Nearness $\Gamma$-Near Rings

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Abstract: The aim of this study is to introduce nearness $\Gamma$-near ring, nearness $\Gamma$-subnear ring and nearness $\Gamma$-ideal. Moreover, some properties of these structures are investigated.

Keywords: Nearness ring, nearness $\Gamma$-near ring, nearness $\Gamma$-ideal.

1. Introduction

A generalization of rough sets, near sets and near approximation spaces were introduced in 2007 [12, 20]. The selection of probe functions that provide a basis for defining and distinguishing affinities between objects is the first step in near set theory. A probe function is a real-valued function representing a feature of objects such as images.

Instead of abstract points, the sets in the nearness approximation space are mainly composed of perceptual objects (non-abstract points). Perceptual objects are featured points. Feature vectors can be used to describe these points [12]. The upper approximation of a set is determined by matching descriptions of objects in the set of perceptual objects. The consideration of upper approximations of perceptual object subsets is a fundamental method in algebraic structures built on nearness approximation space. In a nearness groupoid, the binary operation has the closeness property in upper approximation of set instead of set.

In 1936, Zassenhaus defined the near-ring as a generalization of ring [21]. The most basic source in near ring theory is Pilz’s book titled Near Rings [15].

Nobusawa defined the idea of a $\Gamma$-ring that is more general than a ring [9]. Barnes weakened the axioms in Nobusawa’s description of the $\Gamma$-ring [1]. Barnes, Kyuno [6] and Luh [7] investigated the structure of $\Gamma$-rings and discovered a number of generalizations that are analogous to ring theory.

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Satyanarayana defined the $\Gamma$-near ring as a generalization of near-ring and $\Gamma$-ring [16].

In 2012, İnan and Öztürk [3, 4] investigated the nearness groups. In 2013, nearness group of weak cosets was introduced [11]. In 2015, İnan et al. [5] also investigated the nearness semigroups. In 2019, nearness ring was introduced as well [10].

The aim of this study is to introduce nearness $\Gamma$-near ring, nearness $\Gamma$-subnear ring and nearness $\Gamma$-ideal. Moreover, some properties of these structures are investigated.

2. Preliminaries

Perceptual objects are points that are describable with feature vectors. Let $O$ be a set of perceptual objects, $X \subseteq O$, $F$ be a set of probe functions and $\Phi : O \rightarrow \mathbb{R}^L$ be a mapping, where the description length is $|\Phi| = L$.

$$\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \cdots, \varphi_i(x), \cdots, \varphi_L(x))$$

is an object description of $x \in X$ such that each $\varphi_i \in B \subseteq F$ ($\varphi_i : O \rightarrow \mathbb{R}$) is a probe function that represents features of sample objects $X \subseteq O$ [12].

Sample objects are near each other if and only if the objects have similar descriptions. Recall that each $\varphi_i$ defines a description of an object. $\Delta_{\varphi_i}$ is defined by $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$, where $x, x' \in O$.

Let $x, x' \in O$ and $B \subseteq F$.

$$\sim_B = \{(x, x') \in O \times O | \Delta_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $O$, where description length is $i \leq |\Phi|$ [12].

**Definition 2.1** [8] Let $O$ be a set of perceptual objects, $\Phi$ be an object description and $A \subseteq O$. Then the set description of $A$ is defined as

$$Q(A) = \{\Phi(a) | a \in A\}.$$

**Definition 2.2** [8, 14] Let $O$ be a set of perceptual objects and $A$, $B \subseteq O$. Then the descriptive (set) intersection of $A$ and $B$ is defined as

$$A \cap_{\Phi} B = \{x \in A \cup B | \Phi(x) \in Q(A) \text{ and } \Phi(x) \in Q(B)\}.$$ 

If $Q(A) \cap Q(B) \neq \emptyset$, then $A$ is called descriptively near $B$ and denoted by $A \delta_{\Phi} B$. Also, $\xi_{\Phi}(A) = \{B \in \mathcal{P}(O) | A \delta_{\Phi} B\}$ is a descriptive nearness collection [13].

**Definition 2.3** [12] Let $X \subseteq O$ and $x \in X$.

$$[x]_{B_{r}} = \{x' \in O | x \sim_{B_{r}} x'\}$$

47
is called nearness class of \( x \in X \).

