Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (5) (2023), 1206 – 1218
DOI : 10.15672/hujms.1032768

Research Article

Semi-slant submanifolds in a locally conformal Kaehler space form

V. Bonanzinga*[1], K. Matsumoto[2]

[1] *D.I.I.E.S. University of Reggio Calabria, 89100 via Graziella, Reggio Calabria, Italy*
[2] *2-3-65 Nishi-Odori, Yonezawa, Yamagata, 992-0059, Japan*

Abstract

In this paper, we consider semi-slant submanifolds in a locally conformal Kaehler manifold and a locally conformal Kaehler space form. We give inequalities for the length of the second fundamental form and the mean curvature and, using Gauss and Codazzi equations, we get many useful results in locally conformal Kaehler space form.

Mathematics Subject Classification (2020). 53C40, 53B25, 53C15

Keywords. locally conformal Kaehler manifold, locally conformal Kaehler space form, semi-slant submanifold, mean curvature

1. Introduction

In 1990, B.Y. Chen introduced slant submanifolds of an almost Hermitian manifold [10, 11]. Since then, many geometers have studied slant submanifolds in various ambient spaces [1, 5, 18]. Some generalizations of both slant and CR-submanifolds have also been defined in different ambient spaces, such as semi-slant [6, 17, 21, 22] and hemi-slant [19, 20, 22]. In 1994, N. Papaghiuc introduced the notion of a semi-slant submanifold in a Hermitian manifold [17] which is a generalization of CR and slant submanifold [2, 3, 10, 11, 15, 16]. In particular, he considered this submanifold in a Kaehlerian manifold [17]. Then, in 2007, V. A. Khan and M. A. Khan considered this kind of submanifold in a nearly Kaehler manifold and obtained interesting results [13]. In this paper, we consider semi-slant submanifolds in a locally conformal Kaehler manifold and a locally conformal Kaehler space form.

A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is called a *locally conformal Kaehler* (*an l.c.K.*) *manifold* if each point $x \in \tilde{M}$ has an open neighbourhood U with differentiable function $\rho : U \to \mathcal{R}$ such that $\tilde{g}^* = e^{-2\rho}\tilde{g}_{|U}$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is the almost complex structure, $\tilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\tilde{g}^*$ and $\mathcal{R}$ is a real number space [12, 23]. We recall the following result [14]:

*Corresponding Author.
Email addresses: vittoria.bonanzinga@unirc.it (V. Bonanzinga), tokiko_matsumoto@yahoo.com (K. Matsumoto)
Received: 07.12.2021; Accepted: 12.01.2023

Proposition 1.1. *A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is an l.c.K.-manifold if and only if there exists a global 1-form α which is called Lee form satisfying*

$$d\alpha = 0 \quad (\alpha : \textit{closed}), \tag{1.1}$$

and

$$(\tilde{\nabla}_V J)U = -\tilde{g}(\alpha^\sharp, U)JV + \tilde{g}(V, U)\beta^\sharp + \tilde{g}(JV, U)\alpha^\sharp - \tilde{g}(\beta^\sharp, U)V \tag{1.2}$$

for any $V, U \in \tilde{M}$, where $\tilde{\nabla}$ denotes the covariant differentiation with respect to $\tilde{g}$, $\alpha^\sharp$ is the dual vector field of α, the 1 form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β and $T\tilde{M}$ means the tangent bundle of $\tilde{M}$.

An l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature. Then, the Riemannian curvature tensor $\tilde{R}$ with respect to $\tilde{g}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by [14]

$$\begin{aligned} 4\tilde{R}(X, Y, Z, W) = c\{&\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W) + \tilde{g}(JX, W)\tilde{g}(JY, Z) \\ &-\tilde{g}(JX, Z)\tilde{g}(JY, W) - 2\tilde{g}(JX, Y)\tilde{g}(JZ, W)\} + 3\{P(X, W)\tilde{g}(Y, Z) \\ &-P(X, Z)\tilde{g}(Y, W) + \tilde{g}(X, W)P(Y, Z) - \tilde{g}(X, Z)P(Y, W)\} \\ &-\tilde{P}(X, W)\tilde{g}(JY, Z) + \tilde{P}(X, Z)\tilde{g}(JY, W) - \tilde{g}(JX, W)\tilde{P}(Y, Z) \\ &+\tilde{g}(JX, Z)\tilde{P}(Y, W) + 2\{\tilde{P}(X, Y)\tilde{g}(JZ, W) + \tilde{g}(JX, Y)\tilde{P}(Z, W)\} \end{aligned} \tag{1.3}$$

for any $X, Y, Z, W \in T\tilde{M}$, where P and $\tilde{P}$ are respectively defined by

$$P(X, Y) = -(\tilde{\nabla}_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2} \parallel \alpha \parallel^2 \tilde{g}(X, Y), \tag{1.4}$$

and

$$\tilde{P}(X, Y) = P(JX, Y) \tag{1.5}$$

for any $X, Y \in T\tilde{M}$, where $\parallel \alpha \parallel$ is the length of the Lee form α.

From the above equation, we can easily have

$$P(JX, JY) = P(X, Y) \tag{1.6}$$

for any $X, Y \in T\tilde{M}$.

2. Semi-slant-submanifolds in an almost Hermitian manifold

In general, between a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its Riemannian submanifold (M, g), we know the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{2.2}$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$, where ∇ is the covariant differentiation with respect to g, σ is the second fundamental form and A_ξ is the shape operator or the fundamental tensor of Weingarten with respect to ξ [7–9]. The second fundamental form σ and the shape operator A are related by

$$\tilde{g}(A_\xi Y, X) = \tilde{g}(\sigma(Y, X), \xi)$$

for any $Y, X \in TM$ and $\xi \in T^\perp M$.

A submanifold M is called totally geodesic if the second fundamental form vanishes on M.

The Gauss equation is given by

$$\begin{aligned} R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) &+ \tilde{g}(\sigma(X, W), \sigma(Y, Z)) \\ &-\tilde{g}(\sigma(X, Z), \sigma(Y, W)), \end{aligned} \tag{2.3}$$

for any $X, Y, Z, W \in TM$, where R is the curvature tensor with respect to g.

The Codazzi equation is given by

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X\sigma)(Y,Z) - (\bar{\nabla}_Y\sigma)(X,Z) \tag{2.4}$$

for any $X, Y, Z \in TM$, where $\bar{\nabla}\sigma$ is defined by

$$(\bar{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{2.5}$$

for any $X, Y, Z \in TM$ and $(\tilde{R}(X,Y)Z)^{\perp}$ means the normal part of $\tilde{R}(X,Y)Z$.

The second fundamental form is called *parallel* if it satisfies $\bar{\nabla}\sigma = 0$, identically.

Let M be an n-dimensional submanifold in an m-dimensional almost Hermitian manifold $\tilde{M}$.

For a vector field $X \in TM$, the angle between JX and TM is called the *Wirtinger angle* of X [10].

A differentiable distribution $\mathcal{D}^{\theta} : x \to \mathcal{D}^{\theta}{}_x$ on M is said to be a slant distribution if for each $U_x \in \mathcal{D}^{\theta}{}_x$, the Wirtingar angle θ of U_x is constant for any $x \in M$. In this case, the Wirtingar angle is said to be the *slant angle*. In particular, if TM is slant, then the submanifold is called a *slant* one. A slant submanifold is holomorphic (resp. totally real) if its slant angle $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$). A slant submanifold is said to be *proper* if it is not holomorphic nor totally real.

A submanifold M in $\tilde{M}$ is called a *semi-slant submanifold* if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ on M satisfying the following conditions:

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $\mathcal{D}^{\theta} : x \to \mathcal{D}^{\theta}{}_x \subset T_xM$ is slant with slant angle θ, where T_xM means the tangent vector space of M at x [13, 17].

Remark 2.1. [2–4, 15, 16] A semi-slant submanifold is a CR-submanifold if the slant angle is equal to $\frac{\pi}{2}$.

