n, r )</th>
<th>( \theta_1, \theta_2 )</th>
<th>( P(ND \mid H_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 1)</td>
<td>(1, 3)</td>
<td>0.2872</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(2.5)</td>
<td>0.2808</td>
</tr>
<tr>
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<td>(3.5)</td>
<td>0.2768</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(4.5)</td>
<td>0.267</td>
</tr>
<tr>
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<td>(5, 5)</td>
<td>0.2558</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(5, 2)</td>
<td>0.4666</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(5, 3)</td>
<td>0.3504</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(5, 4)</td>
<td>0.3092</td>
</tr>
<tr>
<td>(10, 1)</td>
<td>(5, 6)</td>
<td>0.2624</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(1.5)</td>
<td>0.7242</td>
</tr>
<tr>
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<td>0.7238</td>
</tr>
<tr>
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<td>(3.5)</td>
<td>0.7198</td>
</tr>
<tr>
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</tr>
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<td>(5, 2)</td>
<td>0.8402</td>
</tr>
<tr>
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<td>(5, 3)</td>
<td>0.7662</td>
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<tr>
<td>(10, 2)</td>
<td>(5, 4)</td>
<td>0.73</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>(5, 6)</td>
<td>0.695</td>
</tr>
</tbody>
</table>

In Tables 11-15, one can observe that as \( n \) increases, \( P(ND \mid H_1) \) increases for fixed values of \( r \) and dependence parameters. Furthermore for fixed values of \( r \) and the dependence parameters, as \( r \) increases \( P(ND \mid H_1) \) decreases. In Tables 11 and 14, when
$\theta_1 \ (\theta_3)$ increases $P(ND \mid H_1)$ decreases. In Table 12 for fixed values of $\theta_1 \ (\theta_2)$, $n$ and $r$ when $\theta_2 \ (\theta_1)$ increases $P(ND \mid H_1)$ decreases. In Table 13, for fixed values of $n$, $r$ and $\theta_3$ when $\theta_1$ increases, $P(ND \mid H_1)$ decreases. Similarly for fixed values of $n$, $r$ and $\theta_1$ when $\theta_3$ increases, $P(ND \mid H_1)$ decreases. Likewise, in Table 15 one can observe that for fixed values of $\theta_3 \ (\theta_4)$, $n$ and $r$, as $\theta_4 \ (\theta_3)$ increases $P(ND \mid H_1)$ decreases.

<table>
<thead>
<tr>
<th>$(n, r)$</th>
<th>$\theta_1$</th>
<th>$P(ND \mid H_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(10, 1)$</td>
<td>2</td>
<td>0.4924</td>
</tr>
<tr>
<td>$(10, 1)$</td>
<td>3</td>
<td>0.3656</td>
</tr>
<tr>
<td>$(10, 1)$</td>
<td>4</td>
<td>0.3274</td>
</tr>
<tr>
<td>$(10, 1)$</td>
<td>5</td>
<td>0.2932</td>
</tr>
<tr>
<td>$(10, 2)$</td>
<td>2</td>
<td>0.863</td>
</tr>
<tr>
<td>$(10, 2)$</td>
<td>3</td>
<td>0.7958</td>
</tr>
<tr>
<td>$(10, 2)$</td>
<td>4</td>
<td>0.7566</td>
</tr>
<tr>
<td>$(10, 2)$</td>
<td>5</td>
<td>0.7316</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(n, r)$</th>
<th>$(\theta_3, \theta_4)$</th>
<th>$P(ND \mid H_1)$</th>
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4. Conclusion

In this paper, a test statistic $T(n, r)$ is proposed for testing the distribution of bivariate sample $Z$ based on bivariate samples $Z_1$ and $Z_2$. The consistency of $T(n, n - 1)$ test is discussed by considering the probability of type I, II errors and probability of making no decision under null and alternative hypotheses. These probabilities involve copulas of underlying distributions. Also unbiased and consistent estimators for probability of making no decision under the null/alternative hypotheses are proposed. This approach can help the investigators in prediction of probability of making no decision. If probability of making no decision is high, the test can not be performed or it can be repeated with different samples. Therefore, the researcher avoids making wrong decision. Furthermore, under particular hypotheses a simulation study is performed. In simulation study we observe that the probability of making no decision under the null/alternative hypotheses is considerably high for large values of $n$. Therefore, the use of $T(n, r)$ is preferable for small sample size.