**Definition 2.4** [12] Let \( X \subseteq \mathcal{O} \).

\[
N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \emptyset} [x]_{B_r}
\]

is called upper approximation of \( X \).

A nearness approximation space is \((\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})\), where \( \mathcal{O} \) is a set of perceptual objects, \( \mathcal{F} \) is a set of probe functions, \( \sim_{B_r} \) is an indiscernibility relation relative to \( B_r \subseteq B \subseteq \mathcal{F} \), \( N_r(B) \) is a collection of partitions and \( \nu_{N_r} : \mathcal{F} \times \mathcal{F} \to [0,1] \) is an overlap function that maps a pair of sets to \([0,1]\) representing the degree of nearness between sets. The subscript \( r \) denotes the cardinality of the restricted subset \( B_r \).

**Definition 2.5** [3] Let \((\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})\) be a nearness approximation space and \( \cdot \) be a binary operation defined on \( \mathcal{O} \). \( G \subseteq \mathcal{O} \) is called a nearness group if the following properties are satisfied:

1. \((NG_1)\) For all \( x, y \in G \), \( x \cdot y \in N_r(B)^* G \),
2. \((NG_2)\) For all \( x, y, z \in G \), \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \) property holds in \( N_r(B)^* G \),
3. \((NG_3)\) There exists \( e_G \in N_r(B)^* G \) such that \( x \cdot e_G = e_G \cdot x = x \) for all \( x \in G \) (\( e_G \) is called the near identity element of \( G \)),
4. \((NG_4)\) There exists \( y \in G \) such that \( x \cdot y = y \cdot x = e_G \) for all \( x \in G \) (\( y \) is called the inverse of \( x \) in \( G \) and denoted as \( x^{-1} \)).

Additionally, if the property \( x \cdot y = y \cdot x \) is satisfied in \( N_r(B)^* G \) for all \( x, y \in G \), then \( G \) is said to be a commutative nearness group.

Also, \( S \subseteq \mathcal{O} \) is called a nearness semigroup if \( x \cdot y \in N_r(B)^* S \) for all \( x, y \in S \) and \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \) property is satisfied in \( N_r(B)^* (S) \) for all \( x, y, z \in S \).

**Theorem 2.6** [4] Let \( G \) be a nearness group, \( H \) be a nonempty subset of \( G \) and \( N_r(B)^* H \) be a groupoid. Then \( H \subseteq G \) is a subnearness group of \( G \) if and only if \( x^{-1} \in H \) for all \( x \in H \).

**Definition 2.7** [10] Let \((\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})\) be a nearness approximation space and \( + \) and \( \cdot \) be binary operations defined on \( \mathcal{O} \). \( R \subseteq \mathcal{O} \) is called a nearness ring if the following properties are satisfied:

1. \((NR_1)\) \( R \) is a commutative nearness group with binary operation \( + \),
2. \((NR_2)\) \( R \) is a nearness semigroup with binary operation \( \cdot \).
\((NR_3)\) For all \(x, y, z \in R\),
\[
    x \cdot (y + z) = (x \cdot y) + (x \cdot z) \quad \text{and} \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z)
\]
properties hold in \(N, (B)^* R\).

In addition,
\((NR_4)\) \(R\) is said to be a commutative nearness ring if \(x \cdot y = y \cdot x\) for all \(x, y \in R\),
\((NR_5)\) \(R\) is said to be a nearness ring with identity if \(N, (B)^* R\) contains an element \(1_R\) such that \(1_R \cdot x = x \cdot 1_R = x\) for all \(x \in R\).

**Definition 2.8** [15, 21] Let \(N\) be a nonempty set and “+” and “.” be binary operations defined on \(N\). \(N\) is called a (right) near ring if the following properties are satisfied:

\((N_1)\) \(N\) is a group with binary operation “+” (It does not need to be commutative),
\((N_2)\) \(N\) is a semigroup with binary operation “.”,
\((N_3)\) For all \(x, y, z \in N\), \((x + y) \cdot z = (x \cdot z) + (y \cdot z)\) properties hold in \(N, (B)^* N\).