A semi-slant submanifold M is said to be *proper* if it is neither CR, nor holomorphic, nor totally real.

In a submanifold M of an almost Hermitian manifold $(\tilde{M}, J, \tilde{g})$, for any $U \in TM$ and $\xi \in T^{\perp}M$, we write

$$JU = TU + FU, \tag{2.6}$$

and

$$J\xi = t\xi + f\xi, \tag{2.7}$$

where TU (resp. FU) means the tangential (resp. normal) component of JU and $t\xi$ (resp. $f\xi$) means the tangential (resp. normal) component of $J\xi$.

We can easily obtain the following formulas:

$$\begin{cases} T^2 + tF = -I, & f^2 + Ft = -I, \\ FT + fF = 0, & tf + Tt = 0. \end{cases} \tag{2.8}$$

For a semi-slant submanifold M of an almost Hermitian manifold $\tilde{M}$, the tangent bundle TM and the normal bundle $T^{\perp}M$ of M are decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^{\theta}, \tag{2.9}$$

and

$$T^{\perp}M = F\mathcal{D}^{\theta} \oplus \nu, \tag{2.10}$$

where ν denotes the orthogonal complementary distribution of $F\mathcal{D}^{\theta}$ in $T^{\perp}M$.

Now, we define a new orthonormal frame in $\tilde{M}$. Let M be a semi-slant submanifold with distributions $\mathcal{D}$ and $\mathcal{D}^{\theta}$ in an almost Hermitian manifold $\tilde{M}$ and $\dim \mathcal{D} = 2p$, $\dim \mathcal{D}^{\theta} = q$ and $\dim \nu = 2s$. Then we take the following local frame in $\tilde{M}$;

(1) $\{e_1, e_2, ..., e_p, e_1^*, e_2^*, ..., e_p^*\}$ is an orthonormal frame of $\mathcal{D}$, where we put $e_i^* = Je_i$ for $i \in \{1, 2, ..., p\}$;

(2) $\{e_{2p+1}, e_{2p+2}, ..., e_{2p+q}\}$ is an orthonormal frame of $\mathcal{D}^\theta$ such that Fe_{2p+1}, $Fe_{2p+2}, ..., Fe_{2p+q}$ are orthogonal to TM and belong to $F\mathcal{D}^\theta$;

(3) $\{e_{n+q+1}, e_{n+q+2}, ..., e_{n+q+s}, e^*_{n+q+1}, e^*_{n+q+2}, ..., e^*_{n+q+s}\}$ is an orthonormal frame of ν, where $e^*_{n+q+a} = Je_{n+q+a}$ for $a \in \{1, 2, ..., s\}$.

We put $e^*_{2p+a} = \dfrac{Fe_{2p+a}}{\|Fe_{2p+a}\|}$ for any $a \in \{1, 2, ..., q\}$ and then

- (i) $\{e_1, e_2, ..., e_p, e^*_1, e^*_2, ..., e^*_p, e_{2p+1}, e_{2p+2}, ..., e_{2p+q}\}$ is an orthonormal frame of TM and
- (ii) $\{e^*_{2p+1}, e^*_{2p+2}, ..., e^*_{2p+q}, e_{n+q+1}, e_{n+q+2}, ..., e_{n+q+s}, e^*_{n+q+1}, e^*_{n+q+2}, ..., e^*_{n+q+s}\}$ is an orthonormal frame of $T^\perp M$. We call the above orthonormal frame of $T\tilde{M}$ a *generalized adopted frame* of $\tilde{M}$ and we denote it by

$$\{e_1, e_2, ..., e_{2p}, e_{2p+1}, ..., e_n, ..., e_{n+q}, e_{n+q+1}, ..., e_{n+q+s}, ..., e_m\}.$$

Remark 2.2. In particular, if our submanifold is a CR-one, the generalized adapted frame is adapted. Since, in our case, $e^*_{2p+a} \neq Je_{2p+a}$ for each $a \in \{1, 2, ..., q\}$. So, we have $\tilde{g}(e^*_{2p+b}, e^*_{2p+b}) \neq g(e_{2p+b}, e_{2p+a})$ for any $a, b \in \{1, 2, ..., q\}$.

3. Semi-slant submanifolds in an l.c.K.-space form

In this section, we consider the Riemannian curvature tensor of an l.c.K.-space form using the generalized adopted frame which was defined in the last section.

With respect to the generalized adopted frame, we write

$$\tilde{R}_{\omega\nu\mu\lambda} = \tilde{R}(e_\omega, e_\nu, e_\mu, e_\lambda) \tag{3.1}$$

for any $\omega, \nu, \mu, \lambda \in \{1, 2, ..., m\}$.

Let $\tilde{M}(c)$ be an l.c.K.-space form with the constant holomorphic sectional curvature c. By virtue of (1.3) and (3.1), the Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ will be written as

$$\begin{aligned} 4R_{\omega\nu\mu\lambda} &= c(\tilde{g}_{\omega\lambda}\tilde{g}_{\nu\mu} - \tilde{g}_{\omega\mu}\tilde{g}_{\nu\lambda} + \tilde{g}_{\omega^*\lambda}\tilde{g}_{\nu^*\mu} - \tilde{g}_{\omega^*\mu}\tilde{g}_{\nu^*\lambda} - 2\tilde{g}_{\omega^*\nu}\tilde{g}_{\mu^*\lambda}) \\ &+3(P_{\omega\lambda}\tilde{g}_{\nu\mu} - P_{\omega\mu}\tilde{g}_{\nu\lambda} + P_{\nu\mu}\tilde{g}_{\omega\lambda} - P_{\nu\lambda}\tilde{g}_{\omega\mu}) - \tilde{P}_{\omega\lambda}\tilde{g}_{\nu^*\mu} + \tilde{P}_{\omega\mu}\tilde{g}_{\nu^*\lambda}, \\ &-\tilde{P}_{\nu\mu}\tilde{g}_{\omega^*\lambda} + \tilde{P}_{\nu\lambda}\tilde{g}_{\omega^*\mu} + 2(\tilde{P}_{\omega\nu}\tilde{g}_{\mu^*\lambda} + \tilde{P}_{\mu\lambda}\tilde{g}_{\omega^*\nu}). \end{aligned} \tag{3.2}$$

From now on, we use the generalized adapted frame.

The Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ with respect to the adapted frame is decomposed in (3.4) and (3.5) using the Gauss equation and the Codazzi equation, respectively.

In a semi-slant submanifold, we put

$$Je_a = S_a{}^c e_c + (FS)_a{}^c e^*_c, \tag{3.3}$$

where $S_a{}^c e_c$ (resp. $(FS)_a{}^c e^*_c$) is the $\mathcal{D}^\theta \subset TM$ (resp. $F\mathcal{D}^\theta \subset T^\perp M$) part of Je_a. In particular, if our submanifold is CR, then $(S_b{}^a) = 0_q$ and $((FS)_b{}^a) = I_q$, where 0_q (resp. I_q) denotes the q-th order 0 (resp. unit) matrix. Then we have from (1.3)

