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Automatic Continuity of Almost Derivations on Frechet Q-Algebras

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Abstract
In 1971 R. L. Carpenter proved that every derivation on a semisimple commutative Frechet algebra with identity is continuous. The concept of almost derivations on Frechet algebras is introduced in this article. Also, R. L. Carpenter result motivates us to ask an open question: Is every almost derivation on semisimple commutative Frechet algebras continuous? Moreover, a partial answer to this open question is derived in the sense that every almost derivation T on semisimple commutative Frechet Q-algebras Λ, with an additional condition on Λ, is continuous. Furthermore, an example is provided to illustrate our main result.

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1. Introduction

We provide a brief outline of definitions and known outcomes in this section. For more details, one may refer to [2, 7]. All vector spaces are considered over the complex field, and we assume that all algebras are unital.

Definition 1.1. *A normed algebra Λ is an algebra with a norm $||.||$, which also satisfies $||p.q|| \leq ||p||.||q||$, $\forall\ p,q \in \Lambda$. A complete normed algebra is called a Banach algebra.*

Definition 1.2. *An algebra with a Hausdorff topology is called a topological algebra if all algebraic operations are jointly continuous.*

Definition 1.3. *[2] The Jacobson radical $rad(\Lambda)$ of an algebra Λ is the intersection of all maximal right(or left) ideals. An algebra is said to be semisimple if $rad(\Lambda) = \{0\}$.*

Definition 1.4. *[2] The spectrum $\sigma_\Lambda(p)$ of an element p of an algebra Λ is the set of all complex numbers λ such that $\lambda.1 - p$ is not invertible in Λ. The spectral radius $r_\Lambda(p)$ of an element $p \in \Lambda$ is defined by $r_\Lambda(p) = \sup\{|\lambda| : \lambda \in \sigma_\Lambda(p)\}$.*

If $(\Lambda, ||.||)$ is a Banach algebra, then $r_\Lambda(p) = \lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$. Also, for any algebra Λ, we have $rad(\Lambda) = \{p \in \Lambda : r_\Lambda(pq) = 0$, for every $q \in \Lambda\}$. See [2].

Definition 1.5. *[2] If $T : \Lambda \to \Gamma$ is a linear map from a Banach algebra Λ to a Banach algebra Γ, then the separating space of T is defined as the set*
$S(T) = \{q \in \Gamma :$ there exists $(p_n)_{n=1}^{\infty}$ in Λ such that $p_n \to 0$ and $Tp_n \to q\}$.

Also, $S(T)$ is a closed linear subspace of Γ and moreover, by the closed graph theorem, T is continuous if and only if $S(T)=\{0\}$. For a proof, see [2].

A complete metrizable topological algebra is called an F-algebra. A topological algebra Λ is said to be a LMC algebra if its topology is induced by a separating family of submultiplicative seminorms. A Frechet algebra is a LMC algebra which is also an F-algebra. A Q-algebra is a topological algebra in which the set of all invertible elements is open. A metrizable LMC algebra is written in the form $(\Lambda,(p_n)_{n=1}^{\infty})$, where $(p_n)_{n=1}^{\infty}$ is a separating sequence and each p_n is a submultiplicative seminorm (i.e. $p_n(x.y)\leq p_n(x).p_n(y), \forall x,y\in\Lambda$) satisfying $p_n(x)\leq p_{n+1}(x), \forall n, \forall\, x\in\Lambda$, in which the topology on Λ is induced by the seminorms $p_n, n=1,2,\ldots$. Also, a sequence (x_k) in the Frechet algebra $(\Lambda,(p_n))$ converges to $x\in\Lambda$ if and only if $p_n(x_k-x)\to 0$ for each $n\in N$, as $k\to\infty$. In a Frechet Q-algebra, spectral radius of every element is a finite number. Every Banach algebra is a Frechet Q-algebra.

Definition 1.6. *[6] Let Λ be an algebra. A linear map $T:\Lambda\to\Lambda$ is called derivation if $T(p.q)=p.T(q)+T(p).q, \forall p,q\in\Lambda$.*

Next, we introduce almost derivations on Frechet algebras.

Definition 1.7. *Let $(\Lambda,(p_n))$ be a Frechet algebra. A linear map $T:\Lambda\to\Lambda$ is called an almost derivation if there are $\varepsilon_n\geq 0$ such that $p_n(T(p.q)-p.T(q)-T(p).q)\leq\varepsilon_n p_n(p)\ p_n(q)$; $\forall n\in N, \forall p,q\in\Lambda$.*

Remark 1.8. *If $\varepsilon_n=0$, for every n, then almost derivations on Λ turn out to be derivation on Λ, because (p_n) is a separating sequence of seminorms on Λ. Also, every derivation is an almost derivation, for every $\varepsilon_n\geq 0$.*

A conjecture of Kaplansky [6] can be stated in the following question form. Is every derivation on semisimple Banach algebra continuous? Kaplansky conjecture was proved by Johnson and Sinclair [5] in 1968. In 1971, R. L. Carpenter [1] proved that every derivation on a semisimple commutative Frechet algebra with identity is continuous. There are some recent articles [8, 9, 10, 11, 12] for automatic continuity of derivations in the theory of topological algebras.

Now, we write an open question for almost derivations on Frechet algebras.

Problem 1.9. *Let $T:(\Lambda,(p_n))\to(\Lambda,(p_n))$ be an almost derivation on a semisimple commutative Frechet algebra $(\Lambda,(p_n))$. Is T continuous*?

Also, we derive a partial solution to this open Problem 1.9. More specifically, we prove that every almost derivation T on a semisimple commutative Frechet Q-algebra $(\Lambda,(p_n))$, with an additional condition on $(\Lambda,(p_n))$, is continuous.