**Definition 2.9** [1] A \(\Gamma\)-ring (in the sense of Barnes) is a pair \((M, \Gamma)\), where \(M\) and \(\Gamma\) are (additive) commutative groups for which exists a \(\gamma : \Gamma \times \Gamma \times M \to M\), the image of \((a, \alpha, b)\) being denoted by \(a \alpha b\) for \(a, b \in M\) and \(\alpha \in \Gamma\), satisfying for all \(a, b, c \in M\) and all \(\alpha, \beta \in \Gamma\):

- \((a + b) \alpha c = a \alpha c + b \alpha c\),
- \(a \alpha (b + c) = a \alpha b + a \alpha c\),
- \((a \alpha b) \beta c = a \alpha (b \beta c)\).

**Definition 2.10** [1] Let \(M\) be a \(\Gamma\)-ring. A left (right) ideal of \(M\) is an additive subgroup \(U\) of \(M\) such that \(M U \subseteq U\) \((U M \subseteq U)\). If \(U\) is both a left and a right ideal, then we say that \(U\) is an ideal of \(M\).

**Definition 2.11** [16] A \(\Gamma\)-near ring is a triple \((M, +, \Gamma)\), where

\((\Gamma N_1)\) \((M, +)\) is a group (need not be commutative),
\((\Gamma N_2)\) \(\Gamma\) is a non-empty set of binary operators on \(M\) such that \((M, +, \gamma)\) is a (right) near ring for each \(\gamma \in \Gamma\),
\((\Gamma N_3)\) \((a \alpha b) \beta c = a \alpha (b \beta c)\) for all \(a, b, c \in M\) and \(\alpha, \beta \in \Gamma\).

**Definition 2.12** [19] Let \((\mathcal{O}, \mathcal{F}, \sim_B, N_r, (B), \nu_N)\) be a nearness approximation space and \(M, \Gamma \subseteq \mathcal{O}\) be an additive commutative nearness groups in \(\mathcal{O}\). \(M \subseteq \mathcal{O}\) is named an \(\Gamma\)-ring in nearness approximation space or shortly, nearness \(\Gamma\)-ring if the followings are provided:

\((NT_1)\) \(a \alpha b \in N_r, (B)^* M\),
\((NT_2)\) \((a \alpha b) \beta c = a \alpha (b \beta c)\) property verify on \(N_r, (B)^* M\),
$(N\Gamma_3)$ $(a + b) \alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = aab + a\beta b$, $a\alpha (b + c) = aab + aoc$ properties verify on $N_r(B)^* M$ for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$.

In addition, $M$ is called a commutative nearness $\Gamma$-ring if $a\alpha b = b\alpha a$ for all $a, b \in M$ and all $\alpha \in \Gamma$.

$M$ is called a nearness $\Gamma$-ring with identity if $N_r(B)^* M$ contains $1_M$ such that $1_M \alpha a = a\alpha$ and all $a \in M$ and all $\alpha \in \Gamma$.

3. Nearness $\Gamma$-near rings

Throughout this section, $O$ considered as a set of perceptual objects in nearness approximation space unless otherwise stated.

**Definition 3.1** Let $N, \Gamma \subseteq O$ be additive nearness groups. If for all $k, \ell, m \in N$ and all $\beta, \gamma \in \Gamma$ the conditions

$(N\Gamma N_1)$ $k\beta \ell \in N_r(B)^* N,$

$(N\Gamma N_2)$ $(k + \ell)\beta m = k\beta m + \ell\beta m$ property provides on $N_r(B)^* N$,

$(N\Gamma N_3)$ $(k\beta \ell)\gamma m = k\beta (\ell\gamma m)$ property provides on $N_r(B)^* N$

are satisfied, then $N$ is called an $\Gamma$-near ring in nearness approximation space or shortly nearness $\Gamma$-near ring.

In addition, if $k\beta \ell = \ell\beta k$ for all $k, \ell \in N$ and all $\beta \in \Gamma$, then $N$ is called a commutative nearness $\Gamma$-near ring.