$$\begin{aligned} 4\tilde{R}_{kjih} &= 4\tilde{R}_{k^*j^*i^*h^*} = c(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) + 3(P_{kh}\delta_{ji} - P_{ki}\delta_{jh} \\ &\quad + P_{ji}\delta_{kh} - P_{jh}\delta_{ki}), \\ 4\tilde{R}_{kjih^*} &= 3(P_{kh^*}\delta_{ji} - P_{jh^*}\delta_{ki}) - P_{ki^*}\delta_{jh} + P_{ji^*}\delta_{kh} - 2P_{kj^*}\delta_{ih}, \\ 4\tilde{R}_{kji^*h^*} &= c(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) + P_{kh}\delta_{ji} - P_{ki}\delta_{jh} + P_{ji}\delta_{kh} - P_{jh}\delta_{ki}, \\ 4\tilde{R}_{kj^*ih^*} &= -c(\delta_{ki}\delta_{jh} + \delta_{kh}\delta_{ji} + 2\delta_{kj}\delta_{ih}) - 3(P_{ki}\delta_{jh} + P_{jh}\delta_{ki}) + P_{kh}\delta_{ji} \\ &\quad + P_{ji}\delta_{kh} + 2(P_{kj}\delta_{ih} + P_{ih}\delta_{kj}), \\ 4\tilde{R}_{kj^*i^*h^*} &= 3(P_{kh^*}\delta_{ji} - P_{ki^*}\delta_{jh}) + P_{ji^*}\delta_{kh} - P_{jh^*}\delta_{ki} - 2P_{ih^*}\delta_{kj}, \\ 4\tilde{R}_{kjia} &= 3(P_{ka}\delta_{ji} - P_{ja}\delta_{ki}), \end{aligned}$$

$$
\begin{aligned}
&4\tilde{R}_{kji^*a} = -P_{k^*a}\delta_{ji} + P_{j^*a}\delta_{ki},\\
&4\tilde{R}_{kj^*ia} = -3P_{j^*a}\delta_{ki} + P_{k^*a}\delta_{ji} + 2P_{i^*a}\delta_{kj}, \qquad (3.4)\\
&4\tilde{R}_{k^*j^*ia} = P_{ja}\delta_{ki} - P_{ka}\delta_{ji},\\
&4\tilde{R}_{kj^*i^*a} = 3P_{ka}\delta_{ji} - P_{ja}\delta_{ki} - 2P_{ia}\delta_{kj},\\
&4\tilde{R}_{k^*j^*i^*a} = 3(P_{k^*a}\delta_{ji} - P_{j^*a}\delta_{ki}),\\
&4\tilde{R}_{kcia} = 4\tilde{R}_{k^*ci^*a} = -c\delta_{ki}\delta_{ca} - 3(P_{ki}\delta_{ca} + P_{ca}\delta_{ki}) - P_{ki^*}S_{ca},\\
&4\tilde{R}_{kci^*a} = -c\delta_{ki}S_{ca} - 3P_{ki^*}\delta_{ca} + P_{ki}S_{ca} + \tilde{P}_{ca}\delta_{ki},\\
&2\tilde{R}_{kjba} = \tilde{R}_{k^*j^*ba} = -P_{kj^*}S_{ba},\\
&2\tilde{R}_{kj^*ba} = \tilde{P}_{ba}\delta_{kj} + (P_{kj} - \delta_{kj})S_{ba},\\
&4\tilde{R}_{kcba} = 3(P_{ka}\delta_{cb} - P_{kb}\delta_{ca}) - P_{k^*a}S_{ba} + P_{k^*b}S_{cb} + 2P_{k^*c}S_{ba},\\
&4\tilde{R}_{k^*cba} = 3(P_{k^*a}\delta_{cb} - P_{k^*b}\delta_{ca}) + P_{ka}S_{cb} - P_{kb}S_{ca} - 2P_{kc}S_{ba},\\
&4\tilde{R}_{dcba} = c(\delta_{da}\delta_{cb} - \delta_{db}\delta_{cb}) + 3(P_{da}\delta_{cb} + P_{cb}\delta_{da} - P_{db}\delta_{ca} - P_{ca}\delta_{db})\\
&\qquad + c(S_{da}S_{cb} - S_{db}S_{ca} + 2S_{dc}S_{ba}) - \tilde{P}_{da}S_{cb} - \tilde{P}_{cb}S_{da}\\
&\qquad + \tilde{P}_{db}S_{ca} + \tilde{P}_{ca}S_{db} + 2(\tilde{P}_{dc}S_{ba} + \tilde{P}_{ba}S_{dc}),
\end{aligned}
$$

for any $k, j, i, h \in \{1, 2, ..., p\}$ and $b, a \in \{2p+1, 2p+2, ..., 2p+q\}$, where we put $S_{ba} = \tilde{g}(S_b{}^c e_c, e_a)$, which is skew-symmetric and $\tilde{P}_{ba} = \tilde{P}(e_b, e_a) = S_b{}^c P_{ca} + (FS)_b{}^c P_{c^*a}$.

Using (1.3) and the Codazzi equation, we obtain the following equations:

$$
\begin{aligned}
&4\tilde{R}_{kjia^*} = 3(P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki}),\\
&4\tilde{R}_{kji^*a^*} = P_{j^*a^*}\delta_{ki} - P_{k^*a^*}\delta_{ji},\\
&4\tilde{R}_{kj^*ia^*} = -3P_{j^*a^*}\delta_{ki} + P_{k^*a^*}\delta_{ji} + 2P_{i^*a^*}\delta_{kj},\\
&4\tilde{R}_{kj^*i^*a^*} = 3P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki} - 2P_{ia^*}\delta_{kj},\\
&4\tilde{R}_{k^*j^*i^*a^*} = 3(P_{k^*a^*}\delta_{ji} - P_{j^*a^*}\delta_{ki}),\\
&2\tilde{R}_{kjba^*} = 2\tilde{R}_{k^*j^*ba^*} = P_{k^*j}(FS)_{ba},\\
&2\tilde{R}_{kj^*ba^*} = (P_{kj} - c\delta_{kj})(FS)_{ba} + \tilde{P}_{ba^*}\delta_{kj},\\
&4\tilde{R}_{kcba^*} = 3P_{ka^*}\delta_{cb} - P_{k^*a^*}S_{cb} + P_{k^*b}(FS)_{ca} + 2P_{k^*c}(FS)_{ba},\\
&4\tilde{R}_{k^*cba^*} = 3P_{k^*a^*}\delta_{cb} + P_{ka^*}S_{cb} - P_{kb}(FS)_{ca} - 2P_{kc}(FS)_{ba},\\
&4\tilde{R}_{dcba^*} = c\{(FS)_{da}S_{cb} - S_{db}(FS)_{ca} - 2S_{dc}(FS)_{ba}\}\\
&\qquad + 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) - \tilde{P}_{da^*}S_{cb} - \tilde{P}_{cb}(FS)_{da}\\
&\qquad + \tilde{P}_{db}(FS)_{ca} + \tilde{P}_{ca^*}S_{db} + 2\{\tilde{P}_{dc}(FS)_{ba} + \tilde{P}_{ba^*}S_{dc}\},\\
&4\tilde{R}_{kji\alpha} = 3(P_{k\alpha}\delta_{ji} - P_{j\alpha}\delta_{ki}),\\
&4\tilde{R}_{kji^*\alpha} = P_{j^*\alpha}\delta_{ki} - P_{k^*\alpha}\delta_{ji},\\
&4\tilde{R}_{kj^*i\alpha} = -3P_{j^*\alpha}\delta_{ki} + 2P_{i^*\alpha}\delta_{kj},\\
&4\tilde{R}_{kj^*i^*\alpha} = 3P_{k\alpha}\delta_{ji} - P_{j\alpha}\delta_{ki} - 2P_{i\alpha}\delta_{kj}, \qquad (3.5)\\
&4\tilde{R}_{k^*j^*i^*\alpha} = 3(P_{k^*\alpha}\delta_{ji} - P_{j^*\alpha}\delta_{ki}),\\
&\tilde{R}_{kja\alpha} = \tilde{R}_{k^*j^*a\alpha} = 0,\\
&2\tilde{R}_{kj^*a\alpha} = -P_{a\alpha^*}\delta_{kj},\\
&4\tilde{R}_{kba\alpha} = 3P_{k\alpha}\delta_{ba} - P_{k^*\alpha}S_{ba},
\end{aligned}
$$