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References


Appendix A

Proof of Theorem 2.2. We have to show that the probabilities of type I and type II errors both tend to infinity as \( n \to \infty \) and the probability of not making a decision under hypotheses \( H_0 \) and \( H_1 \) are less than 1. We use probability integral transformation

\[ F_X(t) = u, \quad F_Y(s) = v \quad \text{and} \quad F(t, s) = C_1(F_X(t), F_Y(s)), \]

in Equation (2.3). It is clear that

\[
|J| = \left| \begin{array}{cc} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{f_X(F_X^{-1}(u))} & 0 \\ 0 & \frac{1}{f_Y(F_Y^{-1}(v))} \end{array} \right| = \frac{1}{f_X(t)f_Y(s)}
\]

and

\[
P_{1-1}(t, s) = \frac{\partial F(t, s)}{\partial t} = \frac{\partial C_1(F_X(t), F_Y(s))}{\partial t} = \frac{\partial C_1(F_X(t), F_Y(s))}{\partial F_X(t)} \cdot dF_X(t) = C_1^{-1}(F_X(t), F_Y(s)) f_X(t), \quad (A.1)
\]

where \( C_1^{-1}(F_X(t), F_Y(s)) = \frac{\partial C_1(F_X(t), F_Y(s))}{\partial F_X(t)} \). Similarly,

\[
P_{1-1}^{-1}(t, s) = C_1^{-1}(F_X(t), F_Y(s)) f_Y(s),
\]

where \( C_1^{-1}(F_X(t), F_Y(s)) = \frac{\partial C_1(F_X(t), F_Y(s))}{\partial F_Y(s)} \). Taking into account Equations (2.2) and (2.3), the probability of type I error can be written as follows:

\[
\alpha_n = \int_0^1 \int_0^1 (C_1(u, v))^n \left\{ \sum_{l_1=a_1}^{a_2} \sum_{l_2=b_1}^{b_2} c_1(u, v) \right\}^{n-1} [u - C_1(u, v)]^{n-1-t_1} [v - C_1(u, v)]^{n-1-t_2} \times \left[ C_1^{-1}(u, v) \right]^{1-t_1} \left[ 1 - C_1^{-1}(u, v) \right]^{t_1} \left[ 1 - C_1^{-1}(u, v) \right]^{1-t_2} dudv,
\]

where \( C_1^{-1}(1-u, 1-v) = 1 - u + C_1(u, v) \) and \( c_1(u, v) = \frac{\partial^2 C_1(u, v)}{\partial u \partial v} \). It is obvious that \( a_1 = c_1 = d_1 = 0 \) and \( a_2 = c_2 = d_2 = 1 \) for \( r = n - 1 \). Then, the probability of type I error
under $H_0$: $C(u, v) = C_1(u, v) = uv$ is given by

$$
\alpha_n = \int_0^1 \int_0^1 (uv)^n \left\{ \left( n(n-1)^2 uv [(1-u)(1-v)]^{n-2} \right) \right.
\left. + \sum_{t_4=0}^1 \sum_{t_2=0}^1 \sum_{t_1=\max(0,4-n-t_2-t_4)}^{\min(1-t_2,1-t_4)} l_2 uv [(1-u)(1-v)]^{n-2} \right\} du dv
$$

$$
= \frac{n(n-1)^2 (n+1)! (n-2)!}{(2n)!} + \frac{(n^2-1)(n-1)! (n+1)! (n!)^2}{((2n)!)^2}
$$

$$
= \frac{n(n-1)^2 (n+1)(n-2)!}{4^n A(n)} + \frac{(n^2-1)(n-1)! (n+1)!}{4^n (A(n))^2},
$$

where

$$
A(n) = \prod_{k=1}^n (2n - (2k - 1)).
$$

For simplicity denote $I'_n = \frac{n(n-1)^2 (n+1)(n-2)!}{4^n A(n)}$ and $I''_n = \frac{(n^2-1)(n-1)! (n+1)!}{4^n (A(n))^2}$. Then,

$$
\lim_{n \to \infty} I'_n = \lim_{n \to \infty} \frac{n(n-1)^2 (n+1)(n-2)!}{4^n A(n)}
$$