**Example 3.2** $O = \{k_{ij} \mid 0 \leq i, j \leq 4\}$ be a set of perceptual objects and $B = \{\varphi\} \subseteq F$ be a set of probe function. Probe function

$$\varphi : O \longrightarrow V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

is given in Table 1.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$k_{00}$</th>
<th>$k_{01}$</th>
<th>$k_{02}$</th>
<th>$k_{03}$</th>
<th>$k_{04}$</th>
<th>$k_{10}$</th>
<th>$k_{11}$</th>
<th>$k_{12}$</th>
<th>$k_{13}$</th>
<th>$k_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$v_5$</td>
<td>$v_4$</td>
<td>$v_4$</td>
<td>$v_6$</td>
<td>$v_6$</td>
<td>$v_7$</td>
<td>$v_8$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$k_{20}$</th>
<th>$k_{21}$</th>
<th>$k_{22}$</th>
<th>$k_{23}$</th>
<th>$k_{24}$</th>
<th>$k_{30}$</th>
<th>$k_{31}$</th>
<th>$k_{32}$</th>
<th>$k_{33}$</th>
<th>$k_{34}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_7$</td>
<td>$v_6$</td>
<td>$v_8$</td>
<td>$v_8$</td>
<td>$v_8$</td>
<td>$v_8$</td>
<td>$v_8$</td>
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<td></td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$k_{40}$</th>
<th>$k_{41}$</th>
<th>$k_{42}$</th>
<th>$k_{43}$</th>
<th>$k_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_8$</td>
<td>$v_5$</td>
<td>$v_7$</td>
<td>$v_8$</td>
<td>$v_3$</td>
<td></td>
</tr>
</tbody>
</table>
Thus

\[
[k_{00}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{00}) = v_1 \} = \{ k_{00} \}, \]

\[
[k_{01}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{01}) = v_2 \} = \{ k_{01} \}, \]

\[
[k_{02}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{02}) = v_3 \} = \{ k_{02}, k_{44} \} = [k_{44}]_\varphi, \]

\[
[k_{03}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{03}) = v_5 \} = \{ k_{03}, k_{04}, k_{12}, k_{41} \} = [k_{04}]_\varphi = [k_{12}]_\varphi = [k_{41}]_\varphi, \]

\[
[k_{10}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{10}) = v_4 \} = \{ k_{10}, k_{11} \} = [k_{11}]_\varphi, \]

\[
[k_{13}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{13}) = v_6 \} = \{ k_{13}, k_{21}, k_{22}, k_{31} \} = [k_{21}]_\varphi = [k_{22}]_\varphi = [k_{31}]_\varphi, \]

\[
[k_{14}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{14}) = v_7 \} = \{ k_{14}, k_{20}, k_{24}, k_{32}, k_{42} \} = [k_{20}]_\varphi = [k_{24}]_\varphi = [k_{32}]_\varphi = [k_{42}]_\varphi, \]

\[
[k_{23}]_\varphi = \{ k \in O \mid \varphi(k) = \varphi(k_{23}) = v_8 \} = \{ k_{23}, k_{30}, k_{33}, k_{34}, k_{40}, k_{43} \} = [k_{30}]_\varphi = [k_{33}]_\varphi = [k_{34}]_\varphi = [k_{40}]_\varphi = [k_{43}]_\varphi. \]

Therefore

\[
\xi_\varphi = \{ [k_{00}]_\varphi, [k_{01}]_\varphi, [k_{02}]_\varphi, [k_{03}]_\varphi, [k_{10}]_\varphi, [k_{13}]_\varphi, [k_{14}]_\varphi, [k_{23}]_\varphi \}. \]

Hence, a set of partitions of \( O \) is \( N_1(B) = \{ \xi_\varphi \} \) for \( r = 1 \). Thus

\[
N_1(B)^\ast N = \bigcup [k]_\varphi \cap N \neq \emptyset \nsubseteq O \]

\[
= \{ k_{00} \} \cup \{ k_{01} \} \cup \{ k_{10}, k_{11} \} = \{ k_{00}, k_{01}, k_{10}, k_{11} \}
\]

and

\[
N_1(B)^\ast \Gamma = \bigcup [k]_\varphi \cap \Gamma \neq \emptyset \nsubseteq O \}

\[
= \{ k_{00}, k_{02}, k_{44} \}.
\]

where \( N = \{ k_{00}, k_{01}, k_{10} \} \), \( \Gamma = \{ k_{00}, k_{02} \} \subseteq O \).