$$
\begin{aligned}
&4R_{k^*ba\alpha} = 3P_{k^*\alpha}\delta_{ba} + P_{k\alpha}S_{ba},\\
&4\tilde{R}_{cba\alpha} = 3(P_{c\alpha}\delta_{ba} - P_{b\alpha}\delta_{ca}) + P_{c\alpha^*}S_{ca} - P_{b\alpha^*}S_{ca} - 2P_{a\alpha^*}S_{cb},\\
&4\tilde{R}_{kji\alpha^*} = 3(P_{k\alpha^*}\delta_{ji} - P_{j\alpha^*}\delta_{ki}),\\
&4\tilde{R}_{kji^*\alpha^*} = P_{j\alpha}\delta_{ki} - P_{k\alpha}\delta_{ji},\\
&4\tilde{R}_{kj^*i\alpha^*} = -3P_{j\alpha}\delta_{ki} - P_{k\alpha}\delta_{ji} + 2P_{i\alpha}\delta_{kj},\\
&4\tilde{R}_{kj^*i^*\alpha^*} = 3P_{k\alpha^*}\delta_{ji} - P_{j\alpha^*}\delta_{ki} - 2P_{i\alpha^*}\delta_{kj},\\
&4\tilde{R}_{k^*j^*i^*\alpha^*} = 3(P_{k\alpha}\delta_{ji} - P_{j\alpha}\delta_{ki}),\\
&\tilde{R}_{kja\alpha^*} = \tilde{R}_{k^*j^*a\alpha^*} = 0,\\
&2\tilde{R}_{kj^*a\alpha^*} = P_{a\alpha}\delta_{kj},\\
&4\tilde{R}_{kba\alpha^*} = 3P_{k\alpha^*}\delta_{ba} - P_{k\alpha^*}S_{ba},\\
&4\tilde{R}_{k^*ba\alpha^*} = 3P_{k\alpha}\delta_{ba} + P_{k\alpha^*}S_{ba},\\
&4\tilde{R}_{cba\alpha^*} = 3(P_{c\alpha^*}\delta_{ba} - P_{b\alpha^*}\delta_{ca}) - P_{c\alpha}S_{ba} + P_{b\alpha}S_{ca} - 2P_{a\alpha}S_{cb},
\end{aligned}
$$

for any $k, j \in \{1, 2, ..., p\}$, $b, a \in \{1, 2, ..., q\}$ and $\alpha \in \{1, 2, ..., s\}$, where we put $(FS)_{ba} = \tilde{g}((FS)_b{}^c e_c^*, e_a^*)$.

4. The Gauss equations

Using (1.3) and (3.4), the Gauss equations in a semi-slant submanifold of $\tilde{M}(c)$ are given by

$$
\begin{aligned}
4R_{kjih} &= c(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) + 3(P_{kh}\delta_{ji} + P_{ji}\delta_{kh} - P_{ki}\delta_{jh} - P_{jh}\delta_{ki})\\
&\quad + 4\{\tilde{g}(\sigma_{kh}, \sigma_{ji}) - \tilde{g}(\sigma_{ki}, \sigma_{jh})\},\\
4R_{kjih^*} &= 3(P_{kh^*}\delta_{ji} - P_{jh^*}\delta_{ki}) + P_{ji^*}\delta_{kh} - P_{ki^*}\delta_{jh} - 2P_{kj^*}\delta_{ih}\\
&\quad + 4\{\tilde{g}(\sigma_{kh^*}, \sigma_{ji}) - \tilde{g}(\sigma_{ki}\sigma_{jh^*}\},\\
4R_{kji^*h^*} &= c(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) + P_{kh}\delta_{ji} + P_{ji}\delta_{kh} - P_{ki}\delta_{jh} - P_{jh}\delta_{ki}\\
&\quad + 4\{\tilde{g}(\sigma_{kh^*}, \sigma_{ji^*}) - \tilde{g}(\sigma_{ki^*}, \sigma_{jh^*})\},\\
4R_{kj^*ih^*} &= -c(\delta_{kh}\delta_{ji} + \delta_{ki}\delta_{jh} + 2\delta_{kj}\delta_{ih}) - 3(P_{ki}\delta_{jh} + P_{jh}\delta_{ki})\\
&\quad + P_{kh}\delta_{ji} + P_{ji}\delta_{kh} + 2(P_{kj}\delta_{ih} + P_{ih}\delta_{kj})\\
&\quad + 4\{\tilde{g}(\sigma_{kh^*}, \sigma_{j^*i}) + \tilde{g}(\sigma_{ki}, \sigma_{j^*i^*})\},\\
4R_{kj^*i^*h^*} &= 3(P_{kh^*}\delta_{ji} - P_{ki^*}\delta_{jh}) + P_{ji^*}\delta_{kh} - P_{jh^*}\delta_{ki} - 2P_{ih^*}\delta_{kj}\\
&\quad + 4\{\tilde{g}(\sigma_{kh^*}, \sigma_{j^*i^*} - \tilde{g}(\sigma_{ki^*}, \sigma_{j^*h^*})\},\\
4R_{k^*j^*i^*h^*} &= c(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) + 3(P_{kh}\delta_{ji} + P_{ji}\delta_{kh} - P_{ki}\delta_{jh}\\
&\quad - P_{jh}\delta_{ki}) + 4\{\tilde{g}(\sigma_{k^*h^*}, \sigma_{j^*i^*}) - \tilde{g}(\sigma_{k^*i^*}, \sigma_{j^*h^*})\},\\
4R_{kjia} &= 3(P_{ka}\delta_{ji} - P_{ja}\delta_{ki}) + 4\{\tilde{g}(\sigma_{ka}, \sigma_{ji}) - \tilde{g}(\sigma_{ki}, \sigma_{ja})\},\\
4R_{kji^*a} &= P_{j^*a}\delta_{ki} - P_{k^*a}\delta_{ji} + 4\{\tilde{g}(\sigma_{ka}, \sigma_{ji^*}) - \tilde{g}(\sigma_{ki^*}, \sigma_{ja})\},\\
4R_{kj^*ia} &= -3P_{j^*a}\delta_{ki} + P_{k^*a}\delta_{ji} + 2P_{i^*a}\delta_{kj}\\
&\quad + 4\{\tilde{g}(\sigma_{ka}, \sigma_{j^*i}) - \tilde{g}(\sigma_{ki}, \sigma_{j^*a})\},\\
4R_{kj^*i^*a} &= 3P_{ka}\delta_{ji} - P_{ja}\delta_{ki} - 2P_{ia}\delta_{kj} \qquad (4.1)\\
&\quad + 4\{\tilde{g}(\sigma_{ka}, \sigma_{j^*i^*}) - \tilde{g}(\sigma_{ki^*}, \sigma_{j^*a})\},\\
4R_{k^*j^*i^*a} &= 3(P_{k^*a}\delta_{ji} - P_{j^*a}\delta_{ki}) + 4\{\tilde{g}(\sigma_{k^*a}, \sigma_{j^*i^*}) - \tilde{g}(\sigma_{k^*i^*}, \sigma_{j^*a})\},\\
2R_{kjba} &= -P_{kj^*}T_{ba} + 2\{\tilde{g}(\sigma_{ka}, \sigma_{jb}) - \tilde{g}(\sigma_{kb}, \sigma_{ja})\},\\
2R_{kj^*ba} &= \tilde{P}_{ba}\delta_{kj} + (P_{kj} - \delta_{kj})T_{ba} + 2\{\tilde{g}(\sigma_{ka}, \sigma_{j^*i}) - \tilde{g}(\sigma_{ki}, \sigma_{j^*a})\},
\end{aligned}
$$