$$
= \lim_{n \to \infty} \frac{n(n+1)(n-2)!}{A(n)} \cdot \lim_{n \to \infty} \frac{(n-1)^2}{4^n}
$$

$$
= 0 \cdot \lim_{n \to \infty} \frac{(n-1)^2}{4^n}
$$

$$
= 0.
$$

Similarly,

$$
\lim_{n \to \infty} I''_n = \lim_{n \to \infty} \frac{(n^2-1)(n-1)! (n+1)!}{4^n A(n)}
$$

$$
= \lim_{n \to \infty} \frac{(n^2-1)(n+1)! (n-1)! n!}{4^n (A(n))^2}
$$

$$
= \lim_{n \to \infty} \frac{(n^2-1)(n+1)!}{4^n} \cdot \lim_{n \to \infty} \frac{(n-1)! n!}{(A(n))^2}
$$

$$
= 0.
$$

It can be easily shown that $\lim_{n \to \infty} \alpha_n = 0$.

Now let us calculate the probability of no decision given that $H_0$ is true. Let $A$ denotes the event that at least $r$ of the sample values in $Z$ are in $(-\infty, X_{1:n}^{(2)}) \times (-\infty, Y_{1:n}^{(2)})$ and $B$ denotes the event at least $r$ of the sample values in $Z$ are in $X_{n:n}^{(1)} \times (Y_{1:n}^{(1)}, \infty)$. The events $A$ and $B$ are mutually exclusive events. Then,

$$
P \{ ND | H_0 \} = 1 - P \{ A \cup B | H_0 \}
$$

$$
= 1 - P \{ A | H_0 \} - P \{ B | H_0 \}.
$$

(A.3)

It is obvious that $P \{ B | H_0 \} = \alpha_n$, then,

$$
P \{ ND | H_0 \} = 1 - \alpha_n - P \{ A | H_0 \}.
$$
\[ P \{ A \mid H_0 \} = P \left\{ \text{at least } r \text{ of the sample values in } Z \text{ are in } (-\infty, X_{1/n}^{(2)}) \times (-\infty, Y_{1/n}^{(2)}) \mid H_0 \right\} \]

\[ = P \left\{ X_{r:n} < X_{1/n}^{(2)}, Y_{r:n} < Y_{1/n}^{(2)} \mid H_0 \right\} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\Theta(t, s)]^n f_{X_{r:n}, Y_{r:n}}(t, s) \, dt \, ds. \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\Theta(t, s)]^n \left( \sum_{t_1=n-3}^{n-2} \tilde{l}_1 [F(t, s)]^{t_1} \left[ (F_X(t) - F(t, s)) (F_Y(s) - F(t, s)) \right]^{n-2-t_1} \right. \]

\[ \times \left[ F(t, s) \right]^{3-n+t_1} f(t, s) + \sum_{t_4=0}^{1} \sum_{t_2=0}^{1} \sum_{t_1=\max(0, n-2-t_2-t_4)}^{n-2-t_4} \tilde{l}_2 [F(t, s)]^{t_1} \right), \]

where

\[ \tilde{l}_1 = \frac{n!}{t_1! [(n-2-t_1)!]^2 (3-n+t_1)!} \]

and

\[ \tilde{l}_2 = \frac{n!}{t_1! (n-2-t_1-t_2)! (n-2-t_1-t_4)! (2-n+t_1+t_2+t_4)!} \]

Similarly by using probability integral transformation we have

\[ P \{ A \mid H_0 \} = \int_{0}^{1} \int_{0}^{1} \left( \hat{C}_2 (1-u, 1-v) \right)^n \]

\[ \times \left\{ \sum_{t_1=n-3}^{n-2} \tilde{l}_1 [C_1(u,v)]^{t_1} \left[ (u - C_1(u,v)) (v - C_1(u,v)) \right]^{n-2-t_1} \right. \]

\[ \times \left[ \hat{C}_1 (1-u, 1-v) \right]^{3-n+t_1} c_1(u,v) \]

\[ + \sum_{t_4=0}^{1} \sum_{t_2=0}^{1} \sum_{t_1=\max(0, n-2-t_2-t_4)}^{n-2-t_1-t_2} \tilde{l}_2 [C_1(u,v)]^{t_1} [u - C_1(u,v)]^{n-2-t_1-t_2} \]

\[ \times [v - C_1(u,v)]^{n-2-t_1-t_4} \left[ \hat{C}_1 (1-u, 1-v) \right]^{2-n+t_1+t_2+t_4} \]

\[ \left. \times \left[ C_1(u,v) \right]^{t_2} [1 - C_1(u,v)]^{1-t_2} \left[ C_1(u,v) \right]^{t_4} [1 - C_1(u,v)]^{1-t_4} \right\} du \, dv \]