Let

\[
+_1 : O \times O \to O
\]

\[
(k_{ij}, k_{mn}) \mapsto k_{ij} +_1 k_{mn}
\]
be a binary operation (first addition) on $O$ such that

$$k_{ij} +_1 k_{mn} = k_{pr}, \quad i + m \equiv p \pmod{2} \text{ ve } j + n \equiv r \pmod{2}.$$ 

Then $(N, +_1)$ is a nearness group.

Furthermore, let

$$+_2 : O \times O \rightarrow O \quad (k_{ij}, k_{mn}) \mapsto k_{ij} +_2 k_{mn}$$

be a binary operation (second addition) on $O$ such that

$$k_{ij} +_2 k_{mn} = k_{st}, \quad i + m \equiv s \pmod{4} \text{ ve } j + n \equiv t \pmod{4}.$$ 

Then $(\Gamma, +_2)$ is a nearness group.

Since $k_{01} + k_{10} = k_{11} \notin N$, $N \subseteq O$ is not a group with binary operation “$+_1$” and so $N$ is not a $\Gamma$-near ring.

Let

$$O \times \Gamma \times O \rightarrow O \quad (k_{ij}, k_{uv}, k_{mn}) \mapsto k_{ij}k_{uv}k_{mn} = k_{ij}$$

be an operation on $O$.

From Definition 3.1, it is easily shown that

$$(N \Gamma N_1) \quad k\beta\ell \in N_r(B)^* N,$$

$$(N \Gamma N_2) \quad (k +_1 \ell)\beta m = k\beta m +_1 \ell\beta m \text{ property provides on } N_r(B)^* N,$$

$$(N \Gamma N_3) \quad (k\beta\ell)\gamma m = k\beta(\ell\gamma m) \text{ property provides on } N_r(B)^* N$$

for all $k, \ell, m \in N$ and all $\beta, \gamma \in \Gamma$.

Consequently, $N$ is a nearness $\Gamma$-near ring.

Lemma 3.3 is obvious since $N \subseteq N_r(B)^* N$.

**Lemma 3.3** Every $\Gamma$-near ring is a nearness $\Gamma$-near ring.

From definition of nearness $\Gamma$-ring, it is clear that Lemma 3.4 is true.

**Lemma 3.4** Every nearness $\Gamma$-ring is a nearness $\Gamma$-near ring.

A nearness $\Gamma$-near ring is not always a $\Gamma$-near ring, and also a nearness $\Gamma$-near ring is not always a nearness $\Gamma$-ring.

Examples 3.5 and 3.6 are show that the opposites of the Lemma 3.3 and Lemma 3.4 are not true.
Example 3.5 From Example 3.2 $N$ is a nearness $\Gamma$-near ring. But $N$ is not a $\Gamma$-near ring because of $k_{01} + k_{10} = k_{11} \notin N$ for $k_{01}, k_{10} \in N$.

Example 3.6 From Example 3.2 $N$ is a nearness $\Gamma$-near ring. But $N$ is not a nearness $\Gamma$-ring because of $k_{10}k_{02}(k_{01} + k_{10}) \neq (k_{10}k_{02}k_{01}) + (k_{10}k_{02}k_{10})$ for $k_{10}, k_{01} \in N$ and $k_{02} \in \Gamma$.

Lemma 3.7 Let $N \subseteq O$ be a nearness $\Gamma$-near ring and $0 \in N$. If $0k_\gamma N \in N$ then

(i) $0k_\gamma N = 0_N$,

(ii) $(-k)k_\gamma \ell = -(k_\gamma \ell)$ for all $k, \ell \in N$ and all $\gamma \in \Gamma$.

Proof (i) For all $k \in N$ and all $\gamma \in \Gamma$,

$$0_Nk_\gamma = (0_N + 0_N)k_\gamma = 0_Nk_\gamma + 0_Nk_\gamma.$$  
Since the near identity element is unique, $0_Nk_\gamma = 0_N$.

(ii) From (i), $0_Nk_\gamma \ell = 0_N$ for all $k, \ell \in N$ and all $\gamma \in \Gamma$. Then

$$0_N = 0_Nk_\gamma \ell = ((-k)k_\gamma \ell + k_\gamma \ell).$$  
Since the inverse element is unique, $(-k)k_\gamma \ell = -(k_\gamma \ell)$.