$$
\begin{aligned}
2R_{k^*j^*ba} &= -P_{kj^*}T_{ba} + 2\{\tilde{g}(\sigma_{k^*a}, \sigma_{j^*b}) - \tilde{g}(\sigma_{k^*b}, \sigma_{j^*a})\},\\
4R_{kcia} &= -c\delta_{ki}\delta_{ca} - 3(P_{ki}\delta_{ca} + P_{ca}\delta_{ki}) - P_{ki^*}S_{ca}\\
&\quad + 4\{g(\sigma_{ka}, \sigma_{ci}) - \tilde{g}(\sigma_{ki}, \sigma_{ca})\},\\
4R_{kci^*a} &= -c\tilde{S}_{ca} - 3P_{ki^*}\delta_{ca} + P_{ki}S_{ca} + \tilde{P}_{ca}\delta_{ki}\\
&\quad + 4\{\tilde{g}(\sigma_{ka}, \sigma_{ci^*}) - \tilde{g}(\sigma_{ki^*}, \sigma_{ca})\},\\
4R_{k^*ci^*a} &= -c\delta_{ki}\delta_{ca} - 3(P_{ki}\delta_{ca} + P_{ca}\delta_{ki}) - P_{ki^*}S_{ca}\\
&\quad + 4\{g(\sigma_{k^*a}, \sigma_{ci^*}) - \tilde{g}(\sigma_{k^*i^*}, \sigma_{ca})\},\\
4R_{kcba} &= 3(P_{ka}\delta_{cb} - P_{kb}\delta_{ca}) - P_{k^*a}S_{ba} + P_{k^*b}S_{cb}\\
&\quad + 2P_{k^*c}S_{ba} + 4\{\tilde{g}(\sigma_{ka}, \sigma_{cb}) - \tilde{g}(\sigma_{kb}, \sigma_{ca})\},\\
4R_{k^*cba} &= 3(P_{k^*a}\delta_{cb} - P_{k^*b}\delta_{ca}) - P_{ka}S_{cb} + P_{kb}S_{ca}\\
&\quad - 2P_{kc}S_{ba} + 4\{\tilde{g}(\sigma_{k^*a}, \sigma_{cb}) - \tilde{g}(\sigma_{k^*b}, \sigma_{ca})\},\\
4R_{dcba} &= c(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) + 3(P_{da}\delta_{cb} + P_{cb}\delta_{da} - P_{db}\delta_{ca} - P_{ca}\delta_{db})\\
&\quad + c(S_{da}S_{cb} - S_{db}S_{ca} + 2S_{dc}S_{ba}) - \tilde{P}_{da}S_{cb} - \tilde{P}_{cb}S_{da} + \tilde{P}_{db}S_{ca}\\
&\quad + \tilde{P}_{ca}S_{db} + 2(\tilde{P}_{dc}S_{ba} + \tilde{P}_{ba}S_{dc}) + 4\{\tilde{g}(\sigma_{da}, \sigma_{cb}) - \tilde{g}(\sigma_{db}, \sigma_{ca})\},
\end{aligned}
$$

for any $k, j, i, h \in \{1, 2, ..., p\}$ and $d, c, b, a \in \{1, 2, ..., q\}$.

Using (4.1), we calculate the length of the second fundamental form and the mean curvature.

The mean curvature tensor H and the norm of the mean curvature $\|H\|^2$ are respectively given by

$$
H = \frac{1}{n}\sum_{\mu=1}^{n}\sigma_{\mu\mu}, \quad \|H\|^2 = \frac{1}{n^2}\sum_{\mu,\lambda=1}^{n}\tilde{g}(\sigma_{\mu\mu}, \sigma_{\lambda\lambda}) \tag{4.2}
$$

and the length $\|\sigma\|$ of the second fundamental form σ is given by

$$
\|\sigma\|^2 = \sum_{\mu,\lambda=1}^{n}\tilde{g}(\sigma_{\mu\lambda}, \sigma_{\mu\lambda}) = \sum_{\mu,\lambda=1}^{n}\sum_{r=n+1}^{m}\{\tilde{g}(\sigma_{\mu\lambda}, e_r)\}^2 \tag{4.3}
$$

for any orthonormal frame of M.

From the Gauss equation, we have

$$
R_{\mu\lambda\mu\lambda} = \tilde{R}_{\mu\lambda\mu\lambda} + \tilde{g}(\sigma_{\mu\lambda}, \sigma_{\mu\lambda}) - \tilde{g}(\sigma_{\mu\mu}, \sigma_{\lambda\lambda}). \tag{4.4}
$$

So we have

$$
\sum_{\mu,\lambda=1}^{n}(R_{\mu\lambda\mu\lambda} - \tilde{R}_{\mu\lambda\mu\lambda}) = \|\sigma\|^2 - n^2\|H\|^2. \tag{4.5}
$$

The equation $\sum_{\mu,\lambda=1}^{n} R_{\mu\lambda\mu\lambda}$ is decomposed in

$$
\begin{aligned}
\sum_{\mu,\lambda=1}^{n} R_{\mu\lambda\mu\lambda} &= \sum_{j,i=1}^{p}\{R_{jiji} + 2R_{ji^*ji^*} + R_{j^*i^*j^*i^*}\}\\
&\quad +2\sum_{j=1}^{p}\sum_{a=1}^{q}(R_{jaja} + R_{j^*aj^*a}) + \sum_{b,a=1}^{q} R_{baba}.
\end{aligned}
$$

Using (3.5) and (4.1), we obtain

$$
\begin{aligned}
4(R_{jiji} - \tilde{R}_{jiji}) &= 4R_{jiji} - c(\delta_{ji}\delta_{ji} - \delta_{jj}\delta_{ii}) - 6(P_{ji}\delta_{ji} - P_{jj}\delta_{ii}),\\
4(R_{ji^*ji^*} - \tilde{R}_{ji^*ji^*}) &= 4R_{ji^*ji^*} + c(\delta_{jj}\delta_{ii} + 3\delta_{ji}\delta_{ji})\\
&\quad + 6(P_{jj}\delta_{ii} - P_{ji}\delta_{ji}),
\end{aligned}
$$

$$\begin{aligned}
4(R_{j^*i^*j^*i^*} - \tilde{R}_{j^*i^*j^*i^*}) &= 4R_{j^*i\cdot j^*i^*} - c(\delta_{ji}\delta_{ji} - \delta_{jj}\delta_{ii}) \\
&\quad - 6(P_{ji}\delta_{ji} - P_{jj}\delta_{ii}), \\
4(R_{jaja} - \tilde{R}_{jaja}) &= 4R_{jaja} + c\delta_{jj}\delta_{aa} + 3(P_{jj}\delta_{aa} + P_{aa}\delta_{jj}), \\
4(R_{j^*aj^*a} - \tilde{R}_{j^*aj^*a}) &= 4R_{j^*aj^*a} + c\delta_{jj}\delta_{aa} + 3(P_{jj}\delta_{aa} + P_{aa}\delta_{jj}), \\
4(R_{baba} - \tilde{R}_{baba}) &= 4R_{baba} - c(\delta_{ba}\delta_{ba} - \delta_{bb}\delta_{aa}) - 6(P_{ba}\delta_{ba} - P_{bb}\delta_{aa}) \\
&\quad - c(S_{ba})^2 - 6\tilde{P}_{ba}S_{ba},
\end{aligned} \tag{4.6}$$

for any $j, i \in \{1, 2, ..., p\}$ and $b, a \in \{1, 2, ..., q\}$. From (4.6), we can easily obtain