(A.4)

Then, under \( H_0 : C(u,v) = C_1(u,v) = uv \),

\[ P \{ A \mid H_0 \} + P \{ B \mid H_0 \} \geq o(h_1) > 0, \]

where \( o(h_1) = \alpha_n = \frac{n(n-1)^2(n+1)(n-2)!}{4^n(A(n))^2} + \frac{(n^2-1)(n-1)(n+1)!}{4^n(A(n))^2}. \) It is clear that \( P \{ ND \mid H_0 \} < 1. \) Thus Theorem 2.2 is proved. \( \square \)
Appendix B

**Proof of Theorem 2.3.** By using the similar probability integral transformation used in Equation (A.2), Equation (2.4) can be written as follows:

\[
\beta_n = P \{ H_0 \mid H_1 \} \\
= P \{ \text{at least } n - 1 \text{ of the sample values in } Z \text{ are in } \\
\left( -\infty, X_{1:n}^{(2)} \right) \times \left( -\infty, Y_{1:n}^{(2)} \right) \mid H(x, y) = G(x, y) \}
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (G(t, s))^n g_{X_{n-1:n}, Y_{n-1:n}}(t, s) \, dt \, ds.
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (G(t, s))^n \left( \sum_{t_1 = n-3}^{n-2} \tilde{t}_1 [G(t, s)]^{t_1} \left[ (F_X(t) - G(t, s)) (F_Y(s) - G(t, s)) \right]^{n-2-t_1} \right. \\
\times \left. \left[ G(t, s) \right]^{3-n+t_1} g(t, s) + \sum_{t_4=0}^{1} \sum_{t_2=0}^{1} \sum_{t_2=0}^{\min(n-2-t_2,n-2-t_4)} \tilde{t}_2 [G(t, s)]^{t_1} \\
\times \left[ (F_X(t) - G(t, s))^{n-2-t_1-t_2} \left[ (F_Y(s) - G(t, s))^{n-2-t_1-t_4} \right] \right] \\
\times \left[ G(t, s) \right]^{2-n+t_1+t_2+t_4} \left[ G^{-1}(t, s) \right]^{t_2} \left[ f_Y(s) - G^{-1}(t, s) \right]^{1-t_2} \right. \\
\times \left. \left[ G^{1\nu}(t, s) \right]^{t_4} [f_X(t) - G^{1\nu}(t, s)]^{1-t_4} \right) \, dt \, ds,
\]

(A.5)

where \( \tilde{t}_1 \) and \( \tilde{t}_2 \) are

\[
\tilde{t}_1 = \frac{n!}{t_1! \left( (n-2-t_1)! \right)^2 \left( 3-n+t_1 \right)!}
\]

\[
\tilde{t}_2 = \frac{n!}{t_1! \left( n-2-t_1-t_2 \right)! \left( n-2-t_1-t_4 \right)! \left( 2-n+t_1+t_2+t_4 \right)!}
\]

By probability integral transformation we have the following equation.

\[
\beta_n = \int_0^1 \int_0^1 \left( \tilde{C}_2(1-u,1-v) \right)^n \left\{ \sum_{t_1=n-3}^{n-2} \tilde{t}_1 \left[ C_2(u,v) \right]^{t_1} [u - C_2(u,v)]^{1-t_1} \right. \\
\times \left. \left[ C_2(1-u,1-v) \right]^{3-n+t_1} c_2(u,v) + \sum_{t_4=0}^{1} \sum_{t_2=0}^{1} \sum_{t_2=0}^{\min(n-2-t_2,n-2-t_4)} \tilde{t}_2 \left[ C_2(u,v) \right]^{t_1} [u - C_2(u,v)]^{n-2-t_1-t_2} \\
\times \left. [v - C_2(u,v)]^{n-2-t_1-t_4} \left[ \tilde{C}_2(1-u,1-v) \right]^{2-n+t_1+t_2+t_4} \right) \\
\times \left. \left[ C_2^{1\nu}(u,v) \right]^{t_4} [1 - C_2^{1\nu}(u,v)]^{1-t_4} \left[ C_2^{1\nu}(u,v) \right]^{t_4} [1 - C_2^{1\nu}(u,v)]^{1-t_4} \right) \, du \, dv.
\]
Then, the probability of type II error under $H_1 : C (u, v) = C_2 (u, v) = uv$ is