For all $k, \ell \in N$ and all $\gamma \in \Gamma$, the equalities $k_\gamma 0_N = 0_N$ and $k_\gamma (-\ell) = -(k_\gamma \ell)$ may not be provided.

Definition 3.8 Let $N$ be a nearness $\Gamma$-near ring. The set

$$N_0 = \{ k \in N \mid k_\gamma 0_N = 0_N, \ \gamma \in \Gamma \}$$

is called a zero symmetric part of $N$ and the set

$$N_c = \{ k \in N \mid k_\gamma 0_N = k, \ \gamma \in \Gamma \}$$

is called a constant part of $N$.

If $N = N_0$, then $N$ is called a zero symmetric nearness $\Gamma$-near ring. If $N = N_c$, then $N$ is called a constant nearness $\Gamma$-near ring. The set of all zero symmetric nearness $\Gamma$-near rings is denoted by $N_0$ and the set of all constant nearness $\Gamma$-near rings is denoted by $N_c$.

If the condition $d_\gamma (k + \ell) = d_\gamma k + d_\gamma \ell$ holds in $N_\gamma (B)^* N$ for all $k, \ell \in N$ and all $\gamma \in \Gamma$ then $d$ is called a distributive element. Also, the set of all nearness $\Gamma$-near ring with the identity is represented as $N_1$ and the set of all distributive elements in $N$ is represented as $N_d$. If $N = N_d$, then $N$ is called a distributive nearness $\Gamma$-near ring.
Definition 3.9 Let $N$ be a nearness $\Gamma$-near ring and $(S, +)$ be a subnearness group of $(N, +)$. $S$ is called a nearness $\Gamma$-subnear ring of $N$ if $S \Gamma S \subseteq N_r(B)^* S$.

Example 3.10 Let $N$ be a nearness $\Gamma$-near ring. Then $N_0$ and $N_c$ are nearness $\Gamma$-subnear rings of $N$.

Theorem 3.11 Let $N, \Gamma \subseteq \mathcal{O}$, $N$ be a nearness $\Gamma$-near ring, $S \subseteq N$ and $N_r(B)^* S$ be an additive groupoid and $\Gamma$-groupoid. Then $S$ is a nearness $\Gamma$-subnear ring of $N$ iff $-s \in S$ for all $s \in S$.

Proof (⇒) Let $S$ be a nearness $\Gamma$-subnear ring of $N$. Then $(S, +)$ is a nearness group and hence $-s \in S$ for all $s \in S$.

(⇐) Let $-s \in S$ for all $s \in S$. Since $N_r(B)^* S$ is an additive groupoid, $(S, +)$ is a nearness group from Theorem 2.6. Therefore, since $N_r(B)^* S$ is a $\Gamma$-groupoid and $S \subseteq N$, $p\beta r, r\gamma s \in N_r(B)^* S$ and $(p\beta r) \gamma s = p\beta (r\gamma s)$ property holds in $N_r(B)^* S$ for all $p, r, s \in S$ and all $\beta, \gamma \in \Gamma$.

Furthermore, since $N_r(B)^* S$ is an additive groupoid, $\Gamma$-groupoid and $N$ is a nearness $\Gamma$-near ring, $(p + r)\beta s = (p\beta s) + (r\beta s)$ property holds in $N_r(B)^* S$ for all $p, r, s \in S$ and all $\beta \in \Gamma$.

Consequently, $S$ is a nearness $\Gamma$-subnear ring of $N$. $\square$

Definition 3.12 Let $N$ be a nearness $\Gamma$-near ring and $J$ be a subnearness group of $(N, +)$. Let $N_r(B)^* S$ be an additive groupoid and $\Gamma$-groupoid. Then $J$ is called a nearness $\Gamma$-ideal of $N$ if the following properties are satisfied:

1) $J \Gamma N = \{ x\gamma k : x \in J, \gamma \in \Gamma, k \in N \} \subseteq N_r(B)^* J$,

2) $k\gamma (1 + x) - k\gamma l \in N_r(B)^* J$ for all $k, l \in N$, all $x \in J$ and all $\gamma \in \Gamma$.

Furthermore, $J$ is called a right nearness $\Gamma$-ideal of $N$ if only the condition (1) is satisfied. Also, $J$ is called a left nearness $\Gamma$-ideal of $N$ if only the condition (2) is satisfied.