$$\begin{aligned}
4\sum_{j,i=1}^{p}(R_{jiji} - \tilde{R}_{jiji}) &= 4\sum_{j,i=1}^{p} R_{jiji} + cp(p-1) + 6(p-1)\sum_{j=1}^{p} P_{jj}, \\
4\sum_{j,i=1}^{p}(R_{ji^*ji^*} - \tilde{R}_{ji^*ji^*}) &= 4\sum_{j,i=1}^{p} R_{ji^*ji^*} + cp(p+3) + 6(p-1)\sum_{j=1}^{p} P_{jj}, \\
4\sum_{j,i=1}^{p}(R_{j^*i^*j^*i^*} - \tilde{R}_{j^*i^*j^*i^*}) &= 4\sum_{j,i=1}^{p} R_{j^*i^*j^*i^*} + cp(p-1) \\
&\quad + 6(p-1)\sum_{j=1}^{p} P_{jj}, \\
4\sum_{j=1}^{p}\sum_{a=1}^{q}(R_{jaja} - \tilde{R}_{jaja}) &= 4\sum_{j=1}^{p}\sum_{a=1}^{q} R_{jaja} + cpq \\
&\quad + 3(q\sum_{j=1}^{p} P_{jj} + p\sum_{a=1}^{q} P_{aa}), \\
4\sum_{j=1}^{p}\sum_{a=1}^{q}(R_{j^*aj^*a} - \tilde{R}_{j^*aj^*a}) &= 4\sum_{j=1}^{p}\sum_{a=1}^{q} R_{j^*aj^*a} + cpq \\
&\quad + 3(q\sum_{j=1}^{p} P_{jj} + p\sum_{a=1}^{q} P_{aa}), \\
4\sum_{b,a=1}^{q}(R_{baba} - \tilde{R}_{baba}) &= 4\sum_{b,a=1}^{q} R_{baba} + cq(q-1) + 6(q-1)\sum_{b=1}^{q} P_{bb} \\
&\quad - c\sum_{b,a=1}^{q}(S_{ba})^2 - 6\sum_{b,a=1}^{q}\tilde{P}_{ba}S_{ba}.
\end{aligned} \tag{4.7}$$

By virtue of (4.5) and the above equations, we obtain

$$-4r + (n^2 + 4p - q)c + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} - 6\sum_{j=1}^{2p} P_{jj} \tag{4.8}$$

$$-c\sum_{b,a=1}^{q}(S_{ba})^2 - 6\sum_{b,a=1}^{q}\tilde{R}_{ba}S_{ba} = 4\{\|\sigma\|^2 - n^2\|H\|^2\},$$

where r means the scalar curvature with respect to the induced metric on M.

Thus we have the following theorem:

Theorem 4.1. *For a semi-slant submanifold in an l.c.K.-space form $\tilde{M}(c)$, we have*

(**I**) *The squared norm of the second fundamental form satisfies the following inequality*

$$\|\sigma\|^2 \geq \frac{1}{4}\{-4r + (n^2 + 4p - q)c + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} \tag{4.9}$$

$$-6\sum_{j=1}^{2p} P_{jj} - c\sum_{b,a=1}^{q} (S_{ba})^2 - 6\sum_{b,a=1}^{q} \tilde{P}_{ba} S_{ba}\}.$$

In particular, if the equality case of (4.9) *holds, that the submanifold is minimal, i.e.* $H = 0$.

(**II**) *The squared norm of the mean curvature vector field satisfies the following inequality*

$$\|H\|^2 \geq \frac{1}{4n^2}\{4r - c(n^2 + 4p - q) + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} \tag{4.10}$$

$$+6\sum_{j=1}^{2p} P_{jj} + c\sum_{b,a=1}^{q} (S_{ba})^2 + 6\sum_{b,a=1}^{q} \tilde{P}_{ba} S_{ba}\}.$$

In particular, if the equality case of (4.10) *holds, then the submanifold is totally geodesic and the scalar curvature* r *satisfies the following relation:*

$$r = \frac{1}{4}\{(n^2 + 4p - q)c + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} \tag{4.11}$$

$$-6\sum_{j=1}^{2p} P_{jj} - c\sum_{b,a=1}^{q} (S_{ba})^2 - 6\sum_{b,a=1}^{q} \tilde{P}_{ba} S_{ba}\}.$$

If our submanifold is CR, then $Te_a = 0$ for any $a \in \{1, 2, ..., q\}$. Thus we have the following result:

Corollary 4.2. *In a CR-submanifold in an l.c.K.-space form* $\tilde{M}(c)$, *we have*

(**I**) *the squared norm of the second fundamental form* σ *satisfies the following inequality*

$$\|\sigma\|^2 \geq \frac{1}{4}\{-4r + (n^2 + 4p - q)c + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} - 6\sum_{j=1}^{2p} P_{jj}\}. \tag{4.12}$$

In particular, if the equality case of (4.12) *holds then the submanifold is minimal, i.e.* $\|H\| = 0$.

(**II**) *The squared norm of the mean curvature vector field satisfies the inequality*

$$\|H\|^2 \geq \frac{1}{4n^2}\{4r - c(n^2 + 4p - q) + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} + 6\sum_{j=1}^{2p} P_{jj}\}. \tag{4.13}$$

In particular, if the equality case of (4.13) *holds, then the submanifold is totally geodesic and the scalar curvature* r *satisfies the following relation*

$$r = \frac{1}{4}\{(n^2 + 4p - q)c + 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} - 6\sum_{j=1}^{2p} P_{jj}\}. \tag{4.14}$$