$$
\beta_n = \int_0^1 \int_0^1 [(1 - u) (1 - v)]^n (uv) n (n - 1)^2 + (uv) n (n - 1) (1 - u) (1 - v) \sum_{t_1=0}^1 \sum_{t_2=0}^1 \sum_{t_3=0}^1 \sum_{t_4=0}^1 \tilde{t}_2 dudv = \frac{n (n - 1)^2 [(n - 2)!]^2 [(n + 1)!]^2}{[(2n)!]^2} + \frac{(n - 1) (n - 1)! n! [(n + 1)!]^2}{[(2n)!]^2}.
$$

Denote $K_n' = \frac{n(n - 1)^2[(n - 2)!]^2[(n + 1)!]^2}{[(2n)!]^2}$ and $K_n'' = \frac{(n - 1)(n - 1)!n!(n + 1)!^2}{[(2n)!]^2}$. Then,

$$
\lim_{n \to \infty} K_n' = \lim_{n \to \infty} \frac{n (n - 1)^2 [(n - 2)!]^2 [(n + 1)!]^2}{[(2n)!]^2} = \lim_{n \to \infty} \frac{n (n + 1)^2 [(n - 1)!]^2 (n!)^2}{[(2n)!]^2} = \lim_{n \to \infty} \frac{n (n + 1)^2 [(n - 1)!]^2}{4^n (A(n))^2} = \lim_{n \to \infty} \frac{n (n + 1)^2}{4^n} \cdot \lim_{n \to \infty} \left[ \frac{(n - 1)!^2}{A(n)} \right] = \lim_{n \to \infty} K_n' = 0.
$$

Similarly,

$$
\lim_{n \to \infty} K_n'' = \lim_{n \to \infty} \frac{(n - 1) (n - 1)! n! [(n + 1)!]^2}{[(2n)!]^2} = \lim_{n \to \infty} \frac{(n - 1) (n + 1)^2 (n - 1)! n! (n!)}{[(2n)!]^2} = \lim_{n \to \infty} \frac{(n - 1) (n + 1)^2 (n - 1)! n!}{[(2n) (2n - 1) (2n - 2) \cdots 3.2.1]^2} = \lim_{n \to \infty} \frac{(n - 1) (n + 1)^2 (n - 1)! n!}{4^n [(2n - 1) (2n - 3) (2n - 5) \cdots 5.3.1]^2} = \lim_{n \to \infty} \frac{(n - 1) (n + 1)^2}{4^n} \cdot \lim_{n \to \infty} \frac{(n - 1)! n!}{[(2n - 1) (2n - 3) (2n - 5) \cdots 5.3.1]^2} = 0 \cdot \lim_{n \to \infty} \frac{(n - 1)! n!}{[(2n - 1) (2n - 3) (2n - 5) \cdots 5.3.1]^2} = 0.
$$

Thus $\lim_{n \to \infty} \beta_n = 0$.

Now consider $P \{ ND \mid H_1 \}$ under $H_1 : C (u, v) = C_2 (u, v) = uv$,

$$
P \{ ND \mid H_1 \} = 1 - P \{ A \cup B \mid H_1 \} = 1 - P \{ A \mid H_1 \} - P \{ B \mid H_1 \}.
$$

(A.6)

It is obvious that $\beta_n = P \{ A \mid H_1 \}$, then,

$$
P \{ A \mid H_1 \} + P \{ B \mid H_1 \} \geq o(h_2) > 0,$$

where $o(h_2) = \frac{n(n - 1)^2[(n - 2)!]^2[(n + 1)!]^2}{[(2n)!]^2} + \frac{(n - 1)(n - 1)!n!(n + 1)!^2}{[(2n)!]^2}$ and $P \{ ND \mid H_1 \} < 1$. Thus Theorem 2.3 is proved.