Example 3.13 From Example 3.2, let we consider nearness $\Gamma$-near ring $N = \{ k_{00}, k_{01}, k_{10} \}$ and $J = N$. Since $J$ is an additive nearness subgroup of $N$, $J \Gamma N = J$ by definition of the operation $\mathcal{O} \times \Gamma \times \mathcal{O} \rightarrow \mathcal{O}$ from Example 3.2 and $J \subseteq N_r(B)^* J$, $J$ is a right nearness $\Gamma$-ideal of $N$. Also, since $k\gamma (1 + x) - k\gamma l \in N_r(B)^* J$ for all $k, l \in N$, all $x \in J$ and all $\gamma \in \Gamma$, $J$ is a left nearness $\Gamma$-ideal of $N$. Hence $J$ is a nearness $\Gamma$-ideal of $N$.

Remark 3.14 Every nearness $\Gamma$-ideal of $N$ is also a nearness $\Gamma$-subnear ring of $N$. 54
Theorem 3.15 Let $N \subseteq \mathcal{O}$ be a nearness $\Gamma$-near ring, $I, J \subseteq N$ and $N_r(B)^* I$, $N_r(B)^* J$ be additive groupoids and $\Gamma$-groupoids. If $I$, $J$ are both right (left) nearness $\Gamma$-ideals of $N$ and $(N_r(B)^* I) \cap (N_r(B)^* J) = N_r(B)^*(I \cap J)$, then $I \cap J$ is also a right (left) nearness $\Gamma$-ideal of $N$.

**Proof** Since $I$ and $J$ are both right nearness $\Gamma$-ideals of $N$, $\Gamma I \subseteq N$ and $\Gamma J \subseteq N_r(B)^* (I \cap J)$.

\[
(I \cap J) \Gamma N = \{ x \gamma k \mid k \in N, \gamma \in \Gamma, x \in I \cap J \}
\]

\[
= \{ x \gamma k \mid k \in N, \gamma \in \Gamma, x \in I \text{ and } x \in J \}
\]

\[
= \{ x \gamma k \mid k \in N, \gamma \in \Gamma, x \in I \} \cap \{ x \gamma k \mid k \in N, \gamma \in \Gamma, x \in J \}
\]

\[
= \Gamma I N \cap J \Gamma N
\]

\[
\subseteq (N_r(B)^* I) \cap (N_r(B)^* J)
\]

\[
= N_r(B)^*(I \cap J).
\]

Therefore $(I \cap J) \Gamma N \subseteq N_r(B)^*(I \cap J)$, that is, $I \cap J$ is a right nearness $\Gamma$-ideal of $N$.

Let $x \in I \cap J$. Since $I$ and $J$ are both left nearness $\Gamma$-ideals of $N$, then $k \gamma (l + x) - k \gamma l \in N_r(B)^* I$ and $k \gamma (l + x) - k \gamma l \in N_r(B)^* J$ for all $k, l \in N$ and all $\gamma \in \Gamma$. Therefore $k \gamma (l + x) - k \gamma l \in (N_r(B)^* I) \cap (N_r(B)^* J)$ and so $k \gamma (l + x) - k \gamma l \in N_r(B)^*(I \cap J)$ from the hypothesis. Hence $I \cap J$ is a left nearness $\Gamma$-ideal of $N$.

\[\square\]

Corollary 3.16 Let $N \subseteq \mathcal{O}$ be a nearness $\Gamma$-near ring, $J_i \subseteq N \ (1 \leq i \leq n, \ n \geq 2)$ and $N_r(B)^* J_i$ be additive groupoids and $\Gamma$-groupoids. If $J_i$ are right (left) nearness $\Gamma$-ideals of $N$ and $\bigcap_{1 \leq i \leq n} N_r(B)^* J_i = N_r(B)^* (\bigcap_{1 \leq i \leq n} J_i)$, then $\bigcap_{1 \leq i \leq n} J_i$ is also a right (left) nearness $\Gamma$-ideal of $N$.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors’ Contributions

Author [Baki Çokakoğlu]: Collected the data, contributed to research method or evaluation of data (%40).

Author [Mustafa Uçkun]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%60).

Conflict of Interest

The authors declare no conflicts of interest.
References