5. The Codazzi equations

By virtue of (2.4) and (4.5), the Codazzi equation is written as

$$
\begin{aligned}
&3(P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki}) = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{ji}, e_a^*) - \tilde{g}((\bar{\nabla}_j\sigma)_{ki}, e_a^*)\},\\
&P_{j^*a^*}\delta_{ki} - P_{k^*a^*}\delta_{ji} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{ji^*}, e_a^*) - \tilde{g}((\bar{\nabla}_j\sigma)_{ki^*}, e_a^*)\},\\
&-3P_{j^*a^*}\delta_{ki} + P_{k^*a^*}\delta_{ji} + 2P_{i^*a^*}\delta_{kj} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*i}, e_a^*)\\
&\qquad - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{ki}, e_a^*)\},\\
&3P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki} - 2P_{ia^*}\delta_{kj} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*i^*}, e_a^*)\\
&\qquad - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{ki^*}, e_a^*)\},\\
&3(P_{k^*a^*}\delta_{ji} - P_{j^*a^*}\delta_{ki}) = 4\{\tilde{g}((\bar{\nabla}_{k^*}\sigma)_{j^*i^*}, a^*) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{k^*i^*}, e_a^*)\},\\
&P_{k^*j}(FS)_{ba} = 2\{\tilde{g}((\bar{\nabla}_k\sigma)_{jb}, a^*) - \tilde{g}((\bar{\nabla}_j\sigma)_{kb}, e_a^*)\},\\
&(P_{kj} - c\delta_{kj})(FS)_{ba} + \tilde{P}_{ba^*}\delta_{kj} = 2\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*b}, e_a^*) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{kb}, e_a^*)\},\\
&3P_{ka^*}\delta_{cb} - P_{k^*a^*}S_{cb} + P_{k^*b}(FS)_{ca} + 2P_{k^*c}(FS)_{ba}\\
&\qquad = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{cb}, e_a^*) - \tilde{g}((\bar{\nabla}_c\sigma)_{kb}, e_a^*)\},\\
&3P_{k^*a^*}\delta_{cb} + P_{ka^*}S_{cb} - P_{kb}\tilde{g}(FS)_{ca} - 2P_{kc}(FS)_{ba}\\
&\qquad = 4\{\tilde{g}((\bar{\nabla}_{k^*}\sigma)_{cb}, e_a^*) - \tilde{g}((\bar{\nabla}_c\sigma)_{k^*b}, e_a^*)\},\\
&c\{(FS)_{da}S_{cb} - S_{db}(FS)_{ca} - 2S_{dc}(FS)_{ba}\} + 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db})\\
&\qquad - \tilde{P}_{da^*}S_{cb} - \tilde{P}_{cb}(FS)_{da} + \tilde{P}_{db}(FS)_{ca} + \tilde{P}_{ca^*}S_{db} + 2(\tilde{P}_{cd}(FS)_{ba}\\
&\qquad + \tilde{P}_{ba^*}S_{dc}) = 4\{\tilde{g}((\bar{\nabla}_d\sigma)_{cb}, e_a^*) - \tilde{g}((\bar{\nabla}_c\sigma)_{db}, e_a^*)\},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&3(P_{k\alpha}\delta_{ji} - P_{j\alpha}\delta_{ki}) = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{ji}, e_\alpha - \tilde{g}((\bar{\nabla}_j\sigma)_{ki}, e_\alpha)\},\\
&P_{k\alpha^*}\delta_{ji} - P_{j\alpha^*}\delta_{ki} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{ji^*}, e_\alpha) - \tilde{g}((\bar{\nabla}_j\sigma)_{ki^*}, e_\alpha)\},\\
&3P_{j\alpha^*}\delta_{ki} - 2P_{i\alpha^*}\delta_{kj} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*i}, e_\alpha) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{ki}, e_\alpha)\},\\
&3P_{k\alpha}\delta_{ji} - P_{j\alpha}\delta_{ki} - 2P_{i\alpha}\delta_{kj} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*i^*}, e_\alpha) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{ki^*}, e_\alpha)\},\\
&3(P_{j\alpha^*}\delta_{ki} - P_{k\alpha^*}\delta_{ji}) = 4\{\tilde{g}((\bar{\nabla}_{k^*}\sigma)_{j^*i^*}, e_\alpha) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{k^*i^*}, e_\alpha)\},\\
&3P_{k\alpha^*}\delta_{ki} - P_{j\alpha^*}\delta_{ki} - 2P_{i\alpha^*}\delta_{kj} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*i^*}, e_\alpha^*)\\
&\qquad - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{ki^*}, e_\alpha^*)\},\\
&\tilde{g}((\bar{\nabla}_k\sigma)_{ja}, e_\alpha^*) - \tilde{g}((\bar{\nabla}_j\sigma)_{ka}, e_\alpha^*) = 0,\\
&P_{a\alpha}\delta_{kj} = 2\{\tilde{g}((\bar{\nabla}_k\sigma)_{j^*a}, e_\alpha^*) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{ka}, e_\alpha^*)\},\\
&\tilde{g}(\bar{\nabla}_{k^*}\sigma)_{j^*a}, e_\alpha^*) - \tilde{g}((\bar{\nabla}_{j^*}\sigma)_{k^*a}, e_\alpha^*) = 0,\\
&3P_{k\alpha^*}\delta_{ba} - P_{k\alpha^*}S_{ba} = 4\{\tilde{g}((\bar{\nabla}_k\sigma)_{ba}, e_\alpha^*)\\
&\qquad - \tilde{g}((\bar{\nabla}_b\sigma)_{ka}, e_\alpha^*)\},\\
&3P_{k\alpha}\delta_{ba} + P_{k\alpha^*}S_{ba} = 4\{\tilde{g}((\bar{\nabla}_{k^*}\sigma)_{ba}, e_\alpha^*) - \tilde{g}((\bar{\nabla}_b\sigma)_{k^*a}, e_\alpha^*)\},\\
&3(P_{c\alpha^*}\delta_{ba} - P_{b\alpha^*}\delta_{ca}) - P_{c\alpha}S_{ba} + P_{b\alpha}S_{ca} - 2P_{a\alpha}S_{cb}\\
&\qquad = 4\{\tilde{g}((\bar{\nabla}_c\sigma)_{ba}, e_\alpha^*) - \tilde{g}((\bar{\nabla}_b\sigma)_{ca}, e_\alpha^*)\}.
\end{aligned}
\tag{5.2}
$$

Let the second fundamental form σ be parallel. Then we have from (5.1),

$$
\begin{aligned}
&P_{ka^*} = P_{k^*a^*} = 0, \quad P_{k^*j}F_b{}^c\delta_{ca} = 0,\\
&(P_{kj} - c\delta_{kj})(FS)_{ba} + \{S_b{}^cP_{ca^*} + (FS)_b{}^cP_{c^*a^*}\}\delta_{kj} = 0,\\
&P_{k^*b}(FS)_{ca} + 2P_{k^*c}(FS)_{ba} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&P_{kb}(FS)_{ca} + 2P_{kc}(FS)_{ba} = 0, \\
&c\{(FS)_{da}S_{cb} - S_{db}(FS)_{ca} - 2S_{dc}(FS)_{ba}\} \\
&+ 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) - \{S_d{}^e P_{ea^*} + (FS)_d{}^e P_{e^*a^*})T_{cb} \\
&- \{S_c{}^e P_{eb} + (FS)_c{}^e P_{e^*b}\}F_{da} + \{S_d{}^e P_{eb} + (FS)_d{}^e P_{e^*b}\}(FS)_{ca} \\
&+ \{S_c{}^e P_{ea^*} + (FS)_c{}^e P_{e^*a^*}\}S_{db} + 2\{S_d{}^e P_{ec} + (FS)_d{}^e P_{e^*c}\}F_{ba} \\
&+ \{S_b{}^e P_{ea^*} + (FS)_b{}^e P_{e^*a^*}\}S_{dc} = 0, \\
&P_{k\alpha} = P_{k\alpha^*} = P_{a\alpha} = P_{a\alpha^*} = 0,
\end{aligned} \tag{5.3}
$$

for any $k, j \in \{1, 2, ..., p\}$, $b, a \in \{1, 2, ..., q\}$ and $\alpha \in \{1, 2, ..., s\}$.

The 6 equations in (5.2) are denoted by $(5.2)_i$ for $i = 1, \dots, 6$. The equation $(5.2)_2$ means $P_{j^*i} = P_{ji^*} = 0$.

Next, we have from $(5.2)_3$

$$
\tilde{P}_{ba^*} = S_b{}^c P_{ca^*} + (FS)_b{}^c P_{c^*a^*} = -\frac{1}{p}(\sum_{i=1}^{p} P_{ii} - cp)(FS)_{ba}. \tag{5.4}
$$

Substituting the above equation into $(5.2)_3$, we get

$$
P_{kj} = \gamma\delta_{kj}, \tag{5.5}
$$

where we put $\gamma = \frac{1}{p}\sum_{i=1}^{p} P_{ii}$.

From $(5.2)_4$, we can easily get

$$
P_{i^*a} = 0. \tag{5.6}
$$

The equation $(5.2)_5$ means $P_{ia} = P_{ai} = 0$.

Substituting (5.3) into $(5.2)_6$, we obtain

$$
\begin{aligned}
\gamma\{(FS)_{da}S_{cb} - S_{db}(FS)_{ca} - 2S_{dc}(FS)_{ba}\} + 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) \\
-\tilde{P}_{cb}(FS)_{da} + \tilde{P}_{db}(FS)_{ca} + 2\tilde{P}_{dc}(FS)_{ba} = 0,
\end{aligned} \tag{5.7}
$$

and, we get

$$
\tilde{P}_{ba} = \gamma S_{ba} - \frac{3}{2(q+1)}\{P_{be^*}(FS)_a{}^e - P_{ae^*}(FS)_b{}^e\}. \tag{5.8}
$$

Substituting (5.7) into (5.6), we have

$$
\begin{aligned}
P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db} = -\frac{1}{2(q+1)}[\{P_{ce^*}(FS)_b{}^e - P_{be^*}(FS)_c{}^e\}(FS)_{da} \\
-\{P_{de^*}(FS)_b{}^e - P_{be^*}(FS)_d{}^e\}(FS)_{ca} - 2\{P_{de^*}(FS)_c{}^e - P_{ce^*}(FS)_d{}^e\}(FS)_{ba}].
\end{aligned} \tag{5.9}
$$

Contraction of (5.8) by b and c gives us

$$
P_{ba^*} = -\frac{3}{2q^2 - 5} P_{de^*}(FS)_b{}^e (FS)^d{}_a. \tag{5.10}
$$

The above equation means $P_{ba^*} = P_{b^*a} = 0$ for any $b, a \in \{1, 2, ..., q\}$.