2. Main Result

Definition 2.1. *[4] If $T:\Lambda\to\Gamma$ is a linear map from a Frechet algebra Λ to a Frechet algebra Γ, then the separating space of T is defined by*
$S(T)=\{q\in\Gamma$: there exists $(q_n)_{n=1}^{\infty}$ in Λ such that $q_n\to 0$ and $Tq_n\to q\}$.

Theorem 2.2. *Let $(\Lambda,(p_n))$ be a Frechet algebra. If $T:\Lambda\to\Lambda$ is an almost derivation, then the separating space $S(T)$ is a closed two sided ideal in $(\Lambda,(p_n))$.*

Proof. Obviously $S(T)$ is a closed linear subspace of $(\Lambda,(p_n))$.

Now, we prove that $S(T)$ is an ideal in Λ. Let $b\in S(T)$ and $c\in\Lambda$. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ in Λ such that $a_n\to 0$, and $T(a_n)\to b$. Let $w=T(c)$. Also we have $p_k(c.a_n)\leq p_k(c).p_k(a_n)\to 0, \forall k$, as $n\to\infty$. Since T is almost derivation, we have

$$\begin{aligned}p_k(T(c.a_n)-c.b) &\leq p_k(T(c.a_n)-c.T(a_n)-T(c).a_n)+p_k(c.T(a_n)+w.a_n-c.b)\\ &\leq p_k(T(c.a_n)-c.T(a_n)-T(c).a_n)+p_k(c.T(a_n)-c.b)+p_n(w.a_n)\\ &\leq \varepsilon_k p_k(c)\ p_k(a_n)+p_k(c)\ p_k(T(a_n)-b)+p_k(w.a_n).\end{aligned}$$

Since $p_k(T(a_n)-b)\to 0$, $p_k(a_n)\to 0$ and $p_k(w.a_n)\leq p_k(w).p_k(a_n)\to 0, \forall k$, as $n\to\infty$, we have $p_k(T(c.a_n)-c.b)\to 0$, and hence $T(c.a_n)\to c.b$, when $c.a_n\to 0$. Therefore, we conclude that $c.b\in S(T)$. Similarly $b.c\in S(T)$. Hence $S(T)$ is a two sided ideal in Λ. □

Theorem 2.3. *Let $(\Lambda,(p_n))$ be a Frechet Q-algebra such that Λ is semisimple, and r_Λ is continuous on Λ. If $T:\Lambda\to\Lambda$ is an almost derivation with $r_\Lambda(Ta)\leq r_\Lambda(a), \forall a\in\Lambda$, then T is continuous.*

Proof. Let $b\in S(T)$. Then there exists $(a_n)_{n=1}^{\infty}$ in Λ such that $a_n\to 0$ and $Ta_n\to b$. Since $r_\Lambda(Ta)\leq r_\Lambda(a)$ and $r_\Lambda(a_n)\to 0$, we have $r_\Lambda(Ta_n)\to 0$. Also, we have $r_\Lambda(Ta_n)\to r_\Lambda(b)$. So, we conclude that $r_\Lambda(b)=0$. By Theorem 2.2, $S(T)$ is an ideal in Λ. For every $c\in\Lambda, b.c\in S(T)$. Therefore $r_\Lambda(b.c)=0$. Also, $rad(\Lambda)=\{a_1\in\Lambda : r_\Lambda(a_1.a_2)=0, \forall a_2\in\Lambda\}$, and hence $b\in rad(\Lambda)$. So, $S(T)\subseteq rad(\Lambda)$. Since Λ is semisimple, $S(T)=\{0\}$. Therefore T is continuous, by the closed graph theorem. □

Corollary 2.4. *Let $(\Lambda,(p_n))$ be a commutative Frechet Q-algebra such that Λ is semisimple. If $T:\Lambda\to\Lambda$ is an almost derivation with $r_\Lambda(Ta)\le r_\Lambda(a), \forall a\in\Lambda$, then T is continuous.*

Proof. If Λ is a commutative Frechet Q-algebra, then the spectral radius function r_Λ is uniformly continuous. See, for example ([3], Theorem 6.18). □

This Corollary 2.4 is a partial solution to the Problem 1.9.

Corollary 2.5. *Let Λ be a commutative semisimple Banach algebra. If $T:\Lambda\to\Lambda$ is an almost derivation with $r_\Lambda(Ta)\le r_\Lambda(a), \forall a\in\Lambda$, then T is continuous.*

Proof. If Λ is a commutative Banach algebra, then spectral radius function r_Λ is continuous on Λ. □

Example 2.6. *Let $(\Lambda,(p_n))$ be a semisimple commutative Frechet Q-algebra. A linear map $T:\Lambda\to\Lambda$ is defined by $T(a)=\beta a$, $\forall a\in\Lambda$ where $(\varepsilon_n=)\beta\in(0,\infty)$. Since*

$$p_n(T(p.q)-p.T(q)-T(p).q)=p_n(\beta p.q-p.\beta q-\beta p.q)=p_n(-\beta p.q)\le|-\beta|p_n(p).p_n(q),$$

T is an almost derivation but not a derivation on $(\Lambda,(p_n))$. Since Λ is a Q-algebra, there exists $k\in N$ such that $r_\Lambda(a)=\lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}}$, $\forall a\in\Lambda$. See, for example ([3], Theorem 6.18). So

$$r_\Lambda(Ta)=r_\Lambda(\beta a)=\lim_{n\to\infty}(p_k((\beta a)^n))^{\frac{1}{n}}=|\beta|\lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}}\le r_\Lambda(a).$$

All hypotheses of Corollary 2.4 are satisfied, so T is continuous.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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