Therefore, we have the following theorem:

Theorem 5.1. *If a semi-slant submanifold in an l.c.K.-space form $\tilde{M}(c)$ has a parallel second fundamental form, then the tensor field $P_{\mu\lambda}$ satisfies*

$$
(P_{\mu\lambda}) = \begin{pmatrix}
P_{ji} & P_{ji^*} & P_{ja} & P_{ja^*} & P_{j\alpha} & P_{j\alpha^*} \\
P_{j^*i} & P_{j^*i^*} & P_{j^*a} & P_{j^*a^*} & P_{j^*\alpha} & P_{j^*\alpha^*} \\
P_{bi} & P_{bi^*} & P_{ba} & P_{ba^*} & P_{b\alpha} & P_{b\alpha^*} \\
P_{b^*i} & P_{b^*i^*} & P_{b^*a} & P_{b^*a^*} & P_{b^*\alpha} & P_{b^*\alpha^*} \\
P_{\beta i} & P_{\beta i^*} & P_{\beta^a} & P_{\beta a^*} & P_{\beta\alpha} & P_{\beta\alpha^*} \\
P_{\beta^*i} & P_{\beta^*i^*} & P_{\beta^*a} & P_{\beta^*a^*} & P_{\beta^*\alpha} & P_{\beta^*\alpha^*}
\end{pmatrix} \tag{5.11}
$$

$$= \begin{pmatrix} \gamma\delta_{ji} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma\delta_{ji} & 0 & 0 & 0 & 0 \\ 0 & 0 & P_{ba} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{b^*a^*} & P_{b^*\alpha} & P_{b^*\alpha^*} \\ 0 & 0 & 0 & P_{\beta a^*} & P_{\beta\alpha} & P_{\beta\alpha^*} \\ 0 & 0 & 0 & P_{\beta^* a^*} & P_{\beta^*\alpha} & P_{\beta\alpha} \end{pmatrix},$$

where we put

$$\gamma = \frac{1}{p}\sum_{i=1}^{p} P_{ii}. \tag{5.12}$$

In particular, for a CR-submanifold in the l.c.K.-space form $\tilde{M}(c)$, we know $T = 0$, $Fe_a{}^* = e_a^*$, $\tilde{P}_{ba} = P_{b^*a}$ and $\tilde{P}_{ba^*} = P_{ba}$ for any $a, b \in \{1, 2, ..., q\}$. Moreover, (5.3) can be written as

$$P_{ba} = -\frac{1}{p}(\sum_{i=1}^{p} P_{ii} - cp)\delta_{ba}. \tag{5.13}$$

In $(5.2)_5$, we put $T = 0$ and $(F_b{}^a) = I_q$, and we can easily obtain

$$P_{ba^*} = 0. \tag{5.14}$$

Then, we have the following result:

Corollary 5.2. *If a CR-submanifold in an l.c.K.-space form $\tilde{M}(c)$ has a parallel second fundamental form, then $T = 0$ and $Fe_a = e_a^*$ for any $a \in \{1, 2, ..., q\}$. So, the tensor field $P_{\mu\lambda}$ satisfies the following condition*

$$(P_{\mu\lambda}) = \begin{pmatrix} P_{ji} & P_{ji^*} & P_{ja} & P_{ja^*} & P_{j\alpha} & P_{j\alpha^*} \\ P_{j^*i} & P_{j^*i^*} & P_{j^*a} & P_{j^*a^*} & P_{j^*\alpha} & P_{j^*\alpha^*} \\ P_{bi} & P_{bi^*} & P_{ba} & P_{ba^*} & P_{b\alpha} & P_{b\alpha^*} \\ P_{b^*i} & P_{b^*i^*} & P_{b^*a} & P_{b^*a^*} & P_{b^*\alpha} & P_{b^*\alpha^*} \\ P_{\beta i} & P_{\beta i^*} & P_{\beta^a} & P_{\beta a^*} & P_{\beta\alpha} & P_{\beta\alpha^*} \\ P_{\beta^*i} & P_{\beta^*i^*} & P_{\beta^*a} & P_{\beta^*a^*} & P_{\beta^*\alpha} & P_{\beta^*\alpha^*} \end{pmatrix} \tag{5.15}$$

$$= \begin{pmatrix} \gamma\delta_{ji} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma\delta_{ji} & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\gamma - c)\delta_{ba} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\gamma - c)\delta_{ba} & P_{b^*\alpha} & P_{b^*\alpha^*} \\ 0 & 0 & 0 & P_{\beta a^*} & P_{\beta\alpha} & P_{\beta\alpha^*} \\ 0 & 0 & 0 & P_{\beta^*a^*} & P_{\beta^*\alpha} & P_{\beta\alpha} \end{pmatrix}.$$

Acknowledgment. This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA-INDAM).The authors thank the National Group and the referees for their very useful suggestions.

References

[1] P. Alegre, *Slant submanifolds of Lorentzian Sasakian and Para Sasakian manifold*, Taiwan. J. Math. **17** (3), 897 - 910, 2013.

[2] A. Bejancu, *CR-submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc. **69**, 134 -142, 1978.

[3] A. Bejancu, *CR-submanifolds of a Kaehler manifold II*, Trans. Amer. Math. Soc. **250**, 333-345, 1979.

[4] A. Bejancu, *Geomerty of CR-submanifolds*, D. Reidel Publishing Company, 1986.

[5] J.L. Cabrerizo, A. Carriazo, L. M. Fernández, M. Fernández, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J. **42** (1), 125 -138, 2000.

[6] J.L. Cabrerizo, A. Carriazo, L. M. Fernández, M. Fernández, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata **78**, 183-199, 1999.
[7] B.Y. Chen, *CR-submanifolds of a Kaehler manifold I*, J. of Differ. Geom. **16**, 305-322, 1981.
[8] B.Y. Chen, *CR-submanifolds of a Kaehler manifold II* , J. of Differ. Geom. **16**, 493-509, 1981.
[9] B.Y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York, 1973.
[10] B.Y. Chen, *Geometry of slant submanifolds*, Katholieke Univ. Leuven, 1990.
[11] B.Y. Chen, *Slant immersions*, B. Aust. Math. Soc. **41**, 135-147, 1990.
[12] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Prog. Math. **155**, Birkhäuser Boston, Inc., Boston, MA, 1998.
[13] T. Kashiwada, *Some properties of locally conformal Kähler manifolds*, Hokkaido Math. J. **8**, 191-198, 1979.
[14] V.A. Khan, M. A. Khan, *Semi-slant submanifolds of a nearly Kaehler manifold*, Turk. J. Math. **31**, 341-353, 2007.
[15] K. Matsumoto, *On CR-submanifolds of locally conformal Kähler manifolds I*, J. Korean Math. Soc. **21**, 49-61, 1984.
[16] K. Matsumoto, *On CR-submanifolds of locally conformal Kähler manifolds II*, Tensor N. S. **45**, 144-150, 1987.
[17] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. St. Univ. Al. I. Cuza, Iasi, **40**, 55-61, 1994.
[18] B. Sahin, *Slant submanifolds of an almost product Riemannian manifold*, J. Korean Math. Soc. **43**, 717-732, 2006.
[19] B. Sahin, *Warped product submanifolds of Kaehler manifolds with a slant factor*, Ann. Pol. Math. **95**, 207-226, 2009.
[20] H.M. Taştan, S. Gerdan, *Hemi-slant submanifolds of a locally conformal Kähler manifold*, Int. Electron. J. Geom. **8** (2), 46-56, 2015.
[21] H.M. Taştan, M. M. Tripathi, *Semi-slant submanifolds of a locally conformal Kähler manifold*, Ann. Şt. Al. I. Cuza Univ. Iaşi, (N.S.), **62** (2), 337-347, 2016.
[22] H.M. Taştan, S. Gerdan Aydin, *Hemi-slant and Semi-slant submanifolds of Locally Conformal Kähler Manifolds*, in: B.Y. Chen, M.H. Shahid, F. Al Solamy (eds) Complex Geometry of Slant Submanifolds, Springer, Singapore, 2022.
[23] I. Vaisman, *Locally conformal almost Kähler manifolds*, Isr. J. Math. **24**, 338-351, 1976.